MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #20

LAST TIME

- (BI)MODULES IN $(\mathcal{C}, \otimes, \mathbb{1})$
- MONADS & EM CATEGORIES

TOPICS:

I. OPERATIONS ON ALGEBRAS & (BI)MODULES (§4.5)

II. GENERALIZED EILENBERG-WATTS THEOREM (§4.7.1)

III. GENERALIZED MORITA'S THEOREM (§4.7.2)

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

GIVEN A MONOIDAL CATEGORY $(\mathscr{C}, \otimes, \mathbb{1})$,

AN ALGEBRA IN $\mathscr{C}$ IS $(A,\ m : A \otimes A \to A,\ u : \mathbb{1} \to A)$

OBJECT IN $\mathscr{C}$ MORPHISMS IN $\mathscr{C}$

SATISFYING ASSOCIATIVITY + UNITALITY AXIOMS

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

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AN **ALGEBRA IN $\mathcal{C}$** IS $(A,\ m: A \otimes A \to A,\ u: \mathbb{1} \to A)$

OBJECT IN $\mathcal{C}$ — MORPHISMS IN $\mathcal{C}$

SATISFYING ASSOCIATIVITY + UNITALITY AXIOMS

A **LEFT A-MODULE IN $\mathcal{C}$** IS $(\underset{\mathcal{C}}{\underset{\in}{M}},\ \underset{\mathrm{Hom}_{\mathcal{C}}(A \otimes M, M)}{\underset{\in}{\triangleright}}: A \otimes M \to M)$

SAT. ASSOC. + UNIT. AXIOMS

A **RIGHT A-MODULE IN $\mathcal{C}$** IS $(\underset{\mathcal{C}}{\underset{\in}{M}},\ \underset{\mathrm{Hom}_{\mathcal{C}}(M \otimes A, M)}{\underset{\in}{\triangleleft}}: M \otimes A \to M)$

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AN **ALGEBRA IN $\mathcal{C}$** IS $(A,\ m: A\otimes A \longrightarrow A,\ u: \mathbb{1} \longrightarrow A)$

OBJECT IN $\mathcal{C}$ MORPHISMS IN $\mathcal{C}$

SATISFYING ASSOCIATIVITY + UNITALITY AXIOMS

A **LEFT A-MODULE IN $\mathcal{C}$** IS $(\underset{\mathcal{C}}{\underset{\cap}{M}},\ \underset{\mathrm{Hom}_{\mathcal{C}}(A\otimes M, M)}{\underset{\cap}{\triangleright}}: A\otimes M \longrightarrow M)$

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SAT. ASSOC. + UNIT. AXIOMS

A **(B_1, B_2)-BIMODULE IN $\mathcal{C}$** IS $(M,\ \triangleright: B_1\otimes M \to M,\ \triangleleft: M\otimes B_2 \to M)$

s.t. $(M, \triangleright) \in B_1\text{-Mod}(\mathcal{C})$, $(M, \triangleleft) \in \text{Mod-}B_2(\mathcal{C})$

& $\triangleleft(\triangleright \otimes id_{B_2}) = \triangleright(id_{B_1} \otimes \triangleleft)\, a_{B_1, M, B_2}$

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

GIVEN A MONOIDAL CATEGORY $(\mathcal{C}, \otimes, \mathbb{1})$,

WANT TO CREATE MORE!

AN ALGEBRA IN $\mathcal{C}$ IS $(A,\ m: A\otimes A \to A,\ u: \mathbb{1} \to A)$

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A LEFT A-MODULE IN $\mathcal{C}$ IS $(M,\ \triangleright: A\otimes M \to M)$

$M \in \mathcal{C}$, $\triangleright \in \mathrm{Hom}_{\mathcal{C}}(A\otimes M, M)$

SAT. ASSOC. + UNIT. AXIOMS

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SAT. ASSOC. + UNIT. AXIOMS

A (B_1, B_2)-BIMODULE IN $\mathcal{C}$ IS $(M,\ \triangleright: B_1\otimes M \to M,\ \triangleleft: M\otimes B_2 \to M)$

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I. OPERATIONS ON ALGEBRAS & (BI)MODULES

ASSUME $(\mathcal{C}, \otimes, \mathbb{1})$ ABELIAN MONOIDAL

TAKE MON. CAT. $(\mathcal{C}, \otimes, \mathbb{1})$.

ALGEBRA IN $\mathcal{C}$ $\equiv$ $(A,\ m: A\otimes A \to A,\ u: \mathbb{1} \to A)$ SAT. ASSOC + UNITALITY

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... VIA BIPRODUCTS

TAKE (A_1, m_1, u_1), $(A_2, m_2, u_2) \in \mathsf{Alg}(\mathcal{C})$.

DEF: $(A_1, m_1, u_1) \,\square\, (A_2, m_2, u_2)$

$$:= (A_1 \,\square\, A_2,\ m_\square,\ u_\square) \in \mathsf{Alg}(\mathcal{C})$$

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VIA

$$\begin{aligned}
m_\square :\ & (A_1 \,\square\, A_2) \otimes (A_1 \,\square\, A_2) \\
& \quad \Big\downarrow \wr \\
& (A_1 \otimes A_1) \,\square\, (A_1 \otimes A_2) \,\square\, (A_2 \otimes A_1) \,\square\, (A_2 \otimes A_2) \\
& \quad \Big\downarrow {\scriptstyle m_1 \,\square\, \vec{0} \,\square\, \vec{0} \,\square\, m_2} \\
& A_1 \,\square\, 0 \,\square\, 0 \,\square\, A_2 \\
& \quad \Big\downarrow \wr \\
& A_1 \,\square\, A_2
\end{aligned}$$

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... VIA BIPRODUCTS

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VIA

$$m_\square : (A_1 \square A_2) \otimes (A_1 \square A_2) \xrightarrow{\sim} (A_1 \otimes A_1) \square (A_1 \otimes A_2) \square (A_2 \otimes A_1) \square (A_2 \otimes A_2) \xrightarrow{m_1 \square \vec{0} \square \vec{0} \square m_2} A_1 \square 0 \square 0 \square A_2 \xrightarrow{\sim} A_1 \square A_2$$

UNIV. PROPERTY OF PRODUCTS:

$$\begin{tikzcd}
 & \mathbb{1} \arrow[dl, bend right, "u_1"'] \arrow[d, dashed, "u_\square"] \arrow[dr, bend left, "u_2"] & \\
A_1 & A_1 \square A_2 \arrow[l, "\alpha_1'"] \arrow[r, "\alpha_2'"'] & A_2
\end{tikzcd}$$

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... VIA BIPRODUCTS

TAKE $(A_1, m_1, u_1),\ (A_2, m_2, u_2) \in Alg(\mathcal{C})$.

EXER. 4.36

DEF: $(A_1, m_1, u_1) \,\square\, (A_2, m_2, u_2) := (A_1 \square A_2,\ m_\square,\ u_\square) \in Alg(\mathcal{C})$

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VIA

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... VIA BIPRODUCTS

TAKE $(A,m,u) \in \text{Alg}(\mathcal{C})$ & $(M_1,\triangleright_1), (M_2,\triangleright_2) \in A\text{-Mod}(\mathcal{C})$

DEF: $(M_1,\triangleright_1) \,\Box\, (M_2,\triangleright_2) := (M_1 \,\Box\, M_2,\ \blacktriangleright) \in A\text{-Mod}(\mathcal{C})$

VIA

$$\blacktriangleright : A\otimes (M_1 \,\Box\, M_2) \xrightarrow{\sim} (A\otimes M_1) \,\Box\, (A\otimes M_2) \xrightarrow{\triangleright_1 \,\Box\, \triangleright_2} M_1 \,\Box\, M_2$$

WORKS SIMILARLY ON

RIGHT MODULES & BIMODULES

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... VIA $\otimes$

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$.

TO DEFINE $A \otimes B \in \mathrm{Alg}(\mathcal{C})$, NEED —

$m_{\otimes} : (A \otimes B) \otimes (A \otimes B)$

$$\begin{tikzcd} (A \otimes A) \otimes (B \otimes B) \arrow[r, "m_A \otimes m_B"] & A \otimes B \end{tikzcd}$$

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$$m_\otimes : (A\otimes B)\otimes(A \otimes B) \dashrightarrow (A\otimes A)\otimes(B\otimes B) \xrightarrow{m_A \otimes m_B} A\otimes B$$

USE ASSOCIATIVITY

&

NEED MORPHISM

$B \otimes A \xrightarrow{\sim} A \otimes B$ IN $\mathcal{C}$

... DONE WITH "BRAIDING" ON $\mathcal{C}$ LATER

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... VIA $\otimes$

TAKE $B_1, B_2 \in \mathrm{Alg}(\mathcal{C})$ & $(M, \triangleright) \in B_1\text{-Mod}(\mathcal{C})$, $(N, \triangleleft) \in \text{Mod-}B_2(\mathcal{C})$

DEF: $(M, \triangleright) \otimes (N, \triangleleft) := (M \otimes N,\ \blacktriangleright,\ \blacktriangleleft) \in (B_1, B_2)\text{-Bimod}(\mathcal{C})$

$\blacktriangleright$: ??

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$$\blacktriangleleft : (M\otimes N)\otimes B_2 \xrightarrow[\sim]{a_{M,N,B_2}} M\otimes (N\otimes B_2) \xrightarrow{id \otimes \triangleleft} M\otimes N$$

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... VIA $\otimes_A$

TAKE $A, B_1, B_2 \in \mathrm{Alg}(\mathcal{C})$ & $(M, \triangleright_M, \triangleleft_M) \in (B_1, A)\text{-Bimod}(\mathcal{C})$
$(N, \triangleright_N, \triangleleft_N) \in (A, B_2)\text{-Bimod}(\mathcal{C})$

DEF: $(M, \triangleright_M, \triangleleft_M) \otimes_A (N, \triangleright_N, \triangleleft_N)$
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$M \otimes_A N :=$??

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$$\blacktriangleleft : (M \otimes N) \otimes B_2 \xrightarrow[\sim]{a_{M, N, B_2}} M \otimes (N \otimes B_2) \xrightarrow{id \otimes \triangleleft_N} M \otimes N$$

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$(M,\ \triangleright: A\otimes M\to M)$
SAT. ASSOC + UNITALITY

RIGHT A-MODULE IN $\mathcal{C}$ $\equiv$
$(M,\ \triangleleft: M\otimes A\to M)$
SAT. ASSOC + UNITALITY

(B_1,B_2)-BIMOD. IN $\mathcal{C}$ $\equiv$
$(M,\ \triangleright: B_1\otimes M\to M,$
$\triangleleft: M\otimes B_2\to M)$
s.t. $(M,\triangleright)\in B_1\text{-Mod}(\mathcal{C})$,
$(M,\triangleleft)\in \text{Mod-}B_2(\mathcal{C})$
& $\triangleleft(\triangleright\otimes id_{B_2}) =$
$\triangleright(id_{B_1}\otimes\triangleleft)\,a_{B_1,M,B_2}$

... VIA $\otimes_A$

TAKE $A, B_1, B_2 \in Alg(\mathcal{C})$ & $(M,\triangleright_M,\triangleleft_M)\in(B_1,A)\text{-Bimod}(\mathcal{C})$
$(N,\triangleright_N,\triangleleft_N)\in(A,B_2)\text{-Bimod}(\mathcal{C})$

DEF: $(M,\triangleright_M,\triangleleft_M)\otimes_A(N,\triangleright_N,\triangleleft_N)$
$:= (M\otimes_A N,\ \triangleright,\ \triangleleft)\in(B_1,B_2)\text{-Bimod}(\mathcal{C})$

CF. EXER. 2.11

$$M\otimes_A N := \mathrm{coeq}\left(\begin{tikzcd}(M\otimes A)\otimes N \arrow[r, shift left, "\triangleleft_M\otimes id"] \arrow[r, shift right, "(id\otimes\triangleright_N)\,a_{M,A,N}"'] & M\otimes N\end{tikzcd}\right)$$

$$\triangleright : B_1\otimes(M\otimes N)\xrightarrow[\sim]{a^{-1}_{B_1,M,N}}(B_1\otimes M)\otimes N\xrightarrow{\triangleright_M\otimes id}M\otimes N$$

$$\triangleleft : (M\otimes N)\otimes B_2\xrightarrow[\sim]{a_{M,N,B_2}}M\otimes(N\otimes B_2)\xrightarrow{id\otimes\triangleleft_N}M\otimes N$$

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

ASSUME $(\mathcal{C}, \otimes, \mathbb{1})$ ABELIAN MONOIDAL

TAKE MON. CAT. $(\mathcal{C}, \otimes, \mathbb{1})$.

ALGEBRA IN $\mathcal{C}$ $\equiv$
$(A,\ m: A\otimes A \to A,\ u: \mathbb{1} \to A)$
SAT. ASSOC + UNITALITY

LEFT A-MODULE IN $\mathcal{C}$ $\equiv$
$(M,\ \triangleright: A\otimes M \to M)$
SAT. ASSOC + UNITALITY

RIGHT A-MODULE IN $\mathcal{C}$ $\equiv$
$(M,\ \triangleleft: M\otimes A \to M)$
SAT. ASSOC + UNITALITY

(B_1, B_2)-BIMOD. IN $\mathcal{C}$ $\equiv$
$(M,\ \triangleright: B_1\otimes M \to M,$
$\triangleleft: M\otimes B_2 \to M)$
s.t. $(M, \triangleright) \in B_1\text{-Mod}(\mathcal{C})$,
$(M, \triangleleft) \in \text{Mod-}B_2(\mathcal{C})$
& $\triangleleft(\triangleright \otimes id_{B_2}) =$
$\triangleright(id_{B_1} \otimes \triangleleft)\, a_{B_1,M,B_2}$

... VIA $\otimes_A$

TAKE $A, B_1, B_2 \in \mathrm{Alg}(\mathcal{C})$ & $(M, \triangleright_M, \triangleleft_M) \in (B_1, A)\text{-Bimod}(\mathcal{C})$
$(N, \triangleright_N, \triangleleft_N) \in (A, B_2)\text{-Bimod}(\mathcal{C})$

DEF: $(M, \triangleright_M, \triangleleft_M) \otimes_A (N, \triangleright_N, \triangleleft_N)$
$:= (M \otimes_A N,\ \blacktriangleright,\ \blacktriangleleft) \in (B_1, B_2)\text{-Bimod}(\mathcal{C})$

EXER. 4.37

$$M \otimes_A N := \mathrm{coeq}\left((M\otimes A)\otimes N \underset{(id\otimes \triangleright_N)\, a_{M,A,N}}{\overset{\triangleleft_M \otimes id}{\rightrightarrows}} M\otimes N \right)$$

$$\blacktriangleright : B_1 \otimes (M\otimes N) \xrightarrow[\sim]{a^{-1}_{B_1,M,N}} (B_1 \otimes M)\otimes N \xrightarrow{\triangleright_M \otimes id} M\otimes N$$

$$\blacktriangleleft : (M\otimes N)\otimes B_2 \xrightarrow[\sim]{a_{M,N,B_2}} M\otimes(N\otimes B_2) \xrightarrow{id \otimes \triangleleft_N} M\otimes N$$

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

ASSUME $(\mathcal{C}, \otimes, \mathbb{1})$ ABELIAN MONOIDAL

TAKE MON. CAT. $(\mathcal{C}, \otimes, \mathbb{1})$.

ALGEBRA IN $\mathcal{C}$ ≡ $(A,\ m: A\otimes A \to A,\ u: \mathbb{1} \to A)$ SAT. ASSOC + UNITALITY

LEFT A-MODULE IN $\mathcal{C}$ ≡ $(M,\ \triangleright: A\otimes M \to M)$ SAT. ASSOC + UNITALITY

RIGHT A-MODULE IN $\mathcal{C}$ ≡ $(M,\ \triangleleft: M\otimes A \to M)$ SAT. ASSOC + UNITALITY

(B_1, B_2)-BIMOD. IN $\mathcal{C}$ ≡ $(M,\ \triangleright: B_1 \otimes M \to M,\ \triangleleft: M\otimes B_2 \to M)$

∋. $(M, \triangleright) \in B_1\text{-Mod}(\mathcal{C})$, $(M, \triangleleft) \in \text{Mod-}B_2(\mathcal{C})$

& $\triangleleft(\triangleright \otimes id_{B_2}) = \triangleright(id_{B_1} \otimes \triangleleft)\, a_{B_1, M, B_2}$

... VIA $\otimes_A$

TAKE $A, B_1, B_2 \in \mathrm{Alg}(\mathcal{C})$ & $(M, \triangleright_M, \triangleleft_M) \in (B_1, A)\text{-Bimod}(\mathcal{C})$

$(N, \triangleright_N, \triangleleft_N) \in (A, B_2)\text{-Bimod}(\mathcal{C})$

DEF: $(M, \triangleright_M, \triangleleft_M) \otimes_A (N, \triangleright_N, \triangleleft_N)$

$:= (M \otimes_A N,\ \blacktriangleright,\ \blacktriangleleft) \in (B_1, B_2)\text{-Bimod}(\mathcal{C})$ EXER. 4.37

EXER. 4.38: TAKE $M \in (B_1, \mathbb{1})\text{-Bimod}(\mathcal{C})$,

$N \in (\mathbb{1}, B_2)\text{-Bimod}(\mathcal{C})$.

SHOW: $M \otimes_{\mathbb{1}} N \cong M \otimes N$ ← PREV. CONSTRUCTION

AS (B_1, B_2)-BIMODULES IN $\mathcal{C}$.

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

ASSUME $(\mathcal{C}, \otimes, \mathbb{1})$ ABELIAN MONOIDAL

TAKE MON. CAT. $(\mathcal{C}, \otimes, \mathbb{1})$.

ALGEBRA IN $\mathcal{C}$ $\equiv$
$(A,\ m: A \otimes A \to A,\ u: \mathbb{1} \to A)$
SAT. ASSOC + UNITALITY

LEFT A-MODULE IN $\mathcal{C}$ $\equiv$
$(M,\ \triangleright: A \otimes M \to M)$
SAT. ASSOC + UNITALITY

RIGHT A-MODULE IN $\mathcal{C}$ $\equiv$
$(M,\ \triangleleft: M \otimes A \to M)$
SAT. ASSOC + UNITALITY

(B_1, B_2)-BIMOD. IN $\mathcal{C}$ $\equiv$
$(M,\ \triangleright: B_1 \otimes M \to M,$
$\triangleleft: M \otimes B_2 \to M)$
∋. $(M, \triangleright) \in B_1\text{-Mod}(\mathcal{C})$,
$(M, \triangleleft) \in \text{Mod-}B_2(\mathcal{C})$
& $\triangleleft(\triangleright \otimes id_{B_2}) =$
$\triangleright(id_{B_1} \otimes \triangleleft)\, a_{B_1, M, B_2}$

... VIA $\otimes_A$

TAKE $A, B_1, B_2 \in \text{Alg}(\mathcal{C})$ & $(M, \triangleright_M, \triangleleft_M) \in (B_1, A)\text{-Bimod}(\mathcal{C})$
$(N, \triangleright_N, \triangleleft_N) \in (A, B_2)\text{-Bimod}(\mathcal{C})$

DEF: $(M, \triangleright_M, \triangleleft_M) \otimes_A (N, \triangleright_N, \triangleleft_N)$
$:= (M \otimes_A N,\ \blacktriangleright,\ \blacktriangleleft) \in (B_1, B_2)\text{-Bimod}(\mathcal{C})$

EXER. 4.37

EXER. 4.39: TAKE $M, N, P \in A\text{-Bimod}(\mathcal{C})$.

SHOW:

(a) $(M \otimes_A N) \otimes_A P \cong M \otimes_A (N \otimes_A P)$,

(b) $M \otimes_A A_{reg} \cong M \cong A_{reg} \otimes_A M$,

AS A-BIMODULES IN $\mathcal{C}$.

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

ASSUME $(\mathcal{C}, \otimes, \mathbb{1})$ ABELIAN MONOIDAL

TAKE MON. CAT. $(\mathcal{C}, \otimes, \mathbb{1})$.

ALGEBRA IN $\mathcal{C}$ $\equiv$
$(A,\ m: A \otimes A \to A,\ u: \mathbb{1} \to A)$
SAT. ASSOC + UNITALITY

LEFT A-MODULE IN $\mathcal{C}$ $\equiv$
$(M,\ \triangleright: A \otimes M \to M)$
SAT. ASSOC + UNITALITY

RIGHT A-MODULE IN $\mathcal{C}$ $\equiv$
$(M,\ \triangleleft: M \otimes A \to M)$
SAT. ASSOC + UNITALITY

(B_1, B_2)-BIMOD. IN $\mathcal{C}$ $\equiv$
$(M,\ \triangleright: B_1 \otimes M \to M,$
$\triangleleft: M \otimes B_2 \to M)$
$\ni$. $(M, \triangleright) \in B_1\text{-Mod}(\mathcal{C})$,
$(M, \triangleleft) \in \text{Mod-}B_2(\mathcal{C})$
& $\triangleleft(\triangleright \otimes id_{B_2}) =$
$\triangleright(id_{B_1} \otimes \triangleleft)\, a_{B_1, M, B_2}$

... VIA Hom

... BUT $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ IS NOT NECESSARILY AN OBJECT IN $\mathcal{C}$

SO LET'S NOT CONSIDER THIS OPERATION HERE.

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

TAKE MON. CAT. $(\mathcal{C}, \otimes, \mathbb{1})$.

ALGEBRA IN $\mathcal{C}$ $\equiv$
$(A,\ m: A\otimes A \to A,\ u: \mathbb{1} \to A)$
SAT. ASSOC + UNITALITY

LEFT A-MODULE IN $\mathcal{C}$ $\equiv$
$(M,\ \triangleright: A\otimes M \to M)$
SAT. ASSOC + UNITALITY

RIGHT A-MODULE IN $\mathcal{C}$ $\equiv$
$(M,\ \triangleleft: M\otimes A \to M)$
SAT. ASSOC + UNITALITY

(B_1, B_2)-BIMOD. IN $\mathcal{C}$ $\equiv$
$(M,\ \triangleright: B_1\otimes M \to M,$
$\triangleleft: M\otimes B_2 \to M)$
∋. $(M, \triangleright) \in B_1\text{-Mod}(\mathcal{C})$,
$(M, \triangleleft) \in \text{Mod-}B_2(\mathcal{C})$
& $\triangleleft(\triangleright \otimes id_{B_2}) =$
$\triangleright(id_{B_1} \otimes \triangleleft)\, a_{B_1, M, B_2}$

... VIA DUALS

ASSUME $(\mathcal{C}, \otimes, \mathbb{1})$ ABELIAN, MONOIDAL, RIGID

RECALL $(-)^*,\ {}^*(-): \mathcal{C} \to \mathcal{C}$ ARE CONTRAVARIANT

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

TAKE MON. CAT. $(\mathcal{C}, \otimes, \mathbb{1})$.

ALGEBRA IN $\mathcal{C}$ $\equiv$
$(A,\ m: A \otimes A \to A,\ u: \mathbb{1} \to A)$
SAT. ASSOC + UNITALITY

LEFT A-MODULE IN $\mathcal{C}$ $\equiv$
$(M,\ \triangleright: A \otimes M \to M)$
SAT. ASSOC + UNITALITY

RIGHT A-MODULE IN $\mathcal{C}$ $\equiv$
$(M,\ \triangleleft: M \otimes A \to M)$
SAT. ASSOC + UNITALITY

(B_1, B_2)–BIMOD. IN $\mathcal{C}$ $\equiv$
$(M,\ \triangleright: B_1 \otimes M \to M,$
$\triangleleft: M \otimes B_2 \to M)$
$\ni$. $(M, \triangleright) \in B_1\text{-Mod}(\mathcal{C})$,
$(M, \triangleleft) \in \text{Mod-}B_2(\mathcal{C})$
& $\triangleleft(\triangleright \otimes id_{B_2}) =$
$\triangleright(id_{B_1} \otimes \triangleleft)\, a_{B_1, M, B_2}$

... VIA DUALS

ASSUME $(\mathcal{C}, \otimes, \mathbb{1})$ ABELIAN, RIGID MONOIDAL

RECALL $(-)^*,\ {}^*(-): \mathcal{C} \to \mathcal{C}$ ARE CONTRAVARIANT

GIVEN $(A, m, u) \in \text{Alg}(\mathcal{C})$,
THE OBJECT A^* IS NATURALLY
A "COALGEBRA":

$$\Big(A^*,\quad \underset{\text{"COMULTIPLICATION"}}{\Delta: A^* \longrightarrow A^* \otimes A^*},\quad \underset{\text{"COUNIT"}}{\varepsilon: A^* \longrightarrow \mathbb{1}}\Big)$$

SATISFYING "COASSOCIATIVITY" & "COUNITALITY"

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

ASSUME $(\mathcal{C}, \otimes, \mathbb{1})$ ABELIAN, MONOIDAL, RIGID

TAKE MON. CAT. $(\mathcal{C}, \otimes, \mathbb{1})$.

ALGEBRA IN $\mathcal{C}$ $\equiv$ $(A,\ m: A\otimes A \to A,\ u: \mathbb{1} \to A)$ SAT. ASSOC + UNITALITY

LEFT A-MODULE IN $\mathcal{C}$ $\equiv$ $(M,\ \triangleright: A\otimes M \to M)$ SAT. ASSOC + UNITALITY

RIGHT A-MODULE IN $\mathcal{C}$ $\equiv$ $(M,\ \triangleleft: M\otimes A \to M)$ SAT. ASSOC + UNITALITY

(B_1, B_2)-BIMOD. IN $\mathcal{C}$ $\equiv$ $(M,\ \triangleright: B_1 \otimes M \to M,\ \triangleleft: M\otimes B_2 \to M)$ s.t. $(M, \triangleright) \in B_1\text{-Mod}(\mathcal{C})$, $(M, \triangleleft) \in \text{Mod-}B_2(\mathcal{C})$ & $\triangleleft(\triangleright \otimes id_{B_2}) = \triangleright(id_{B_1} \otimes \triangleleft)\, a_{B_1, M, B_2}$

... VIA DUALS

RECALL $(-)^*, {}^*(-) : \mathcal{C} \to \mathcal{C}$ ARE CONTRAVARIANT

GIVEN $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$, THE OBJECT A^* IS NATURALLY A "COALGEBRA":

$$\left(A^*,\quad \overset{\text{"COMULTIPLICATION"}}{\Delta : A^* \longrightarrow A^* \otimes A^*},\quad \overset{\text{"COUNIT"}}{\varepsilon : A^* \longrightarrow \mathbb{1}}\right)$$

$$\Delta:\ A^* \xrightarrow{m^*} (A\otimes A)^* \xrightarrow{\sim} A^* \otimes A^*, \qquad \varepsilon:\ A^* \xrightarrow{u^*} \mathbb{1}^* \xrightarrow{\sim} \mathbb{1}$$

SATISFYING "COASSOCIATIVITY" & "COUNITALITY"

... MORE LATER

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

ASSUME $(\mathcal{C}, \otimes, \mathbb{1})$ ABELIAN, MONOIDAL, RIGID

TAKE MON. CAT. $(\mathcal{C}, \otimes, \mathbb{1})$.

ALGEBRA IN $\mathcal{C}$ $\equiv$

$(A,\ m: A\otimes A \to A,\ u: \mathbb{1} \to A)$

SAT. ASSOC + UNITALITY

LEFT A-MODULE IN $\mathcal{C}$ $\equiv$

$(M,\ \triangleright: A\otimes M \to M)$

SAT. ASSOC + UNITALITY

RIGHT A-MODULE IN $\mathcal{C}$ $\equiv$

$(M,\ \triangleleft: M\otimes A \to M)$

SAT. ASSOC + UNITALITY

(B_1, B_2)-BIMOD. IN $\mathcal{C}$ $\equiv$

$(M,\ \triangleright: B_1 \otimes M \to M,$

$\triangleleft: M\otimes B_2 \to M)$

$\ni.$ $(M, \triangleright) \in B_1\text{-Mod}(\mathcal{C})$,

$(M, \triangleleft) \in \text{Mod-}B_2(\mathcal{C})$

& $\triangleleft(\triangleright \otimes id_{B_2}) =$

$\triangleright(id_{B_1} \otimes \triangleleft)\, a_{B_1, M, B_2}$

... VIA DUALS

RECALL $(-)^*, {}^*(-): \mathcal{C} \to \mathcal{C}$ ARE CONTRAVARIANT

EXERCISE 4.41: TAKE $(M, \triangleright) \in A\text{-Mod}(\mathcal{C})$.

THEN (IN STRICT CASE)

$(M^*, \blacktriangleleft) \in \text{Mod-}A(\mathcal{C})$ VIA:

$$\blacktriangleleft : M^* \otimes A \dashrightarrow M^*.$$

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

ASSUME $(\mathcal{C}, \otimes, \mathbb{1})$ ABELIAN, MONOIDAL, RIGID

TAKE MON. CAT. $(\mathcal{C}, \otimes, \mathbb{1})$.

ALGEBRA IN $\mathcal{C}$ ≡
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$(M,\ \triangleleft: M\otimes A \to M)$
SAT. ASSOC + UNITALITY

(B_1, B_2)–BIMOD. IN $\mathcal{C}$ ≡
$(M,\ \triangleright: B_1\otimes M \to M,$
$\triangleleft: M\otimes B_2 \to M)$
∋. $(M, \triangleright) \in B_1\text{-Mod}(\mathcal{C})$,
$(M, \triangleleft) \in \text{Mod-}B_2(\mathcal{C})$
& $\triangleleft(\triangleright \otimes id_{B_2}) =$
$\triangleright(id_{B_1} \otimes \triangleleft)\, a_{B_1, M, B_2}$

... VIA DUALS

RECALL $(-)^*,\ {}^*(-): \mathcal{C} \to \mathcal{C}$ ARE CONTRAVARIANT

EXERCISE 4.41: TAKE $(M, \triangleright) \in A\text{-Mod}(\mathcal{C})$.

THEN (IN STRICT CASE)

$(M^*, \blacktriangleleft) \in \text{Mod-}A(\mathcal{C})$ VIA:

$$\blacktriangleleft:\ M^* \otimes A \xrightarrow{id_{M^*} \otimes id_A \otimes coev^L_M} M^* \otimes A \otimes M \otimes M^*$$

M^*.

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

TAKE MON. CAT. $(\mathcal{C}, \otimes, \mathbb{1})$.

ALGEBRA IN $\mathcal{C}$ ≡
$(A,\ m: A\otimes A \to A,\ u: \mathbb{1} \to A)$
SAT. ASSOC + UNITALITY

LEFT A-MODULE IN $\mathcal{C}$ ≡
$(M,\ \triangleright: A\otimes M \to M)$
SAT. ASSOC + UNITALITY

RIGHT A-MODULE IN $\mathcal{C}$ ≡
$(M,\ \triangleleft: M\otimes A \to M)$
SAT. ASSOC + UNITALITY

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$(M,\ \triangleright: B_1\otimes M \to M,$
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$(M,\triangleleft) \in \text{Mod-}B_2(\mathcal{C})$
& $\triangleleft(\triangleright \otimes id_{B_2}) =$
$\triangleright(id_{B_1}\otimes \triangleleft)\, a_{B_1,M,B_2}$

... VIA DUALS

ASSUME $(\mathcal{C}, \otimes, \mathbb{1})$ ABELIAN, MONOIDAL, RIGID

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EXERCISE 4.41: TAKE $(M, \triangleright) \in A\text{-Mod}(\mathcal{C})$.

THEN (IN STRICT CASE)

$$(M^*, \triangleleft) \in \text{Mod-}A(\mathcal{C}) \quad \text{VIA:}$$

$$\triangleleft:\ M^*\otimes A \xrightarrow{id_{M^*}\otimes id_A \otimes coev_M^L} M^*\otimes A\otimes M\otimes M^* \xrightarrow{id_{M^*}\otimes \triangleright \otimes id_{M^*}} M^*\otimes M\otimes M^*$$

$$M^*.$$

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

TAKE MON. CAT. $(\mathcal{C}, \otimes, \mathbb{1})$.

ALGEBRA IN $\mathcal{C}$ ≡ $(A,\ m: A\otimes A \to A,\ u: \mathbb{1} \to A)$ SAT. ASSOC + UNITALITY

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(B_1, B_2)–BIMOD. IN $\mathcal{C}$ ≡ $(M,\ \triangleright: B_1 \otimes M \to M,\ \triangleleft: M \otimes B_2 \to M)$ ∍. $(M, \triangleright) \in B_1\text{-Mod}(\mathcal{C})$, $(M, \triangleleft) \in \text{Mod-}B_2(\mathcal{C})$ & $\triangleleft(\triangleright \otimes id_{B_2}) = \triangleright(id_{B_1} \otimes \triangleleft)\, a_{B_1, M, B_2}$

ASSUME $(\mathcal{C}, \otimes, \mathbb{1})$ ABELIAN, MONOIDAL, RIGID

... VIA DUALS

RECALL $(-)^*, {}^*(-): \mathcal{C} \to \mathcal{C}$ ARE CONTRAVARIANT

EXERCISE 4.41: TAKE $(M, \triangleright) \in A\text{-Mod}(\mathcal{C})$.

THEN (IN STRICT CASE)

$(M^*, \blacktriangleleft) \in \text{Mod-}A(\mathcal{C})$ VIA:

$$\blacktriangleleft: M^* \otimes A \xrightarrow{id_{M^*} \otimes id_A \otimes coev_M^L} M^* \otimes A \otimes M \otimes M^*$$

$$\xrightarrow{id_{M^*} \otimes \triangleright \otimes id_{M^*}} M^* \otimes M \otimes M^*$$

$$\xrightarrow{ev_M^L \otimes id_{M^*}} M^*.$$

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

ASSUME $(\mathcal{C}, \otimes, \mathbb{1})$ ABELIAN, MONOIDAL, RIGID

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$(M,\ \triangleright : B_1 \otimes M \to M,$
$\triangleleft : M \otimes B_2 \to M)$
$\ni$. $(M, \triangleright) \in B_1\text{-Mod}(\mathcal{C})$,
$(M, \triangleleft) \in \text{Mod-}B_2(\mathcal{C})$
& $\triangleleft(\triangleright \otimes id_{B_2}) =$
$\triangleright(id_{B_1} \otimes \triangleleft)\, a_{B_1, M, B_2}$

... VIA DUALS

RECALL $(-)^*, {}^*(-) : \mathcal{C} \to \mathcal{C}$ ARE CONTRAVARIANT

EXERCISE 4.41: TAKE $(M, \triangleright) \in A\text{-Mod}(\mathcal{C})$.

THEN (IN STRICT CASE)

$$(M^*, \blacktriangleleft) \in \text{Mod-}A(\mathcal{C}) \quad \text{VIA:}$$

$$\blacktriangleleft : M^* \otimes A \xrightarrow{id_{M^*} \otimes id_A \otimes coev_M^L} M^* \otimes A \otimes M \otimes M^*$$
$$\xrightarrow{id_{M^*} \otimes \triangleright \otimes id_{M^*}} M^* \otimes M \otimes M^*$$
$$\xrightarrow{ev_M^L \otimes id_{M^*}} M^*.$$

LIKEWISE, $(N, \triangleleft) \in \text{Mod-}A(\mathcal{C})$

$\rightsquigarrow ({}^*N, \blacktriangleright) \in A\text{-Mod}(\mathcal{C})$ FOR SOME $\blacktriangleright$.

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

TAKE MON. CAT. $(\mathcal{C}, \otimes, \mathbb{1})$.

ALGEBRA IN $\mathcal{C}$ $\equiv$ $(A,\ m: A\otimes A \to A,\ u: \mathbb{1} \to A)$ SAT. ASSOC + UNITALITY

LEFT A-MODULE IN $\mathcal{C}$ $\equiv$ $(M,\ \triangleright: A\otimes M \to M)$ SAT. ASSOC + UNITALITY

RIGHT A-MODULE IN $\mathcal{C}$ $\equiv$ $(M,\ \triangleleft: M\otimes A \to M)$ SAT. ASSOC + UNITALITY

(B_1, B_2)-BIMOD. IN $\mathcal{C}$ $\equiv$ $(M,\ \triangleright: B_1 \otimes M \to M,\ \triangleleft: M\otimes B_2 \to M)$ $\ni$. $(M,\triangleright) \in B_1\text{-Mod}(\mathcal{C})$, $(M,\triangleleft) \in \text{Mod-}B_2(\mathcal{C})$ & $\triangleleft(\triangleright \otimes id_{B_2}) = \triangleright(id_{B_1} \otimes \triangleleft)\, a_{B_1,M,B_2}$

... VIA DUALS

ASSUME $(\mathcal{C}, \otimes, \mathbb{1})$ ABELIAN, MONOIDAL RIGID

RECALL $(-)^*,\ {}^*(-): \mathcal{C} \to \mathcal{C}$ ARE CONTRAVARIANT

EXERCISE 4.41: TAKE $(M, \triangleright) \in A\text{-Mod}(\mathcal{C})$.

THEN (IN STRICT CASE)

$(M^*, \blacktriangleleft) \in \text{Mod-}A(\mathcal{C})$ VIA:

$$\blacktriangleleft : M^* \otimes A \xrightarrow{id_{M^*} \otimes id_A \otimes coev^L_M} M^* \otimes A \otimes M \otimes M^* \xrightarrow{id_{M^*} \otimes \triangleright \otimes id_{M^*}} M^* \otimes M \otimes M^* \xrightarrow{ev^L_M \otimes id_{M^*}} M^*.$$

LET'S EXPLORE THIS ☺

LIKEWISE, $(N, \triangleleft) \in \text{Mod-}A(\mathcal{C})$

$\rightsquigarrow ({}^*N, \blacktriangleright) \in A\text{-Mod}(\mathcal{C})$ FOR SOME $\blacktriangleright$.

I. OPERATIONS ON ALGEBRAS & (BI)MODULES

TAKE MON. CAT. $(\mathcal{C}, \otimes, \mathbb{1})$.

ALGEBRA IN $\mathcal{C}$ $\equiv$ $(A,\ m: A\otimes A \to A,\ u: \mathbb{1} \to A)$ SAT. ASSOC + UNITALITY

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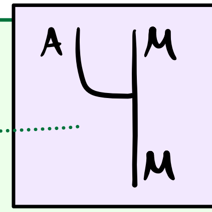
(B_1, B_2)-BIMOD. IN $\mathcal{C}$ $\equiv$ $(M,\ \triangleright: B_1 \otimes M \to M,\ \triangleleft: M \otimes B_2 \to M)$

∋. $(M, \triangleright) \in B_1\text{-Mod}(\mathcal{C})$, $(M, \triangleleft) \in \text{Mod-}B_2(\mathcal{C})$

& $\triangleleft(\triangleright \otimes id_{B_2}) = \triangleright(id_{B_1} \otimes \triangleleft)\, a_{B_1, M, B_2}$

... VIA DUALS

ASSUME $(\mathcal{C}, \otimes, \mathbb{1})$ ABELIAN, RIGID MONOIDAL



EXER. 4.41: TAKE $(M, \triangleright) \in A\text{-Mod}(\mathcal{C})$.

THEN $(M^*, \triangleleft) \in \text{Mod-}A(\mathcal{C})$ VIA:

$$\triangleleft : M^* \otimes A \xrightarrow{id_{M^*} \otimes id_A \otimes coev^L_M} M^* \otimes A \otimes M \otimes M^*$$

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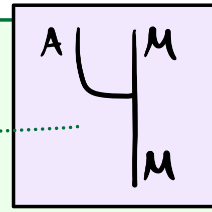

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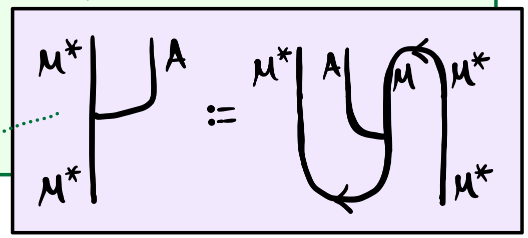

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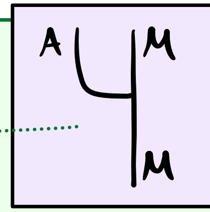

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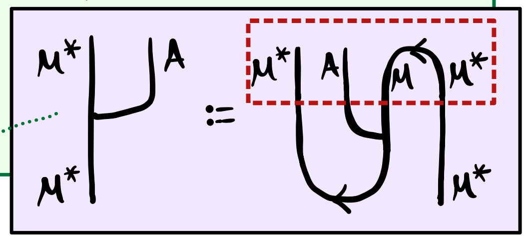

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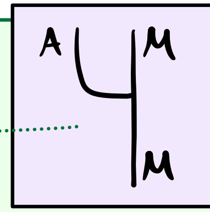

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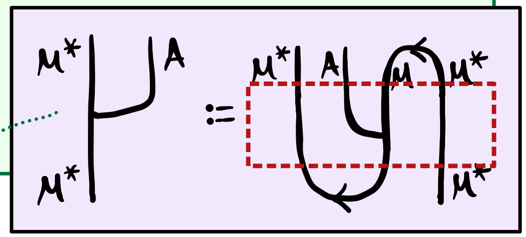

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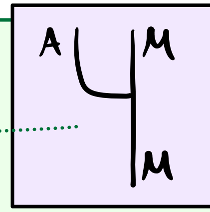

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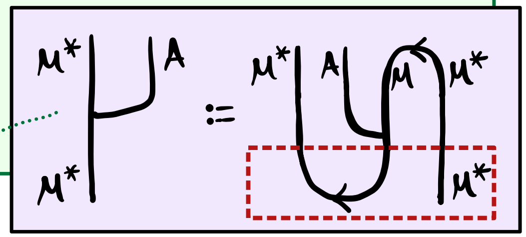

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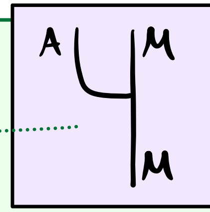

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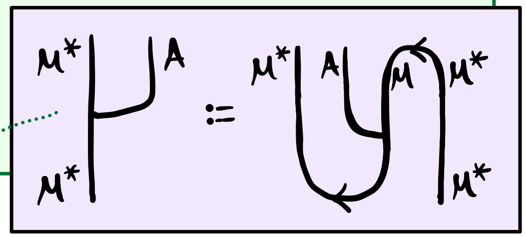
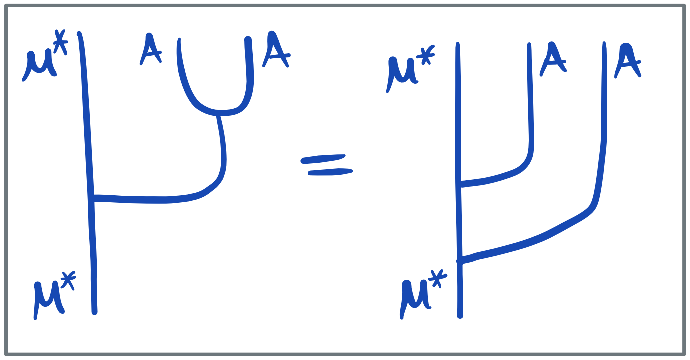
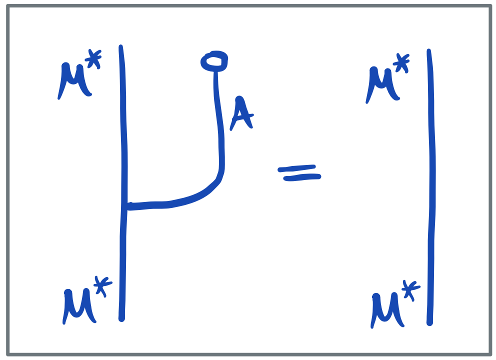


SHOW :





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WE'VE CONSTRUCTED...

BIPROD. $\square$	MON. PROD. $\otimes$	TENS. PROD. $\otimes_A$	HOM	DUAL $(-)^*$, ${}^*(-)$	OF
✓	NOT YET	NOT YET	N/A	N/A	ALGS IN $\mathcal{C}$
✓	✓	✓	N/A	✓	MODS IN $\mathcal{C}$
✓	✓	✓	N/A	THINK ABOUT THIS	BIMODS. IN $\mathcal{C}$

II. Generalized Eilenberg-Watts Theorem

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Take $\Bbbk$-algs. A, B. & $Q = {}_BQ_A$ bimodule.

Get additive functors:

Right ex. $\boxed{Q \otimes_A -} : A\text{-Mod} \longrightarrow B\text{-Mod}$

$\boxed{\mathrm{Hom}_{B\text{-Mod}}(Q, -)} : B\text{-Mod} \longrightarrow A\text{-Mod}$

Left ex.

With $(Q \otimes_A -) \dashv (\mathrm{Hom}_{B\text{-Mod}}(Q, -))$

Recall from Lecture #11

Eilenberg–Watts Theorem

Take finite dim'l $\Bbbk$-algebras A, B.

For $\Bbbk$-linear

$F : A\text{-FdMod} \longrightarrow B\text{-FdMod}$, get:

F left exact	F right exact
$\Updownarrow$	$\Updownarrow$
F has a left adjoint	F has a right adjoint
$\Updownarrow$	$\Updownarrow$
$F \cong \mathrm{Hom}_{A\text{-FdMod}}(P, -)$	$F \cong Q \otimes_A -$
for some bimod.	for some bimod.
$P = {}_AP_B$.	$Q = {}_BQ_A$.

II. GENERALIZED EILENBERG–WATTS THEOREM

TAKE $\Bbbk$-ALGS. A, B. & $Q = {}_BQ_A$ BIMODULE.

GET ADDITIVE FUNCTORS:

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LEFT EX.

WITH $(Q \otimes_A -) \dashv (\mathrm{Hom}_{B\text{-Mod}}(Q,-))$

RECALL FROM LECTURE #11

WE WILL GENERALIZE THIS IN THE MONOIDAL SETTING

EILENBERG–WATTS THEOREM

TAKE FINITE DIM'L $\Bbbk$-ALGEBRAS A, B.

FOR $\Bbbk$-LINEAR

$F : A\text{-FdMod} \longrightarrow B\text{-FdMod}$, GET:

F LEFT EXACT	F RIGHT EXACT
$\Updownarrow$	$\Updownarrow$
F HAS A LEFT ADJOINT	F HAS A RIGHT ADJOINT
$\Updownarrow$	$\Updownarrow$
$F \cong \mathrm{Hom}_{A\text{-FdMod}}(P, -)$ FOR SOME BIMOD. $P = {}_AP_B$.	$F \cong Q \otimes_A -$ FOR SOME BIMOD. $Q = {}_BQ_A$.

II. Generalized Eilenberg–Watts Theorem

Assume $\mathcal{C}$ abelian monoidal & $X \otimes -$, $- \otimes X$ are right exact $\forall X \in \mathcal{C}$

Pay attention to hypotheses

Take $\Bbbk$-algs. A, B. & $Q = {}_B Q_A$ bimodule.

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Lemma: Take $A, B \in \mathrm{Alg}(\mathcal{C})$.

Then,

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ASSUME $\mathcal{C}$ ABELIAN MONOIDAL & $X\otimes -$, $-\otimes X$ ARE RIGHT EXACT $\forall X \in \mathcal{C}$

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PF/ STS $(Q \otimes_A -)$ PRESERVES COKERNELS.

TAKE $\phi : M \longrightarrow N \in A\text{-Mod}(\mathcal{C})$.

WANT: $Q \otimes_A \mathrm{coker}(\phi) \cong \mathrm{coker}(\mathrm{id}_Q \otimes_A \phi)$

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Want: $Q\otimes_A \mathrm{coker}(\phi) \cong \mathrm{coker}(\mathrm{id}_Q \otimes_A \phi)$

$$
\begin{tikzcd}
Q\otimes_A M \arrow[r, "\mathrm{id}\otimes_A \phi"] \arrow[drr, bend right, "\vec{0}"'] & Q\otimes_A N \arrow[r, "\alpha_{\mathrm{id}\otimes_A\phi}"] \arrow[dr, "\mathrm{id}\otimes_A \alpha_\phi"] \arrow[d, phantom, "\circlearrowright"] & \mathrm{coker}(\mathrm{id}_Q\otimes_A \phi) \\
 & {} & Q\otimes_A \mathrm{coker}(\phi)
\end{tikzcd}
$$

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 & & Q \otimes_A \mathrm{coker}(\phi)
\end{tikzcd}$$

By UNIV. PROP. OF COKERNELS

II. GENERALIZED EILENBERG–WATTS THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL & $X \otimes -$, $- \otimes X$ ARE RIGHT EXACT $\forall X \in \mathcal{C}$

TAKE $\Bbbk$-ALGS. A, B. & $Q = {}_B Q_A$ BIMODULE.

GET ADDITIVE FUNCTORS:

RIGHT EX. $\boxed{Q \otimes_A -} : A\text{-Mod} \longrightarrow B\text{-Mod}$

$\boxed{\mathrm{Hom}_{B\text{-Mod}}(Q, -)} : B\text{-Mod} \longrightarrow A\text{-Mod}$
LEFT EX.

WITH $\left(Q \otimes_A -\right) \dashv \left(\mathrm{Hom}_{B\text{-Mod}}(Q, -)\right)$

LEMMA: TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$.
THEN,

$Q \otimes_A - : A\text{-Mod}(\mathcal{C}) \longrightarrow B\text{-Mod}(\mathcal{C})$

IS RIGHT EXACT,

$\forall Q \in (B, A)\text{-Bimod}(\mathcal{C})$.

PF/ STS $(Q \otimes_A -)$ PRESERVES COKERNELS.

TAKE $\phi : M \longrightarrow N \in A\text{-Mod}(\mathcal{C})$.

WANT: $Q \otimes_A \mathrm{Coker}(\phi) \cong \mathrm{coker}(\mathrm{id}_Q \otimes_A \phi)$

$$\begin{tikzcd}
Q \otimes_A M \arrow[r, "\mathrm{id} \otimes_A \phi"] \arrow[drr, bend right, "0"'] & Q \otimes_A N \arrow[r, "\alpha_{\mathrm{id} \otimes_A \phi}"] \arrow[dr, "\mathrm{id} \otimes_A \alpha_\phi"'] & \mathrm{coker}(\mathrm{id}_Q \otimes_A \phi) \arrow[d, dashed, "\exists!\, \gamma_1"] \\
 & & Q \otimes_A \mathrm{coker}(\phi)
\end{tikzcd}$$

By UNIV. PROP. OF COKERNELS

ALSO GET $\gamma_1 \in B\text{-Mod}(\mathcal{C})$
SINCE $B \otimes -$ IS RIGHT EXACT.

II. GENERALIZED EILENBERG–WATTS THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL & $X\otimes -$, $-\otimes X$ ARE RIGHT EXACT $\forall X \in \mathcal{C}$

TAKE $\Bbbk$-ALGS. A, B. & $Q = {}_BQ_A$ BIMODULE.

GET ADDITIVE FUNCTORS:

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$\boxed{\mathrm{Hom}_{B\text{-Mod}}(Q,-)} : B\text{-Mod} \longrightarrow A\text{-Mod}$

LEFT EX.

WITH $(Q\otimes_A -) \dashv (\mathrm{Hom}_{B\text{-Mod}}(Q,-))$

LEMMA: TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$.

THEN,

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& & Q\otimes_A \mathrm{coker}(\phi)
\end{tikzcd}$$

By UNIV. PROP. OF COKERNELS

ALSO GET $\gamma_1 \in B\text{-Mod}(\mathcal{C})$

SINCE $B\otimes -$ IS RIGHT EXACT.

ON THE OTHER HAND, GET

$\gamma_2 : Q\otimes_A \mathrm{coker}(\phi) \longrightarrow \mathrm{coker}(\mathrm{id}_Q\otimes_A\phi)$
$\in B\text{-Mod}(\mathcal{C})$

VIA THE UNIV. PROP. OF COEQUALIZERS.

II. GENERALIZED EILENBERG–WATTS THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL & $X\otimes -$, $-\otimes X$ ARE RIGHT EXACT $\forall X\in\mathcal{C}$

TAKE $\Bbbk$-ALGS. A, B. & $Q = {}_B Q_A$ BIMODULE.

GET ADDITIVE FUNCTORS:

RIGHT EX. $\boxed{Q\otimes_A -}$: A-Mod $\longrightarrow$ B-Mod

$\boxed{\mathrm{Hom}_{B\text{-Mod}}(Q,-)}$: B-Mod $\longrightarrow$ A-Mod

LEFT EX.

WITH $(Q\otimes_A -) \dashv (\mathrm{Hom}_{B\text{-Mod}}(Q,-))$

LEMMA: TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$.

THEN,

$Q\otimes_A -$: $A\text{-Mod}(\mathcal{C}) \longrightarrow B\text{-Mod}(\mathcal{C})$

IS RIGHT EXACT,

$\forall Q \in (B,A)\text{-Bimod}(\mathcal{C})$.

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 & & Q\otimes_A \mathrm{coker}(\phi)
\end{tikzcd}$$

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ALSO GET $\gamma_1 \in B\text{-Mod}(\mathcal{C})$

SINCE $B\otimes -$ IS RIGHT EXACT.

ON THE OTHER HAND, GET

$\gamma_2 : Q\otimes_A \mathrm{coker}(\phi) \longrightarrow \mathrm{coker}(id_Q\otimes_A\phi)$

$\in B\text{-Mod}(\mathcal{C})$

VIA THE UNIV. PROP. OF COEQUALIZERS.

CHECK γ_1, γ_2 MUTUALLY INVERSE. ///

II. GENERALIZED EILENBERG-WATTS THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL & $X \otimes -$, $- \otimes X$ ARE RIGHT EXACT $\forall X \in \mathcal{C}$

EILENBERG-WATTS THEOREM

TAKE FINITE DIM'L $\Bbbk$-ALGEBRAS A, B.

FOR $\Bbbk$-LINEAR

$F : A\text{-FdMod} \longrightarrow B\text{-FdMod}$, GET:

F LEFT EXACT	F RIGHT EXACT
$\Updownarrow$	$\Updownarrow$
F HAS A LEFT ADJOINT	F HAS A RIGHT ADJOINT
$\Updownarrow$	$\Updownarrow$
$F \cong \mathrm{Hom}_{A\text{-FdMod}}(P, -)$ FOR SOME BIMOD. $P = {}_A P_B$.	$\boxed{F \cong Q \otimes_A -}$ FOR SOME BIMOD. $Q = {}_B Q_A$.

LEMMA: TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN,

$Q \otimes_A - : A\text{-Mod}(\mathcal{C}) \longrightarrow B\text{-Mod}(\mathcal{C})$

IS RIGHT EXACT,

$\forall Q \in (B, A)\text{-Bimod}(\mathcal{C})$.

II. GENERALIZED EILENBERG–WATTS THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL & $X\otimes -$, $-\otimes X$ ARE RIGHT EXACT $\forall X \in \mathcal{C}$

GENERALIZED EW THEOREM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$.

RECALL $A\text{-Mod}(\mathcal{C})$, $B\text{-Mod}(\mathcal{C}) \in \text{Mod-}\mathcal{C}$

VIA $(M, \triangleright) \triangleleft_{A \text{ or } B} X := (M \otimes X, \triangleright \otimes \mathrm{id}_X)$.

LEMMA: TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN,

$$Q \otimes_A - : A\text{-Mod}(\mathcal{C}) \longrightarrow B\text{-Mod}(\mathcal{C})$$

IS RIGHT EXACT,

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EILENBERG–WATTS THEOREM

TAKE FINITE DIM'L $\Bbbk$-ALGEBRAS A, B.

FOR $\Bbbk$-LINEAR

$F : A\text{-FdMod} \longrightarrow B\text{-FdMod}$, GET:

F LEFT EXACT	F RIGHT EXACT
$\Updownarrow$	$\Updownarrow$
F HAS A LEFT ADJOINT	F HAS A RIGHT ADJOINT
$\Updownarrow$	$\Updownarrow$
$F \cong \mathrm{Hom}_{A\text{-FdMod}}(P, -)$	$F \cong Q \otimes_A -$
FOR SOME BIMOD.	FOR SOME BIMOD.
$P = {}_A P_B$.	$Q = {}_B Q_A$.

II. Generalized Eilenberg-Watts Theorem

Assume $\mathcal{C}$ abelian monoidal & $X \otimes -$, $- \otimes X$ are right exact $\forall X \in \mathcal{C}$

Generalized EW Theorem

Take $A, B \in \mathrm{Alg}(\mathcal{C})$.

Recall $A\text{-Mod}(\mathcal{C}), B\text{-Mod}(\mathcal{C}) \in \mathrm{Mod}\text{-}\mathcal{C}$

via $(M, \triangleright) \triangleleft_{A \text{ or } B} X := (M \otimes X, \triangleright \otimes \mathrm{id}_X)$.

Then, we get an equiv. of cats:

$$\mathrm{Rex}_{\mathrm{Mod}\text{-}\mathcal{C}}(A\text{-Mod}(\mathcal{C}), B\text{-Mod}(\mathcal{C})) \simeq (B, A)\text{-Bimod}(\mathcal{C}).$$

Right exact mod. cat. functors

Lemma: Take $A, B \in \mathrm{Alg}(\mathcal{C})$. Then,

$Q \otimes_A - : A\text{-Mod}(\mathcal{C}) \longrightarrow B\text{-Mod}(\mathcal{C})$

is right exact, $\forall Q \in (B, A)\text{-Bimod}(\mathcal{C})$.

Eilenberg-Watts Theorem

Take finite dim'l $\Bbbk$-algebras A, B.

For $\Bbbk$-linear

$F : A\text{-FdMod} \longrightarrow B\text{-FdMod}$, get:

F left exact	F right exact
$\Updownarrow$	$\Updownarrow$
F has a left adjoint	F has a right adjoint
$\Updownarrow$	$\Updownarrow$
$F \cong \mathrm{Hom}_{A\text{-FdMod}}(P, -)$ for some bimod. $P = {}_A P_B$.	$F \cong Q \otimes_A -$ for some bimod. $Q = {}_B Q_A$.

II. GENERALIZED EILENBERG–WATTS THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL & $X\otimes -$, $-\otimes X$ ARE RIGHT EXACT $\forall X \in \mathcal{C}$

PF/

GENERALIZED EW THEOREM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$.

RECALL $A\text{-Mod}(\mathcal{C})$, $B\text{-Mod}(\mathcal{C}) \in \text{Mod-}\mathcal{C}$

VIA $(M, \triangleright) \triangleleft_{A \text{ or } B} X := (M \otimes X, \triangleright \otimes \mathrm{id}_X)$.

THEN, WE GET AN EQUIV. OF CATS:

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RIGHT EXACT MOD. CAT. FUNCTORS

LEMMA: TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN,

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RECALL $A\text{-Mod}(\mathcal{C}), B\text{-Mod}(\mathcal{C}) \in \text{Mod-}\mathcal{C}$

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RIGHT EXACT MOD. CAT. FUNCTORS

LEMMA: TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN,

$Q \otimes_A - : A\text{-Mod}(\mathcal{C}) \to B\text{-Mod}(\mathcal{C})$

IS RIGHT EXACT,

$\forall Q \in (B, A)\text{-Bimod}(\mathcal{C})$.

PF/

$$[F : A\text{-Mod}(\mathcal{C}) \to B\text{-Mod}(\mathcal{C})] \xmapsto{\Phi} F({}_A A_{reg})$$

II. GENERALIZED EILENBERG–WATTS THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL
& $X \otimes -$, $- \otimes X$ ARE RIGHT EXACT $\forall X \in \mathcal{C}$

GENERALIZED EW THEOREM

TAKE $A, B \in \mathsf{Alg}(\mathcal{C})$.

RECALL $A\text{-}\mathsf{Mod}(\mathcal{C})$, $B\text{-}\mathsf{Mod}(\mathcal{C}) \in \mathsf{Mod}\text{-}\mathcal{C}$

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RIGHT EXACT MOD. CAT. FUNCTORS

LEMMA: TAKE $A, B \in \mathsf{Alg}(\mathcal{C})$. THEN,

$Q \otimes_A - : A\text{-}\mathsf{Mod}(\mathcal{C}) \longrightarrow B\text{-}\mathsf{Mod}(\mathcal{C})$

IS RIGHT EXACT,

$\forall Q \in (B, A)\text{-}\mathsf{Bimod}(\mathcal{C})$.

PF/

$[F : A\text{-}\mathsf{Mod}(\mathcal{C}) \longrightarrow B\text{-}\mathsf{Mod}(\mathcal{C})] \overset{\Phi}{\longmapsto} F({}_A A_{\mathrm{reg}})$

- HAVE $F(A) \in B\text{-}\mathsf{Mod}(\mathcal{C})$.

II. Generalized Eilenberg–Watts Theorem

Assume $\mathcal{C}$ abelian monoidal & $X \otimes -$, $- \otimes X$ are right exact $\forall X \in \mathcal{C}$

Generalized EW Theorem

Take $A, B \in \mathrm{Alg}(\mathcal{C})$.

Recall $A\text{-}\mathrm{Mod}(\mathcal{C}), B\text{-}\mathrm{Mod}(\mathcal{C}) \in \mathrm{Mod}\text{-}\mathcal{C}$

via $(M, \triangleright) \triangleleft_{A \text{ or } B} X := (M \otimes X, \triangleright \otimes \mathrm{id}_X)$.

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$$\mathrm{Rex}_{\mathrm{Mod}\text{-}\mathcal{C}}(A\text{-}\mathrm{Mod}(\mathcal{C}), B\text{-}\mathrm{Mod}(\mathcal{C})) \simeq (B, A)\text{-}\mathrm{Bimod}(\mathcal{C}).$$

Right exact mod. cat. functors

Lemma: Take $A, B \in \mathrm{Alg}(\mathcal{C})$. Then,

$Q \otimes_A - : A\text{-}\mathrm{Mod}(\mathcal{C}) \to B\text{-}\mathrm{Mod}(\mathcal{C})$

is right exact,

$\forall Q \in (B, A)\text{-}\mathrm{Bimod}(\mathcal{C})$.

PF/

$$[F : A\text{-}\mathrm{Mod}(\mathcal{C}) \to B\text{-}\mathrm{Mod}(\mathcal{C})] \overset{\Phi}{\longmapsto} F({}_A A_{reg})$$

- Have $F(A) \in B\text{-}\mathrm{Mod}(\mathcal{C})$.
- Define

$$\begin{array}{ccc}
\triangleleft_{F(A)} : F(A) \otimes A & \xrightarrow{\quad\quad} & F(A) \\
\text{|||} & \underline{\text{DEF}} & \uparrow F(m_A) \\
F(A) \triangleleft_B A & & F(A \otimes A) \\
\text{||S} & & /\!/\!/ \\
& F(A \triangleleft_A A) &
\end{array}$$

F is a mod. cat. functor $\to$ ||S

To get $F(A) \in \mathrm{Mod}\text{-}A(\mathcal{C})$.

II. GENERALIZED EILENBERG–WATTS THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL & $X \otimes -$, $- \otimes X$ ARE RIGHT EXACT $\forall X \in \mathcal{C}$

GENERALIZED EW THEOREM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$.

RECALL $A\text{-Mod}(\mathcal{C})$, $B\text{-Mod}(\mathcal{C}) \in \mathrm{Mod}\text{-}\mathcal{C}$

VIA $(M, \triangleright) \triangleleft_{A \text{ or } B} X := (M \otimes X, \triangleright \otimes id_X)$.

THEN, WE GET AN EQUIV. OF CATS:

$$\mathrm{Rex}_{\mathrm{Mod}\text{-}\mathcal{C}}\big(A\text{-Mod}(\mathcal{C}), B\text{-Mod}(\mathcal{C})\big) \simeq (B, A)\text{-Bimod}(\mathcal{C}).$$

RIGHT EXACT MOD. CAT. FUNCTORS

LEMMA: TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN,

$Q \otimes_A - : A\text{-Mod}(\mathcal{C}) \longrightarrow B\text{-Mod}(\mathcal{C})$

IS RIGHT EXACT,

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PF/

$$[F : A\text{-Mod}(\mathcal{C}) \longrightarrow B\text{-Mod}(\mathcal{C})] \overset{\Phi}{\longmapsto} F({}_A A_{reg})$$

- HAVE $F(A) \in B\text{-Mod}(\mathcal{C})$.

- DEFINE

$$\begin{tikzcd}
\triangleleft_{F(A)} : F(A) \otimes A \arrow[rr] \arrow[d, phantom, "\cong" rotate=90] & & F(A) \\
F(A) \triangleleft_B A \arrow[r, phantom, "\overset{DEF}{=}"] \arrow[d, phantom, "\cong" rotate=90] & & F(A \otimes A) \arrow[u, "F(m_A)"'] \\
& F(A \triangleleft_A A) \arrow[ur, phantom, "\cong"] &
\end{tikzcd}$$

F IS A MOD. CAT. FUNCTOR

TO GET $F(A) \in \mathrm{Mod}\text{-}A(\mathcal{C})$.

- CHECK $F(A) \in (B, A)\text{-Bimod}(\mathcal{C})$.

II. GENERALIZED EILENBERG–WATTS THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL & $X \otimes -$, $- \otimes X$ ARE RIGHT EXACT $\forall X \in \mathcal{C}$

GENERALIZED EW THEOREM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$.

RECALL $A\text{-Mod}(\mathcal{C}), B\text{-Mod}(\mathcal{C}) \in \mathrm{Mod}\text{-}\mathcal{C}$

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RIGHT EXACT MOD. CAT. FUNCTORS

LEMMA: TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN,

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$$[F : A\text{-Mod}(\mathcal{C}) \longrightarrow B\text{-Mod}(\mathcal{C})] \overset{\Phi}{\longmapsto} F({}_A A_{\mathrm{reg}})$$

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RIGHT EXACT MOD. CAT. FUNCTORS

LEMMA: TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN,

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$$[F : A\text{-Mod}(\mathcal{C}) \to B\text{-Mod}(\mathcal{C})] \overset{\Phi}{\longmapsto} F({}_A A_{\mathrm{reg}})$$

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$$[F : A\text{-Mod}(\mathcal{C}) \longrightarrow B\text{-Mod}(\mathcal{C})] \overset{\Phi}{\longmapsto} F({}_A A_{reg})$$

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THIS IS RIGHT EXACT ✓

THIS IS A $\mathcal{C}$-MODULE FUNCTOR:

$$\begin{aligned} Q \otimes_A (M \triangleleft_A X) &\equiv Q \otimes_A (M \otimes X) \\ &\cong (Q \otimes_A M) \otimes X \\ &\equiv (Q \otimes_A M) \triangleleft_B X, \end{aligned}$$

$\forall X \in \mathcal{C}$, $M \in A\text{-Mod}(\mathcal{C})$.

II. GENERALIZED EILENBERG–WATTS THEOREM

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PF/

$$[F : A\text{-Mod}(\mathcal{C}) \longrightarrow B\text{-Mod}(\mathcal{C})] \xmapsto{\Phi} F({}_A A_{reg})$$

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THIS IS RIGHT EXACT ✓

THIS IS A $\mathcal{C}$–MODULE FUNCTOR:

$$\begin{aligned} Q \otimes_A (M \triangleleft_A X) &= Q \otimes_A (M \otimes X) \\ &\cong (Q \otimes_A M) \otimes X \\ &= (Q \otimes_A M) \triangleleft_B X, \end{aligned}$$

$\forall X \in \mathcal{C}$, $M \in A\text{-Mod}(\mathcal{C})$.

$\therefore$ Ψ IS WELL-DEFINED.

II. Generalized Eilenberg–Watts Theorem

> Assume $\mathcal{C}$ abelian monoidal & $X \otimes -$, $- \otimes X$ are right exact $\forall X \in \mathcal{C}$

Generalized EW Theorem

Take $A, B \in \mathrm{Alg}(\mathcal{C})$.

Recall $A\text{-Mod}(\mathcal{C})$, $B\text{-Mod}(\mathcal{C}) \in \mathrm{Mod}\text{-}\mathcal{C}$

via $(M, \triangleright) \triangleleft_{A \text{ or } B} X := (M \otimes X, \triangleright \otimes \mathrm{id}_X)$.

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$$\mathrm{Rex}_{\mathrm{Mod}\text{-}\mathcal{C}}\big(A\text{-Mod}(\mathcal{C}), B\text{-Mod}(\mathcal{C})\big) \simeq (B, A)\text{-Bimod}(\mathcal{C}).$$

Right exact mod. cat. functors

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$$[F : A\text{-Mod}(\mathcal{C}) \longrightarrow B\text{-Mod}(\mathcal{C})] \overset{\Phi}{\longmapsto} F({}_A A_{reg})$$

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Right exact mod. cat. functors

Pf/

$$[F : A\text{-Mod}(\mathcal{C}) \longrightarrow B\text{-Mod}(\mathcal{C})] \overset{\Phi}{\longmapsto} F({}_A A_{reg})$$

$$[Q \otimes_A - : A\text{-Mod}(\mathcal{C}) \longrightarrow B\text{-Mod}(\mathcal{C})] \overset{\Psi}{\longleftarrow\!\!\shortmid} {}_B Q_A$$

Now $\Phi\Psi(Q) = \Phi(Q \otimes_A -)$

$$= Q \otimes_A {}_A(A_{reg}) \cong Q.$$

II. GENERALIZED EILENBERG–WATTS THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL
& $X \otimes -$, $- \otimes X$ ARE RIGHT EXACT $\forall X \in \mathcal{C}$

GENERALIZED EW THEOREM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$.

RECALL $A\text{-Mod}(\mathcal{C}), B\text{-Mod}(\mathcal{C}) \in \mathrm{Mod}\text{-}\mathcal{C}$

VIA $(M, \triangleright) \triangleleft_{A \text{ or } B} X := (M \otimes X, \triangleright \otimes \mathrm{id}_X)$.

THEN, WE GET AN EQUIV. OF CATS:

$$\mathrm{Rex}_{\mathrm{Mod}\text{-}\mathcal{C}}(A\text{-Mod}(\mathcal{C}), B\text{-Mod}(\mathcal{C})) \simeq (B, A)\text{-Bimod}(\mathcal{C}).$$

RIGHT EXACT
MOD. CAT. FUNCTORS

PF/

$$[F : A\text{-Mod}(\mathcal{C}) \to B\text{-Mod}(\mathcal{C})] \xmapsto{\Phi} F({}_A A_{reg})$$

$$[Q \otimes_A - : A\text{-Mod}(\mathcal{C}) \to B\text{-Mod}(\mathcal{C})] \xleftarrow{\Psi} {}_B Q_A$$

NOW $\Phi\Psi(Q) = \Phi(Q \otimes_A -)$

$$= Q \otimes_A {}_A(A_{reg}) \cong Q.$$

ON THE OTHER HAND, TAKE:

$F : A\text{-Mod}(\mathcal{C}) \to B\text{-Mod}(\mathcal{C}) \in \mathrm{Mod}\text{-}\mathcal{C}$
RIGHT EXACT

CLAIM: $F({}_A A_{reg}) \otimes_A M \cong F(M)$ IN $B\text{-Mod}(\mathcal{C})$ $\forall M \in A\text{-Mod}(\mathcal{C})$

$\Downarrow$

$\Psi\Phi(F) \cong F$

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$= Q \otimes_A {}_A(A_{reg}) \cong Q$.

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$\Downarrow$

$\Psi\Phi(F) \cong F$

PF/ GET: $F(A) \otimes_A M \longrightarrow F(M)$ ← VIA UNIV. PROP. OF COEQ., &

$F(M) \cong F(A \otimes_A M) \longrightarrow F(A) \otimes_A M$ ← F RIGHT EX. & PRES. COEQS.

... CHECK MUTUALLY INV. ///

///

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL & $X\otimes -$, $-\otimes X$ ARE RIGHT EXACT $\forall X \in \mathcal{C}$

MORITA'S THEOREM

TAKE $\Bbbk$-ALGEBRAS A & B. THEN:

A IS MORITA EQUIVALENT TO B

$$\Updownarrow$$

$\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

FROM LECTURE #9

III. GENERALIZED MORITA'S THEOREM

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FROM LECTURE #9

WE WILL GENERALIZE THIS IN THE MONOIDAL SETTING

III. Generalized Morita's Theorem

Assume $\mathcal{C}$ abelian monoidal & $X \otimes -$, $- \otimes X$ are right exact $\forall X \in \mathcal{C}$

Generalized Morita's Theorem

Take $A, B \in \mathrm{Alg}(\mathcal{C})$. Then:

$$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$$

as right $\mathcal{C}$-module categories

$$\Updownarrow$$

$\exists$ bimodules ${}_AP_B$ & ${}_BQ_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ as A-bimodules in $\mathcal{C}$

& $Q \otimes_A P \cong B_{reg}$ as B-bimodules in $\mathcal{C}$.

Morita's Theorem

Take $\Bbbk$-algebras A & B. Then:

A is Morita equivalent to B

$$\Updownarrow$$

$\exists$ bimodules ${}_AP_B$ & ${}_BQ_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ as A-bimodules

& $Q \otimes_A P \cong B_{reg}$ as B-bimodules.

From Lecture #9

We will generalize this in the monoidal setting

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$\exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ s.t.

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SAY A, B ARE **MORITA EQUIVALENT** IN $\mathcal{C}$

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($\Uparrow$) DEFINE:

$$F := Q \otimes_A - : A\text{-}\mathrm{Mod}(\mathcal{C}) \to B\text{-}\mathrm{Mod}(\mathcal{C}),$$

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THEN $\forall M \in A\text{-}\mathrm{Mod}(\mathcal{C})$,

$$GF(M) = P \otimes_B (Q \otimes_A M) \cong (P \otimes_B Q) \otimes_A M$$
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$$\therefore GF \cong \mathrm{Id}_{A\text{-Mod}(\mathcal{C})}$$

LIKEWISE, $FG \cong \mathrm{Id}_{B\text{-Mod}(\mathcal{C})}$

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LIKEWISE, $FG \cong \mathrm{Id}_{B\text{-}\mathrm{Mod}(\mathcal{C})}$

$\therefore$ F IS AN EQUIVALENCE OF CATEGORIES.

ALSO $\forall M \in A\text{-}\mathrm{Mod}(\mathcal{C})$, $X \in \mathcal{C}$:

$$\begin{aligned} F(M \triangleleft_A X) &= F(M \otimes X) \\ &= Q \otimes_A (M \otimes X) \cong (Q \otimes_A M) \otimes X \\ &= F(M) \otimes X = F(M) \triangleleft_B X \end{aligned}$$

III. GENERALIZED MORITA'S THEOREM

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$\therefore$ F IS AN EQUIVALENCE OF MODULE CATEGORIES.

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PF/

$(\Downarrow)$ TAKE QUASI-INVERSES:

$$F : A\text{-Mod}(\mathcal{C}) \longrightarrow B\text{-Mod}(\mathcal{C}),$$

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$\exists \begin{cases} {}_B Q_A \ni. \ F \cong Q \otimes_A - \\ {}_A P_B \ni. \ G \cong P \otimes_B - \end{cases}$

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$\exists \begin{cases} {}_BQ_A \ni. \ F \cong Q \otimes_A - \\ {}_AP_B \ni. \ G \cong P \otimes_B - \end{cases}$

<u>CLAIM</u>

$P \otimes_B Q \cong A$

IN A-Bimod($\mathcal{C}$).

GENERALIZED EW THEOREM

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PF/ HAVE $GF \cong \mathrm{Id}_{A\text{-Mod}(\mathcal{C})}$

VIA NATURAL ISOMORPHISM Φ.

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$P \otimes_B Q \cong A$

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PF/ HAVE $GF \cong \mathrm{Id}_{A\text{-Mod}(\mathcal{C})}$

VIA NATURAL ISOMORPHISM Φ.

THE COMPONENT $\Phi_A : GF(A) \xrightarrow{\sim} A$ DOES THE TRICK... ///

$$GF(A) = P \otimes_B Q \cong P \otimes_B (Q \otimes_A A)$$

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ASSUME $\mathcal{C}$ ABELIAN MONOIDAL & $X \otimes -$, $- \otimes X$ ARE RIGHT EXACT $\forall X \in \mathcal{C}$

GENERALIZED MORITA'S THEOREM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$$

AS RIGHT $\mathcal{C}$-MODULE CATEGORIES

$$\Updownarrow$$

$\exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES IN $\mathcal{C}$

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES IN $\mathcal{C}$.

PF/

$(\Downarrow)$ TAKE QUASI-INVERSES:

$F : A\text{-Mod}(\mathcal{C}) \to B\text{-Mod}(\mathcal{C})$,

$G : B\text{-Mod}(\mathcal{C}) \to A\text{-Mod}(\mathcal{C})$.

THEY'RE EQUIVALENCES,

SO HAVE RIGHT ADJOINTS

SO ARE <u>RIGHT</u> <u>EXACT</u>.

$\exists \begin{cases} {}_B Q_A \ni. \ F \cong Q \otimes_A - \\ {}_A P_B \ni. \ G \cong P \otimes_B - \end{cases}$

CLAIM

$P \otimes_B Q \cong A$

IN A-Bimod($\mathcal{C}$)

PF/ HAVE $GF \cong \mathrm{Id}_{A\text{-Mod}(\mathcal{C})}$

VIA NATURAL ISOMORPHISM Φ.

THE COMPONENT $\Phi_A : GF(A) \xrightarrow{\sim} A$ DOES THE TRICK... ///

$$GF(A) = P \otimes_B (Q \otimes_A A) \cong P \otimes_B Q$$

LIKEWISE, $Q \otimes_A P \cong B$ IN B-Bimod($\mathcal{C}$) ///

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL & $X \otimes -$, $- \otimes X$ ARE RIGHT EXACT $\forall X \in \mathcal{C}$

GENERALIZED MORITA'S THEOREM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

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$\exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ $\ni$.

$\boxed{P \otimes_B Q \cong A_{reg}}$ AS A-BIMODULES IN $\mathcal{C}$

& $\boxed{Q \otimes_A P \cong B_{reg}}$ AS B-BIMODULES IN $\mathcal{C}$.

≡ SHOWING THIS IN PRACTICE ≡

USE UNIVERSAL PROPERTY OF COEQUALIZERS TO GET:

$$P \otimes_B Q \xrightarrow{\phi_A} A \quad \& \quad Q \otimes_A P \xrightarrow{\phi_B} B$$

$\in A\text{-Bimod}(\mathcal{C})$ $\quad \in B\text{-Bimod}(\mathcal{C})$

III. GENERALIZED MORITA'S THEOREM

GENERALIZED MORITA'S THEOREM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$$A\text{-}\mathrm{Mod}(\mathcal{C}) \simeq B\text{-}\mathrm{Mod}(\mathcal{C})$$

AS RIGHT $\mathcal{C}$-MODULE CATEGORIES

$$\Updownarrow$$

$\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

$\boxed{P \otimes_B Q \cong A_{reg}}$ AS A-BIMODULES IN $\mathcal{C}$

& $\boxed{Q \otimes_A P \cong B_{reg}}$ AS B-BIMODULES IN $\mathcal{C}$.

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USE UNIVERSAL PROPERTY OF COEQUALIZERS TO GET:

$$P \otimes_B Q \xrightarrow{\phi_A} A \quad \& \quad Q \otimes_A P \xrightarrow{\phi_B} B$$

$\in A\text{-}\mathrm{Bimod}(\mathcal{C})$ $\quad$ $\in B\text{-}\mathrm{Bimod}(\mathcal{C})$

THEN USE

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL & $X \otimes -$, $- \otimes X$ ARE RIGHT EXACT $\forall X \in \mathcal{C}$

PROP TAKE ${}_AP_B$, ${}_BQ_A$. IF $\exists$ EPIS

$$P \otimes_B Q \xrightarrow{\phi_A} A \quad \& \quad Q \otimes_A P \xrightarrow{\phi_B} B$$

$\in A\text{-}\mathrm{Bimod}(\mathcal{C})$ $\quad$ $\in B\text{-}\mathrm{Bimod}(\mathcal{C})$

s.t.

⋮

THEN ϕ_A, ϕ_B ARE ISOS.

III. GENERALIZED MORITA'S THEOREM

GENERALIZED MORITA'S THEOREM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$$A\text{-}\mathrm{Mod}(\mathcal{C}) \simeq B\text{-}\mathrm{Mod}(\mathcal{C})$$

AS RIGHT $\mathcal{C}$-MODULE CATEGORIES

$$\Updownarrow$$

$\exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ $\ni$.

$\boxed{P \otimes_B Q \cong A_{reg}}$ AS A-BIMODULES IN $\mathcal{C}$

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≡ SHOWING THIS IN PRACTICE ≡

USE UNIVERSAL PROPERTY OF COEQUALIZERS TO GET:

$$P \otimes_B Q \xrightarrow{\phi_A} A \quad \& \quad Q \otimes_A P \xrightarrow{\phi_B} B$$

$\in A\text{-}\mathrm{Bimod}(\mathcal{C})$ $\quad$ $\in B\text{-}\mathrm{Bimod}(\mathcal{C})$

THEN USE

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL & $X \otimes -$, $- \otimes X$ ARE RIGHT EXACT $\forall X \in \mathcal{C}$

PROP TAKE ${}_A P_B$, ${}_B Q_A$. IF $\exists$ EPIS

$$P \otimes_B Q \xrightarrow{\phi_A} A \quad \& \quad Q \otimes_A P \xrightarrow{\phi_B} B$$

$\in A\text{-}\mathrm{Bimod}(\mathcal{C})$ $\quad$ $\in B\text{-}\mathrm{Bimod}(\mathcal{C})$

$\ni$.

$$\begin{tikzcd}
(P \otimes_B Q) \otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A id_P"'] & & P \otimes_B (Q \otimes_A P) \arrow[d, "id_P \otimes_B \phi_B"] \\
A \otimes_A P \arrow[dr, "\sim"'] & \circlearrowleft & P \otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

$$\begin{tikzcd}
(Q \otimes_A P) \otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B id_Q"'] & & Q \otimes_A (P \otimes_B Q) \arrow[d, "id_Q \otimes_A \phi_A"] \\
B \otimes_B Q \arrow[dr, "\sim"'] & \circlearrowleft & Q \otimes_A A \arrow[dl, "\sim"] \\
& Q &
\end{tikzcd}$$

THEN ϕ_A, ϕ_B ARE ISOS.

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL
& $X \otimes -$, $- \otimes X$ ARE RIGHT EXACT $\forall X \in \mathcal{C}$

GEN'D MORITA'S THM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$A\text{-}\mathrm{Mod}(\mathcal{C}) \simeq B\text{-}\mathrm{Mod}(\mathcal{C})$ IN $\mathrm{Mod}\text{-}\mathcal{C}$

$\Updownarrow$

$\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ IN $A\text{-}\mathrm{Bimod}(\mathcal{C})$

& $Q \otimes_A P \cong B_{reg}$ IN $B\text{-}\mathrm{Bimod}(\mathcal{C})$

PROP TAKE ${}_AP_B$, ${}_BQ_A$. IF $\exists$ EPIS

$P \otimes_B Q \xrightarrow{\phi_A} A$ & $Q \otimes_A P \xrightarrow{\phi_B} B$

$\in A\text{-}\mathrm{Bimod}(\mathcal{C})$ $\qquad \in B\text{-}\mathrm{Bimod}(\mathcal{C})$

s.t.

$$\begin{tikzcd}
(P \otimes_B Q) \otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A \mathrm{id}_P"'] & & P \otimes_B (Q \otimes_A P) \arrow[d, "\mathrm{id}_P \otimes_B \phi_B"] \\
A \otimes_A P \arrow[dr, "\sim"'] & \circlearrowleft & P \otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

$$\begin{tikzcd}
(Q \otimes_A P) \otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B \mathrm{id}_Q"'] & & Q \otimes_A (P \otimes_B Q) \arrow[d, "\mathrm{id}_Q \otimes_A \phi_A"] \\
B \otimes_B Q \arrow[dr, "\sim"'] & \circlearrowleft & Q \otimes_A A \arrow[dl, "\sim"] \\
& Q &
\end{tikzcd}$$

THEN ϕ_A, ϕ_B ARE ISOS.

III. Generalized Morita's Theorem

Assume $\mathcal{C}$ abelian $\overset{\vee}{\text{rigid}}$ monoidal
w/ simple $\mathbb{1}$

Gen'd Morita's Thm

Take $A, B \in \mathrm{Alg}(\mathcal{C})$. Then:

$$A\text{-}\mathrm{Mod}(\mathcal{C}) \simeq B\text{-}\mathrm{Mod}(\mathcal{C}) \text{ in } \mathrm{Mod}\text{-}\mathcal{C}$$

$$\Updownarrow$$

$\exists$ bimodules ${}_AP_B$ & ${}_BQ_A$ s.t.

$$P \otimes_B Q \cong A_{reg} \text{ in } A\text{-}\mathrm{Bimod}(\mathcal{C})$$

$$\& \; Q \otimes_A P \cong B_{reg} \text{ in } B\text{-}\mathrm{Bimod}(\mathcal{C})$$

Prop Take ${}_AP_B$, ${}_BQ_A$. If $\exists$ epis $P \otimes_B Q \xrightarrow{\phi_A} A \in A\text{-}\mathrm{Bimod}(\mathcal{C})$ & $Q \otimes_A P \xrightarrow{\phi_B} B \in B\text{-}\mathrm{Bimod}(\mathcal{C})$ s.t.

$$\begin{tikzcd}
(P \otimes_B Q) \otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A id_P"'] & & P \otimes_B (Q \otimes_A P) \arrow[d, "id_P \otimes_B \phi_B"] \\
A \otimes_A P \arrow[dr, "\sim"'] & 2 & P \otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

$$\begin{tikzcd}
(Q \otimes_A P) \otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B id_Q"'] & & Q \otimes_A (P \otimes_B Q) \arrow[d, "id_Q \otimes_A \phi_A"] \\
B \otimes_B Q \arrow[dr, "\sim"'] & 2 & Q \otimes_A A \arrow[dl, "\sim"] \\
& Q &
\end{tikzcd}$$

Then ϕ_A, ϕ_B are isos.

Exer. 4.48 Show $(\mathbb{1}, \; l_{\mathbb{1}} = r_{\mathbb{1}}, \; id_{\mathbb{1}})$ and

$$(X \otimes X^*, \; id_X \otimes ev_X^L \otimes id_{X^*}, \; coev_X^L)$$

are Morita equivalent $\forall X \in \mathcal{C}$.

III. Generalized Morita's Theorem

ASSUME $\mathcal{C}$ ABELIAN RIGID MONOIDAL W/ SIMPLE $\mathbb{1}$

GEN'D MORITA'S THM

TAKE $A, B \in Alg(\mathcal{C})$. THEN:

$A\text{-}Mod(\mathcal{C}) \simeq B\text{-}Mod(\mathcal{C})$ IN $Mod\text{-}\mathcal{C}$

$\Updownarrow$

$\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ IN $A\text{-}Bimod(\mathcal{C})$

& $Q \otimes_A P \cong B_{reg}$ IN $B\text{-}Bimod(\mathcal{C})$

PROP TAKE ${}_AP_B$, ${}_BQ_A$. IF $\exists$ EPIS

$P \otimes_B Q \xrightarrow{\phi_A} A$ & $Q \otimes_A P \xrightarrow{\phi_B} B$

$\in A\text{-}Bimod(\mathcal{C})$ $\quad \in B\text{-}Bimod(\mathcal{C})$

s.t.

$$\begin{tikzcd}
(P\otimes_B Q)\otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A id_P"'] & & P\otimes_B (Q\otimes_A P) \arrow[d, "id_P \otimes_B \phi_B"] \\
A\otimes_A P \arrow[dr, "\sim"'] & 2 & P\otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

$$\begin{tikzcd}
(Q\otimes_A P)\otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B id_Q"'] & & Q\otimes_A (P\otimes_B Q) \arrow[d, "id_Q \otimes_A \phi_A"] \\
B\otimes_B Q \arrow[dr, "\sim"'] & 2 & Q\otimes_A A \arrow[dl, "\sim"] \\
& Q &
\end{tikzcd}$$

THEN ϕ_A, ϕ_B ARE ISOS.

EXER. 4.48 SHOW $(\mathbb{1}, \ l_{\mathbb{1}} = r_{\mathbb{1}}, \ id_{\mathbb{1}})$ AND

$(X \otimes X^*, \ id_X \otimes ev_X^L \otimes id_{X^*}, \ coev_X^L)$

ARE MORITA EQUIVALENT $\forall X \in \mathcal{C}$.

Ex. $\mathcal{C} = FdVec$

HAVE $\Bbbk$ IS MORITA EQUIV. TO $Mat_n(\Bbbk)$

$$\Bbbk = \mathbb{1}_{FdVec} \qquad Mat_n(\Bbbk) \cong X \otimes X^*$$

FOR $X = \Bbbk^{\oplus n}$

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN RIGID MONOIDAL w/ SIMPLE $\mathbb{1}$

GEN'D MORITA'S THM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$ IN $\mathrm{Mod}\text{-}\mathcal{C}$

$\Updownarrow$

$\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ IN $A\text{-Bimod}(\mathcal{C})$

& $Q \otimes_A P \cong B_{reg}$ IN $B\text{-Bimod}(\mathcal{C})$

PROP TAKE ${}_AP_B$, ${}_BQ_A$. IF $\exists$ EPIS

$P \otimes_B Q \xrightarrow{\phi_A} A$ & $Q \otimes_A P \xrightarrow{\phi_B} B$

$\in A\text{-Bimod}(\mathcal{C})$ $\quad \in B\text{-Bimod}(\mathcal{C})$

s.t.

$$\begin{tikzcd}
(P\otimes_B Q)\otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A id_P"'] & & P\otimes_B (Q\otimes_A P) \arrow[d, "id_P \otimes_B \phi_B"] \\
A\otimes_A P \arrow[dr, "\sim"'] & 2 & P\otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

$$\begin{tikzcd}
(Q\otimes_A P)\otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B id_Q"'] & & Q\otimes_A (P\otimes_B Q) \arrow[d, "id_Q \otimes_A \phi_A"] \\
B\otimes_B Q \arrow[dr, "\sim"'] & 2 & Q\otimes_A A \arrow[dl, "\sim"] \\
& Q &
\end{tikzcd}$$

THEN ϕ_A, ϕ_B ARE ISOS.

EXER. 4.48 SHOW $(\mathbb{1}, l_{\mathbb{1}} = r_{\mathbb{1}}, id_{\mathbb{1}})$ AND

$(X \otimes X^*, id_X \otimes ev_X^L \otimes id_{X^*}, coev_X^L)$

ARE MORITA EQUIVALENT $\forall X \in \mathcal{C}$.

Ex. $\mathcal{C} = \mathrm{FdVec}$

HAVE $\Bbbk$ IS MORITA EQUIV. TO $\mathrm{Mat}_n(\Bbbk)$

$\Bbbk = \mathbb{1}_{\mathrm{FdVec}}$ $\qquad$ $\mathrm{Mat}_n(\Bbbk) \cong X \otimes X^*$ FOR $X = \Bbbk^{\oplus n}$

NAMELY, $\Big[$ $E_{i,j}$ ELEMENTARY MATRIX $\updownarrow$ $e_i \otimes_{\Bbbk} e_j^*$ FOR ELEMENTARY BASIS VECTORS $\{e_i\}$ $\Big]$

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN RIGID MONOIDAL W/ SIMPLE $\mathbb{1}$

GEN'D MORITA'S THM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$ IN $\text{Mod-}\mathcal{C}$

$$\Updownarrow$$

$\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ IN $A\text{-Bimod}(\mathcal{C})$

& $Q \otimes_A P \cong B_{reg}$ IN $B\text{-Bimod}(\mathcal{C})$

PROP TAKE ${}_AP_B$, ${}_BQ_A$. IF $\exists$ EPIS

$P \otimes_B Q \xrightarrow{\phi_A} A$ & $Q \otimes_A P \xrightarrow{\phi_B} B$

$\in A\text{-Bimod}(\mathcal{C})$ $\quad \in B\text{-Bimod}(\mathcal{C})$

s.t.

$$\begin{tikzcd}[column sep=small]
(P\otimes_B Q)\otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A \mathrm{id}_P"'] & & P\otimes_B (Q\otimes_A P) \arrow[d, "\mathrm{id}_P \otimes_B \phi_B"] \\
A \otimes_A P \arrow[dr, "\sim"'] & 2 & P \otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
(Q\otimes_A P)\otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B \mathrm{id}_Q"'] & & Q\otimes_A (P\otimes_B Q) \arrow[d, "\mathrm{id}_Q \otimes_A \phi_A"] \\
B \otimes_B Q \arrow[dr, "\sim"'] & 2 & Q \otimes_A A \arrow[dl, "\sim"] \\
& Q &
\end{tikzcd}$$

THEN ϕ_A, ϕ_B ARE ISOS.

EXER. 4.48 SHOW $A = (\mathbb{1}, \ l_{\mathbb{1}} = r_{\mathbb{1}}, \ \mathrm{id}_{\mathbb{1}})$ AND

$B = (X \otimes X^*, \ \mathrm{id}_X \otimes \mathrm{ev}_X^L \otimes \mathrm{id}_{X^*}, \ \mathrm{coev}_X^L)$

ARE MORITA EQUIVALENT $\forall X \in \mathcal{C}$.

GUESS FOR ${}_AP_B$ AND ${}_BQ_A$??

$${}_{\mathbb{1}}P_{X\otimes X^*} := \left(??, \quad \begin{array}{l} \triangleright : \mathbb{1} \otimes P \xrightarrow{??} P, \\ \triangleleft : P \otimes X \otimes X^* \xrightarrow{??} P \end{array} \right)$$

$${}_{X\otimes X^*}Q_{\mathbb{1}} := \left(??, \quad \begin{array}{l} \triangleright : X \otimes X^* \otimes Q \xrightarrow{??} Q, \\ \triangleleft : Q \otimes \mathbb{1} \xrightarrow{??} Q \end{array} \right)$$

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL W/ SIMPLE $\mathbb{1}$ ✓ RIGID

GEN'D MORITA'S THM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$ IN $\mathrm{Mod}\text{-}\mathcal{C}$

$$\Updownarrow$$

$\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ IN $A\text{-Bimod}(\mathcal{C})$

& $Q \otimes_A P \cong B_{reg}$ IN $B\text{-Bimod}(\mathcal{C})$

PROP TAKE ${}_AP_B$, ${}_BQ_A$. IF $\exists$ EPIS

$P \otimes_B Q \xrightarrow{\phi_A} A$ & $Q \otimes_A P \xrightarrow{\phi_B} B$

$\in A\text{-Bimod}(\mathcal{C})$, $\in B\text{-Bimod}(\mathcal{C})$

s.t.

$$\begin{tikzcd}
(P\otimes_B Q)\otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A id_P"'] & & P\otimes_B (Q\otimes_A P) \arrow[d, "id_P \otimes_B \phi_B"] \\
A\otimes_A P \arrow[dr, "\sim"'] & 2 & P\otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

$$\begin{tikzcd}
(Q\otimes_A P)\otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B id_Q"'] & & Q\otimes_A (P\otimes_B Q) \arrow[d, "id_Q \otimes_A \phi_A"] \\
B\otimes_B Q \arrow[dr, "\sim"'] & 2 & Q\otimes_A A \arrow[dl, "\sim"] \\
& Q &
\end{tikzcd}$$

THEN ϕ_A, ϕ_B ARE ISOS.

EXER. 4.48 SHOW $A = (\mathbb{1},\ l_{\mathbb{1}} = r_{\mathbb{1}},\ id_{\mathbb{1}})$ AND

$B = (X \otimes X^*,\ id_X \otimes ev_X^L \otimes id_{X^*},\ coev_X^L)$

ARE MORITA EQUIVALENT $\forall X \in \mathcal{C}$.

GUESS FOR ${}_AP_B$ AND ${}_BQ_A$??

$${}_{\mathbb{1}}P_{X\otimes X^*} := \left(X^*,\ \begin{array}{l} \triangleright : \mathbb{1} \otimes X^* \longrightarrow X^*, \\ \triangleleft : X^* \otimes X \otimes X^* \longrightarrow X^* \end{array} \right)$$

$${}_{X\otimes X^*}Q_{\mathbb{1}} := \left(X,\ \begin{array}{l} \triangleright : X \otimes X^* \otimes X \longrightarrow X, \\ \triangleleft : X \otimes \mathbb{1} \longrightarrow X \end{array} \right)$$

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL w/ SIMPLE $\mathbb{1}$ RIGID

GEN'D MORITA'S THM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$A\text{-}\mathrm{Mod}(\mathcal{C}) \simeq B\text{-}\mathrm{Mod}(\mathcal{C})$ IN $\mathrm{Mod}\text{-}\mathcal{C}$

$\Updownarrow$

$\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ IN $A\text{-}\mathrm{Bimod}(\mathcal{C})$

& $Q \otimes_A P \cong B_{reg}$ IN $B\text{-}\mathrm{Bimod}(\mathcal{C})$

PROP TAKE ${}_AP_B$, ${}_BQ_A$. IF $\exists$ EPIS

$P \otimes_B Q \xrightarrow{\phi_A} A$ & $Q \otimes_A P \xrightarrow{\phi_B} B$

$\in A\text{-}\mathrm{Bimod}(\mathcal{C})$ $\quad \in B\text{-}\mathrm{Bimod}(\mathcal{C})$

s.t.

$$\begin{tikzcd}
(P \otimes_B Q) \otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A id_P"'] & & P \otimes_B (Q \otimes_A P) \arrow[d, "id_P \otimes_B \phi_B"] \\
A \otimes_A P \arrow[dr, "\sim"'] & ② & P \otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

$$\begin{tikzcd}
(Q \otimes_A P) \otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B id_Q"'] & & Q \otimes_A (P \otimes_B Q) \arrow[d, "id_Q \otimes_A \phi_A"] \\
B \otimes_B Q \arrow[dr, "\sim"'] & ② & Q \otimes_A A \arrow[dl, "\sim"] \\
& Q &
\end{tikzcd}$$

THEN ϕ_A, ϕ_B ARE ISOS.

EXER. 4.48 SHOW $A = (\mathbb{1}, l_{\mathbb{1}} = r_{\mathbb{1}}, id_{\mathbb{1}})$ AND

$B = (X \otimes X^*, id_X \otimes ev_X^L \otimes id_{X^*}, coev_X^L)$

ARE MORITA EQUIVALENT $\forall X \in \mathcal{C}$.

GUESS FOR ${}_AP_B$ AND ${}_BQ_A$??

$${}_{\mathbb{1}}P_{X\otimes X^*} := \left(X^*, \begin{array}{l} \triangleright : \mathbb{1} \otimes X^* \xrightarrow{l_{X^*}} X^*, \\ \triangleleft : X^* \otimes X \otimes X^* \xrightarrow{ev_X^L \otimes id} X^* \end{array} \right)$$

$${}_{X\otimes X^*}Q_{\mathbb{1}} := \left(X, \begin{array}{l} \triangleright : X \otimes X^* \otimes X \xrightarrow{id \otimes ev_X^L} X, \\ \triangleleft : X \otimes \mathbb{1} \xrightarrow{r_X} X \end{array} \right)$$

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL RIGID W/ SIMPLE $\mathbb{1}$

GEN'D MORITA'S THM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$ IN $\mathrm{Mod}\text{-}\mathcal{C}$

$\Updownarrow$

$\exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ IN $A\text{-Bimod}(\mathcal{C})$

& $Q \otimes_A P \cong B_{reg}$ IN $B\text{-Bimod}(\mathcal{C})$

PROP TAKE ${}_A P_B$, ${}_B Q_A$. IF $\exists$ EPIS

$P \otimes_B Q \xrightarrow{\phi_A} A$ & $Q \otimes_A P \xrightarrow{\phi_B} B$

$\in A\text{-Bimod}(\mathcal{C})$ $\quad \in B\text{-Bimod}(\mathcal{C})$

s.t.

$$\begin{tikzcd}
(P\otimes_B Q)\otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A id_P"'] & & P\otimes_B (Q\otimes_A P) \arrow[d, "id_P \otimes_B \phi_B"] \\
A\otimes_A P \arrow[dr, "\sim"'] & 2 & P\otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

$$\begin{tikzcd}
(Q\otimes_A P)\otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B id_Q"'] & & Q\otimes_A (P\otimes_B Q) \arrow[d, "id_Q \otimes_A \phi_A"] \\
B\otimes_B Q \arrow[dr, "\sim"'] & 2 & Q\otimes_A A \arrow[dl, "\sim"] \\
& Q &
\end{tikzcd}$$

THEN ϕ_A, ϕ_B ARE ISOS.

EXER. 4.48 SHOW $A = (\mathbb{1},\ l_{\mathbb{1}} = r_{\mathbb{1}},\ id_{\mathbb{1}})$ AND

$B = (X \otimes X^*,\ id_X \otimes ev_X^L \otimes id_{X^*},\ coev_X^L)$

ARE MORITA EQUIVALENT $\forall X \in \mathcal{C}$.

ARE THESE REALLY BIMODULES??

$${}_{\mathbb{1}} P_{X\otimes X^*} := \left(X^*,\ \begin{array}{l} \triangleright : \mathbb{1} \otimes X^* \xrightarrow{l_{X^*}} X^*, \\ \triangleleft : X^* \otimes X \otimes X^* \xrightarrow{ev_X^L \otimes id} X^* \end{array} \right)$$

$${}_{X\otimes X^*} Q_{\mathbb{1}} := \left(X,\ \begin{array}{l} \triangleright : X \otimes X^* \otimes X \xrightarrow{id \otimes ev_X^L} X, \\ \triangleleft : X \otimes \mathbb{1} \xrightarrow{r_X} X \end{array} \right)$$

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN RIGID MONOIDAL
W/ SIMPLE $\mathbb{1}$

GEN'D MORITA'S THM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$ IN $\mathrm{Mod}\text{-}\mathcal{C}$

$$\Updownarrow$$

$\exists$ BIMODULES $_AP_B$ & $_BQ_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ IN $A\text{-Bimod}(\mathcal{C})$

& $Q \otimes_A P \cong B_{reg}$ IN $B\text{-Bimod}(\mathcal{C})$

PROP TAKE $_AP_B$, $_BQ_A$. IF $\exists$ EPIS

$P \otimes_B Q \xrightarrow{\phi_A} A$ & $Q \otimes_A P \xrightarrow{\phi_B} B$
$\in A\text{-Bimod}(\mathcal{C})$ $\quad \in B\text{-Bimod}(\mathcal{C})$

s.t.

$$\begin{tikzcd}
(P \otimes_B Q) \otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A id_P"'] & & P \otimes_B (Q \otimes_A P) \arrow[d, "id_P \otimes_B \phi_B"] \\
A \otimes_A P \arrow[dr, "\sim"'] & 2 & P \otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

$$\begin{tikzcd}
(Q \otimes_A P) \otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B id_Q"'] & & Q \otimes_A (P \otimes_B Q) \arrow[d, "id_Q \otimes_A \phi_A"] \\
B \otimes_B Q \arrow[dr, "\sim"'] & 2 & Q \otimes_A A \arrow[dl, "\sim"] \\
& Q &
\end{tikzcd}$$

THEN ϕ_A, ϕ_B ARE ISOS.

EXER. 4.48 SHOW $A = (\mathbb{1},\ l_{\mathbb{1}} = r_{\mathbb{1}},\ id_{\mathbb{1}})$ AND

$B = (X \otimes X^*,\ id_X \otimes ev^L_X \otimes id_{X^*},\ coev^L_X)$

ARE MORITA EQUIVALENT $\forall X \in \mathcal{C}$.

ARE THESE REALLY BIMODULES?? *You do!*

$$_{\mathbb{1}}P_{X \otimes X^*} := \left(X^*,\ \begin{array}{l} \triangleright : \mathbb{1} \otimes X^* \xrightarrow{l_{X^*}} X^*, \\ \triangleleft : X^* \otimes X \otimes X^* \xrightarrow{ev^L_X \otimes id} X^* \end{array} \right)$$

$$_{X \otimes X^*}Q_{\mathbb{1}} := \left(X,\ \begin{array}{l} \triangleright : X \otimes X^* \otimes X \xrightarrow{id \otimes ev^L_X} X, \\ \triangleleft : X \otimes \mathbb{1} \xrightarrow{r_X} X \end{array} \right)$$

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN, RIGID MONOIDAL W/ SIMPLE $\mathbb{1}$

GEN'D MORITA'S THM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$ IN $\text{Mod-}\mathcal{C}$

$$\Updownarrow$$

$\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ IN $A\text{-Bimod}(\mathcal{C})$

& $Q \otimes_A P \cong B_{reg}$ IN $B\text{-Bimod}(\mathcal{C})$

PROP TAKE ${}_AP_B$, ${}_BQ_A$. IF $\exists$ EPIS

$P \otimes_B Q \xrightarrow{\phi_A} A$ $\in A\text{-Bimod}(\mathcal{C})$ & $Q \otimes_A P \xrightarrow{\phi_B} B$ $\in B\text{-Bimod}(\mathcal{C})$

s.t.

$$\begin{tikzcd}
(P\otimes_B Q)\otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A id_P"'] & & P\otimes_B (Q\otimes_A P) \arrow[d, "id_P \otimes_B \phi_B"] \\
A\otimes_A P \arrow[dr, "\sim"'] & 2 & P\otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

$$\begin{tikzcd}
(Q\otimes_A P)\otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B id_Q"'] & & Q\otimes_A (P\otimes_B Q) \arrow[d, "id_Q \otimes_A \phi_A"] \\
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\end{tikzcd}$$

THEN ϕ_A, ϕ_B ARE ISOS.

EXER. 4.48 SHOW $A = (\mathbb{1},\ l_{\mathbb{1}} = r_{\mathbb{1}},\ id_{\mathbb{1}})$ AND

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$${}_{\mathbb{1}}P_{X\otimes X^*} = (X^*,\ \triangleright = l_{X^*},\ \triangleleft = ev_X^L \otimes id_{X^*})$$

$${}_{X\otimes X^*}Q_{\mathbb{1}} = (X,\ \triangleright = id_X \otimes ev_X^L,\ \triangleleft = r_X)$$

NEED $\phi_{\mathbb{1}} : X^* \otimes_{X\otimes X^*} X \longrightarrow \mathbb{1}$

$\phi_{X\otimes X^*} : X \otimes_{\mathbb{1}} X^* \longrightarrow X \otimes X^*$

SATISFYING

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL W/ SIMPLE $\mathbb{1}$ RIGID

GEN'D MORITA'S THM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$A\text{-}\mathrm{Mod}(\mathcal{C}) \simeq B\text{-}\mathrm{Mod}(\mathcal{C})$ IN $\mathrm{Mod}\text{-}\mathcal{C}$

$\Updownarrow$

$\exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ IN $A\text{-}\mathrm{Bimod}(\mathcal{C})$

& $Q \otimes_A P \cong B_{reg}$ IN $B\text{-}\mathrm{Bimod}(\mathcal{C})$

PROP TAKE ${}_A P_B$, ${}_B Q_A$. IF $\exists$ EPIS

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$\in A\text{-}\mathrm{Bimod}(\mathcal{C})$ $\quad \in B\text{-}\mathrm{Bimod}(\mathcal{C})$

s.t.

$$\begin{tikzcd}
(P\otimes_B Q)\otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A id_P"'] & & P\otimes_B (Q\otimes_A P) \arrow[d, "id_P \otimes_B \phi_B"] \\
A\otimes_A P \arrow[dr, "\sim"'] & 2 & P\otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

$$\begin{tikzcd}
(Q\otimes_A P)\otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B id_Q"'] & & Q\otimes_A (P\otimes_B Q) \arrow[d, "id_Q \otimes_A \phi_A"] \\
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$${}_{X\otimes X^*}Q_{\mathbb{1}} = (X, \triangleright = id_X \otimes ev_X^L, \triangleleft = r_X)$$

NEED $\phi_{\mathbb{1}} : X^* \otimes_{X\otimes X^*} X \longrightarrow \mathbb{1}$

$\phi_{X\otimes X^*} : X \otimes_{\mathbb{1}} X^* \xrightarrow{\sim} X \otimes X^*$

SATISFYING

III. Generalized Morita's Theorem

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL W/ SIMPLE $\mathbb{1}$ RIGID

Gen'd Morita's Thm

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$ IN $\mathrm{Mod}\text{-}\mathcal{C}$

$\Updownarrow$

$\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ IN $A\text{-Bimod}(\mathcal{C})$

& $Q \otimes_A P \cong B_{reg}$ IN $B\text{-Bimod}(\mathcal{C})$

PROP TAKE ${}_AP_B$, ${}_BQ_A$. IF $\exists$ EPIS

$P \otimes_B Q \xrightarrow{\phi_A} A$ & $Q \otimes_A P \xrightarrow{\phi_B} B$

$\in A\text{-Bimod}(\mathcal{C})$ $\quad \in B\text{-Bimod}(\mathcal{C})$

s.t.

$$\begin{tikzcd}
(P \otimes_B Q) \otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A id_P"'] & & P \otimes_B (Q \otimes_A P) \arrow[d, "id_P \otimes_B \phi_B"] \\
A \otimes_A P \arrow[dr, "\sim"'] \arrow[rr, phantom, "\circlearrowleft"] & & P \otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

$$\begin{tikzcd}
(Q \otimes_A P) \otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B id_Q"'] & & Q \otimes_A (P \otimes_B Q) \arrow[d, "id_Q \otimes_A \phi_A"] \\
B \otimes_B Q \arrow[dr, "\sim"'] \arrow[rr, phantom, "\circlearrowleft"] & & Q \otimes_A A \arrow[dl, "\sim"] \\
& Q &
\end{tikzcd}$$

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$$\phi_{X \otimes X^*} : X \otimes_{\mathbb{1}} X^* \xrightarrow{\sim} X \otimes X^*$$

NEED $\phi_{\mathbb{1}} : X^* \otimes_{X \otimes X^*} X \longrightarrow \mathbb{1}$ (EPIC)

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN RIGID MONOIDAL
W/ SIMPLE $\mathbb{1}$

GEN'D MORITA'S THM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$ IN $\text{Mod-}\mathcal{C}$

$$\Updownarrow$$

$\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

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& $Q \otimes_A P \cong B_{reg}$ IN $B\text{-Bimod}(\mathcal{C})$

PROP TAKE ${}_AP_B$, ${}_BQ_A$. IF $\exists$ EPIS

$P \otimes_B Q \xrightarrow{\phi_A} A$ & $Q \otimes_A P \xrightarrow{\phi_B} B$
$\in A\text{-Bimod}(\mathcal{C})$ $\quad \in B\text{-Bimod}(\mathcal{C})$

s.t.

$$\begin{tikzcd}
(P\otimes_B Q)\otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A id_P"'] & & P\otimes_B(Q\otimes_A P) \arrow[d, "id_P \otimes_B \phi_B"] \\
A\otimes_A P \arrow[dr, "\sim"'] & \arrow[phantom, "2"] & P\otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

$$\begin{tikzcd}
(Q\otimes_A P)\otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B id_Q"'] & & Q\otimes_A(P\otimes_B Q) \arrow[d, "id_Q \otimes_A \phi_A"] \\
B\otimes_B Q \arrow[dr, "\sim"'] & \arrow[phantom, "2"] & Q\otimes_A A \arrow[dl, "\sim"] \\
& Q &
\end{tikzcd}$$

THEN ϕ_A, ϕ_B ARE ISOS.

EXER. 4.48 SHOW $A = (\mathbb{1},\ l_{\mathbb{1}} = r_{\mathbb{1}},\ id_{\mathbb{1}})$ AND

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ARE MORITA EQUIVALENT $\forall X \in \mathcal{C}$.

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$${}_{X\otimes X^*}Q_{\mathbb{1}} = (X,\ \triangleright = id_X \otimes ev^L_X,\ \triangleleft = r_X)$$

$$\phi_{X\otimes X^*} : X \otimes_{\mathbb{1}} X^* \xrightarrow{\sim} X \otimes X^*$$

NEED $\phi_{\mathbb{1}} : X^* \otimes_{X\otimes X^*} X \longrightarrow \mathbb{1}$ (EPIC)

REVIEW DEFN OF COEQUALIZER TO DERIVE THIS

III. Generalized Morita's Theorem

Assume $\mathcal{C}$ abelian rigid monoidal w/ simple $\mathbb{1}$

Gen'd Morita's Thm

Take $A, B \in \mathrm{Alg}(\mathcal{C})$. Then:

$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$ in $\text{Mod-}\mathcal{C}$

$\Updownarrow$

$\exists$ bimodules ${}_A P_B$ & ${}_B Q_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ in $A\text{-Bimod}(\mathcal{C})$

& $Q \otimes_A P \cong B_{reg}$ in $B\text{-Bimod}(\mathcal{C})$

Prop Take ${}_A P_B$, ${}_B Q_A$. If $\exists$ epis

$P \otimes_B Q \xrightarrow{\phi_A} A \in A\text{-Bimod}(\mathcal{C})$ & $Q \otimes_A P \xrightarrow{\phi_B} B \in B\text{-Bimod}(\mathcal{C})$

s.t.

$$\begin{tikzcd}
(P\otimes_B Q)\otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A \mathrm{id}_P"'] & \arrow[d, phantom, "2"] & P\otimes_B (Q\otimes_A P) \arrow[d, "\mathrm{id}_P \otimes_B \phi_B"] \\
A\otimes_A P \arrow[dr, "\sim"'] & & P\otimes_B B \arrow[dl, "\sim"] \\
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\end{tikzcd}$$

$$\begin{tikzcd}
(Q\otimes_A P)\otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B \mathrm{id}_Q"'] & \arrow[d, phantom, "2"] & Q\otimes_A (P\otimes_B Q) \arrow[d, "\mathrm{id}_Q \otimes_A \phi_A"] \\
B\otimes_B Q \arrow[dr, "\sim"'] & & Q\otimes_A A \arrow[dl, "\sim"] \\
& Q &
\end{tikzcd}$$

Then ϕ_A, ϕ_B are isos.

Exer. 4.48 Show $A = (\mathbb{1}, \, l_{\mathbb{1}} = r_{\mathbb{1}}, \, \mathrm{id}_{\mathbb{1}})$ and

$B = (X \otimes X^*, \, \mathrm{id}_X \otimes ev_X^L \otimes \mathrm{id}_{X^*}, \, coev_X^L)$

are Morita equivalent $\forall X \in \mathcal{C}$.

$${}_{\mathbb{1}}P_{X\otimes X^*} = (X^*, \ \triangleright = l_{X^*}, \ \triangleleft = ev_X^L \otimes \mathrm{id}_{X^*})$$

$${}_{X\otimes X^*}Q_{\mathbb{1}} = (X, \ \triangleright = \mathrm{id}_X \otimes ev_X^L, \ \triangleleft = r_X)$$

$$\phi_{X\otimes X^*} : X \otimes_{\mathbb{1}} X^* \xrightarrow{\sim} X \otimes X^*$$

Need $\phi_{\mathbb{1}} : X^* \otimes_{X\otimes X^*} X \longrightarrow \mathbb{1}$ (epic)

Omitting assoc

$$\begin{tikzcd}
X^*\otimes X \otimes X^* \otimes X \arrow[r, shift left, "\triangleleft \otimes \mathrm{id} \otimes \mathrm{id}"] \arrow[r, shift right, "\mathrm{id}\otimes\mathrm{id}\otimes\triangleright"'] & X^* \otimes X \arrow[r, "\alpha"] & X^* \otimes_{X\otimes X^*} X
\end{tikzcd}$$

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL W/ SIMPLE $\mathbb{1}$, RIGID

GEN'D MORITA'S THM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$A\text{-}\mathrm{Mod}(\mathcal{C}) \simeq B\text{-}\mathrm{Mod}(\mathcal{C})$ IN $\mathrm{Mod}\text{-}\mathcal{C}$

$\Updownarrow$

$\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ IN $A\text{-}\mathrm{Bimod}(\mathcal{C})$

& $Q \otimes_A P \cong B_{reg}$ IN $B\text{-}\mathrm{Bimod}(\mathcal{C})$

PROP TAKE ${}_AP_B$, ${}_BQ_A$. IF $\exists$ EPIS

$P \otimes_B Q \xrightarrow{\phi_A} A$ & $Q \otimes_A P \xrightarrow{\phi_B} B$

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$$\begin{tikzcd}
(P\otimes_B Q)\otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A id_P"'] & & P\otimes_B(Q\otimes_A P) \arrow[d, "id_P \otimes_B \phi_B"] \\
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\end{tikzcd}$$

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(Q\otimes_A P)\otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B id_Q"'] & & Q\otimes_A(P\otimes_B Q) \arrow[d, "id_Q \otimes_A \phi_A"] \\
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\end{tikzcd}$$

THEN ϕ_A, ϕ_B ARE ISOS.

EXER. 4.48 SHOW $A = (\mathbb{1}, \ l_{\mathbb{1}} = r_{\mathbb{1}}, \ id_{\mathbb{1}})$ AND

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ARE MORITA EQUIVALENT $\forall X \in \mathcal{C}$.

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$$\phi_{X\otimes X^*} : X \otimes_{\mathbb{1}} X^* \xrightarrow{\sim} X \otimes X^*$$

NEED $\phi_{\mathbb{1}} : X^* \otimes_{X\otimes X^*} X \longrightarrow \mathbb{1}$ (EPIC)

OMITTING ASSOC

$$X^*\otimes X\otimes X^*\otimes X \underset{\substack{id\otimes id\otimes \triangleright \\ \| \\ id\otimes id\otimes ev_X^L}}{\overset{\substack{ev_X^L\otimes id\otimes id \\ \| \\ \triangleleft\otimes id\otimes id}}{\rightrightarrows}} X^*\otimes X \xrightarrow{\alpha} X^*\otimes_{X\otimes X^*} X$$

III. Generalized Morita's Theorem

Assume $\mathcal{C}$ abelian $\overset{\text{rigid}}{\vee}$ monoidal w/ simple $\mathbb{1}$

Gen'd Morita's Thm

Take $A, B \in \mathrm{Alg}(\mathcal{C})$. Then:

$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$ in $\mathrm{Mod}\text{-}\mathcal{C}$

$\Updownarrow$

$\exists$ bimodules ${}_AP_B$ & ${}_BQ_A$ s.t.

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$$\begin{tikzcd}
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A \otimes_A P \arrow[dr, "\sim"'] \arrow[r, phantom, "2"] & P \otimes_B B \arrow[d, "\sim"] \\
& P
\end{tikzcd}$$

$$\begin{tikzcd}
(Q\otimes_A P)\otimes_B Q \arrow[r, "\sim"] \arrow[d, "\phi_B \otimes_B id_Q"'] & Q\otimes_A (P\otimes_B Q) \arrow[d, "id_Q \otimes_A \phi_A"] \\
B \otimes_B Q \arrow[dr, "\sim"'] \arrow[r, phantom, "2"] & Q \otimes_A A \arrow[d, "\sim"] \\
& Q
\end{tikzcd}$$

Then ϕ_A, ϕ_B are isos.

Exer. 4.48 Show $A = (\mathbb{1}, \; l_{\mathbb{1}} = r_{\mathbb{1}}, \; id_{\mathbb{1}})$ and

$B = (X \otimes X^*, \; id_X \otimes ev_X^L \otimes id_{X^*}, \; coev_X^L)$

are Morita equivalent $\forall X \in \mathcal{C}$.

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$$\phi_{X\otimes X^*} : X \otimes_{\mathbb{1}} X^* \xrightarrow{\sim} X \otimes X^*$$

NEED $\quad \phi_{\mathbb{1}} : X^* \otimes_{X\otimes X^*} X \longrightarrow \mathbb{1} \quad$ (epic)

Omitting assoc

$$\begin{tikzcd}[column sep=huge]
X^*\otimes X \otimes X^* \otimes X \arrow[r, shift left, "{\substack{ev_X^L \otimes id \otimes id \\ \| \\ \triangleleft \otimes id \otimes id}}"] \arrow[r, shift right, "{\substack{id \otimes id \otimes \triangleright \\ \| \\ id \otimes id \otimes ev_X^L}}"'] & X^* \otimes X \arrow[r, "\alpha"] \arrow[dr, "ev_X^L"'] & X^* \otimes_{X\otimes X^*} X \\
& & \mathbb{1}
\end{tikzcd}$$

III. Generalized Morita's Theorem

Assume $\mathcal{C}$ abelian monoidal w/ simple $\mathbb{1}$, rigid

Gen'd Morita's Thm

Take $A, B \in \mathrm{Alg}(\mathcal{C})$. Then:

$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$ in $\text{Mod-}\mathcal{C}$

$\Updownarrow$

$\exists$ bimodules ${}_AP_B$ & ${}_BQ_A$ s.t.

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$$\begin{tikzcd}
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B\otimes_B Q \arrow[dr, "\sim"'] & 2 & Q\otimes_A A \arrow[dl, "\sim"] \\
& Q &
\end{tikzcd}$$

Then ϕ_A, ϕ_B are isos.

Exer. 4.48 Show $A = (\mathbb{1},\ l_{\mathbb{1}} = r_{\mathbb{1}},\ id_{\mathbb{1}})$ and

$B = (X \otimes X^*,\ id_X \otimes ev_X^L \otimes id_{X^*},\ coev_X^L)$

are Morita equivalent $\forall X \in \mathcal{C}$.

$${}_{\mathbb{1}}P_{X\otimes X^*} = (X^*,\ \triangleright = l_{X^*},\ \triangleleft = ev_X^L \otimes id_{X^*})$$

$${}_{X\otimes X^*}Q_{\mathbb{1}} = (X,\ \triangleright = id_X \otimes ev_X^L,\ \triangleleft = r_X)$$

$$\phi_{X\otimes X^*} : X \otimes_{\mathbb{1}} X^* \xrightarrow{\sim} X \otimes X^*$$

Need $\phi_{\mathbb{1}} : X^* \otimes_{X\otimes X^*} X \longrightarrow \mathbb{1}$ (epic)

Omitting assoc

$$\begin{tikzcd}[column sep=huge]
X^*\otimes X\otimes X^*\otimes X \arrow[r, shift left, "ev_X^L\otimes id\otimes id \,=\, \triangleleft\otimes id\otimes id"] \arrow[r, shift right, "id\otimes id\otimes \triangleright \,=\, id\otimes id\otimes ev_X^L"'] & X^*\otimes X \arrow[r, "\alpha"] \arrow[dr, "ev_X^L"'] & X^*\otimes_{X\otimes X^*} X \arrow[d, dashed, "\exists!\, \phi_{\mathbb{1}}"] \\
& & \mathbb{1}
\end{tikzcd}$$

Univ. Prop of Coeqs

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL w/ SIMPLE $\mathbb{1}$ ✓ RIGID

GEN'D MORITA'S THM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$ IN $\mathrm{Mod}\text{-}\mathcal{C}$

$\Updownarrow$

$\exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ IN $A\text{-Bimod}(\mathcal{C})$

& $Q \otimes_A P \cong B_{reg}$ IN $B\text{-Bimod}(\mathcal{C})$

PROP TAKE ${}_A P_B$, ${}_B Q_A$. IF $\exists$ EPIS

$P \otimes_B Q \xrightarrow{\phi_A} A \in A\text{-Bimod}(\mathcal{C})$ & $Q \otimes_A P \xrightarrow{\phi_B} B \in B\text{-Bimod}(\mathcal{C})$

s.t.

$$\begin{tikzcd}
(P \otimes_B Q) \otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A id_P"'] & & P \otimes_B (Q \otimes_A P) \arrow[d, "id_P \otimes_B \phi_B"] \\
A \otimes_A P \arrow[dr, "\sim"'] & \arrow[phantom, "2"] & P \otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

$$\begin{tikzcd}
(Q \otimes_A P) \otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B id_Q"'] & & Q \otimes_A (P \otimes_B Q) \arrow[d, "id_Q \otimes_A \phi_A"] \\
B \otimes_B Q \arrow[dr, "\sim"'] & \arrow[phantom, "2"] & Q \otimes_A A \arrow[dl, "\sim"] \\
& Q &
\end{tikzcd}$$

THEN ϕ_A, ϕ_B ARE ISOS.

EXER. 4.48 SHOW $A = (\mathbb{1}, \ l_{\mathbb{1}} = r_{\mathbb{1}}, \ id_{\mathbb{1}})$ AND

$B = (X \otimes X^*, \ id_X \otimes ev_X^L \otimes id_{X^*}, \ coev_X^L)$

ARE MORITA EQUIVALENT $\forall X \in \mathcal{C}$.

$${}_{\mathbb{1}}P_{X\otimes X^*} = (X^*, \ \triangleright = l_{X^*}, \ \triangleleft = ev_X^L \otimes id_{X^*})$$

$${}_{X\otimes X^*}Q_{\mathbb{1}} = (X, \ \triangleright = id_X \otimes ev_X^L, \ \triangleleft = r_X)$$

$$\phi_{X\otimes X^*} : X \otimes_{\mathbb{1}} X^* \xrightarrow{\sim} X \otimes X^*$$

NEED $\phi_{\mathbb{1}} : X^* \otimes_{X \otimes X^*} X \longrightarrow \mathbb{1}$ (EPIC) ??

OMITTING ASSOC

$$\begin{tikzcd}
X^* \otimes X \otimes X^* \otimes X \arrow[r, shift left, "{ev_X^L \otimes id \otimes id \,=\, \triangleleft \otimes id \otimes id}"] \arrow[r, shift right, "{id \otimes id \otimes \triangleright \,=\, id \otimes id \otimes ev_X^L}"'] & X^* \otimes X \arrow[r, "\alpha"] \arrow[dr, "ev_X^L"'] & X^* \otimes_{X \otimes X^*} X \arrow[d, dashed, "\exists ! \, \phi_{\mathbb{1}}"] \\
& & \mathbb{1}
\end{tikzcd}$$

UNIV. PROP OF COEQS

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL W/ SIMPLE $\mathbb{1}$ RIGID

GEN'D MORITA'S THM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

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$\Updownarrow$

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PROP TAKE ${}_AP_B$, ${}_BQ_A$. IF $\exists$ EPIS $P \otimes_B Q \xrightarrow{\phi_A} A \in A\text{-Bimod}(\mathcal{C})$ & $Q \otimes_A P \xrightarrow{\phi_B} B \in B\text{-Bimod}(\mathcal{C})$ s.t.

$$\begin{tikzcd}
(P\otimes_B Q)\otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A id_P"'] & \arrow[d, phantom, "2"] & P\otimes_B(Q\otimes_A P) \arrow[d, "id_P \otimes_B \phi_B"] \\
A \otimes_A P \arrow[dr, "\sim"'] & & P \otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

$$\begin{tikzcd}
(Q\otimes_A P)\otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B id_Q"'] & \arrow[d, phantom, "2"] & Q\otimes_A(P\otimes_B Q) \arrow[d, "id_Q \otimes_A \phi_A"] \\
B \otimes_B Q \arrow[dr, "\sim"'] & & Q \otimes_A A \arrow[dl, "\sim"] \\
& Q &
\end{tikzcd}$$

THEN ϕ_A, ϕ_B ARE ISOS.

EXER. 4.48 SHOW $A = (\mathbb{1},\ l_{\mathbb{1}} = r_{\mathbb{1}},\ id_{\mathbb{1}})$ AND

$B = (X \otimes X^*,\ id_X \otimes ev_X^L \otimes id_{X^*},\ coev_X^L)$

ARE MORITA EQUIVALENT $\forall X \in \mathcal{C}$.

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$$\phi_{X\otimes X^*} : X \otimes_{\mathbb{1}} X^* \xrightarrow{\sim} X \otimes X^*$$

NEED $\phi_{\mathbb{1}} : X^* \otimes_{X\otimes X^*} X \longrightarrow \mathbb{1}$ (EPIC)

OMITTING ASSOC

$$\begin{tikzcd}
X^*\otimes X \otimes X^* \otimes X \arrow[r, shift left, "{\triangleleft \otimes id \otimes id \,=\, ev_X^L \otimes id \otimes id}"] \arrow[r, shift right, "{id \otimes id \otimes \triangleright \,=\, id \otimes id \otimes ev_X^L}"'] & X^* \otimes X \arrow[r, "\alpha"] \arrow[dr, "ev_X^L"'] & X^* \otimes_{X\otimes X^*} X \arrow[d, dashed, "\exists!\, \phi_{\mathbb{1}}"] \\
& & \mathbb{1}
\end{tikzcd}$$

UNIV. PROP OF COEQS

EPIC [$\mathbb{1}$ SIMPLE] $\Rightarrow$ EPIC

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL w/ SIMPLE $\mathbb{1}$ RIGID

GEN'D MORITA'S THM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$ IN $\mathrm{Mod}\text{-}\mathcal{C}$

$\Updownarrow$

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& $Q \otimes_A P \cong B_{reg}$ IN $B\text{-Bimod}(\mathcal{C})$

PROP TAKE ${}_A P_B$, ${}_B Q_A$. IF $\exists$ EPIS

$P \otimes_B Q \xrightarrow{\phi_A} A$ & $Q \otimes_A P \xrightarrow{\phi_B} B$

$\in A\text{-Bimod}(\mathcal{C})$ $\quad \in B\text{-Bimod}(\mathcal{C})$

s.t.

$$\begin{tikzcd}
(P\otimes_B Q)\otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A \otimes_A id_P"'] & & P\otimes_B (Q\otimes_A P) \arrow[d, "id_P \otimes_B \phi_B"] \\
A\otimes_A P \arrow[dr, "\sim"'] & 2 & P\otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

$$\begin{tikzcd}
(Q\otimes_A P)\otimes_B Q \arrow[rr, "\sim"] \arrow[d, "\phi_B \otimes_B id_Q"'] & & Q\otimes_A (P\otimes_B Q) \arrow[d, "id_Q \otimes_A \phi_A"] \\
B\otimes_B Q \arrow[dr, "\sim"'] & 2 & Q\otimes_A A \arrow[dl, "\sim"] \\
& Q &
\end{tikzcd}$$

THEN ϕ_A, ϕ_B ARE ISOS.

EXER. 4.48 SHOW $A = (\mathbb{1},\ l_{\mathbb{1}} = r_{\mathbb{1}},\ id_{\mathbb{1}})$ AND

$B = (X \otimes X^*,\ id_X \otimes ev_X^L \otimes id_{X^*},\ coev_X^L)$

ARE MORITA EQUIVALENT $\forall X \in \mathcal{C}$.

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$\phi_{X\otimes X^*} : X \otimes_{\mathbb{1}} X^* \xrightarrow{\sim} X \otimes X^*$ EPIC

$$\begin{tikzcd}
X^*\otimes X \otimes X^* \otimes X \arrow[r, shift left, "{ev_X^L\otimes id\otimes id \,=\, \triangleleft\otimes id\otimes id}"] \arrow[r, shift right, "{id\otimes id\otimes \triangleright \,=\, id\otimes id\otimes ev_X^L}"'] & X^*\otimes X \arrow[r, "\alpha"] \arrow[dr, "ev_X^L"'] & X^* \otimes_{X\otimes X^*} X \arrow[d, dashed, "\exists!\ \phi_{\mathbb{1}}\ \text{EPIC}"] \\
& & \mathbb{1}
\end{tikzcd}$$

UNIV. PROP OF COEQS

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN RIGID MONOIDAL W/ SIMPLE $\mathbb{1}$

GEN'D MORITA'S THM

TAKE $A,B\in \mathrm{Alg}(\mathcal{C})$. THEN:

$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$ IN $\mathrm{Mod}\text{-}\mathcal{C}$

$\Updownarrow$

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PROP TAKE ${}_AP_B$, ${}_BQ_A$. IF $\exists$ EPIS

$P\otimes_B Q \xrightarrow{\phi_A} A$ $\in A\text{-Bimod}(\mathcal{C})$ & $Q\otimes_A P \xrightarrow{\phi_B} B$ $\in B\text{-Bimod}(\mathcal{C})$

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(P\otimes_B Q)\otimes_A P \arrow[rr, "\sim"] \arrow[d, "\phi_A\otimes_A \mathrm{id}_P"'] & & P\otimes_B(Q\otimes_A P) \arrow[d, "\mathrm{id}_P\otimes_B\phi_B"] \\
A\otimes_A P \arrow[dr, "\sim"'] & \circlearrowright & P\otimes_B B \arrow[dl, "\sim"] \\
& P &
\end{tikzcd}$$

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& Q &
\end{tikzcd}$$

THEN ϕ_A, ϕ_B ARE ISOS.

EXER. 4.48 SHOW $A=(\mathbb{1},\ l_{\mathbb{1}}=r_{\mathbb{1}},\ \mathrm{id}_{\mathbb{1}})$ AND $B=(X\otimes X^*,\ \mathrm{id}_X\otimes ev^L_X\otimes \mathrm{id}_{X^*},\ coev^L_X)$ ARE MORITA EQUIVALENT $\forall X\in\mathcal{C}$.

$${}_{\mathbb{1}}P_{X\otimes X^*} = (X^*,\ \triangleright = l_{X^*},\ \triangleleft = ev^L_X\otimes \mathrm{id}_{X^*})$$

$${}_{X\otimes X^*}Q_{\mathbb{1}} = (X,\ \triangleright = \mathrm{id}_X\otimes ev^L_X,\ \triangleleft = r_X)$$

$\phi_{X\otimes X^*}: X\otimes_{\mathbb{1}} X^* \xrightarrow{\sim} X\otimes X^*$ EPIC

DO THESE DIAGRAMS COMMUTE?

$$\begin{tikzcd}
X^*\otimes X\otimes X^*\otimes X \arrow[r, shift left, "{ev^L_X\otimes \mathrm{id}\otimes \mathrm{id} \,=\, \triangleleft\otimes \mathrm{id}\otimes \mathrm{id}}"] \arrow[r, shift right, "{\mathrm{id}\otimes \mathrm{id}\otimes \triangleright \,=\, \mathrm{id}\otimes \mathrm{id}\otimes ev^L_X}"'] & X^*\otimes X \arrow[r, "\alpha"] \arrow[dr, "ev^L_X"'] & X^*\otimes_{X\otimes X^*} X \arrow[d, dashed, "\exists!\,\phi_{\mathbb{1}}\ \text{EPIC}"] \\
& & \mathbb{1}
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UNIV. PROP OF COEQS

III. GENERALIZED MORITA'S THEOREM

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL W/ SIMPLE $\mathbb{1}$, RIGID

GEN'D MORITA'S THM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$ IN Mod-$\mathcal{C}$

$\Updownarrow$

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$P \otimes_B Q \cong A_{reg}$ IN A-Bimod$(\mathcal{C})$

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PROP TAKE ${}_A P_B$, ${}_B Q_A$. IF $\exists$ EPIS

$P \otimes_B Q \xrightarrow{\phi_A} A$ & $Q \otimes_A P \xrightarrow{\phi_B} B$

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\end{tikzcd}$$

THEN ϕ_A, ϕ_B ARE ISOS.

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$\phi_{X\otimes X^*} : X \otimes_{\mathbb{1}} X^* \xrightarrow{\sim} X \otimes X^*$ EPIC

DO THESE DIAGRAMS COMMUTE?

Youdo!

$$\begin{tikzcd}[column sep=large]
X^*\otimes X\otimes X^*\otimes X \arrow[r, shift left, "{\substack{ev_X^L\otimes id\otimes id \\ \| \\ \triangleleft\otimes id\otimes id}}"] \arrow[r, shift right, "{\substack{id\otimes id\otimes \triangleright \\ \| \\ id\otimes id\otimes ev_X^L}}"'] & X^*\otimes X \arrow[r, "\alpha"] \arrow[dr, "ev_X^L"'] & X^*\otimes_{X\otimes X^*} X \arrow[d, dashed, "\exists!\ \phi_{\mathbb{1}}\ \text{EPIC}"] \\
& & \mathbb{1}
\end{tikzcd}$$

UNIV. PROP OF COEQS

ASSUME $\mathcal{C}$ ABELIAN MONOIDAL & $X \otimes -$, $- \otimes X$ ARE RIGHT EXACT $\forall X \in \mathcal{C}$

GENERALIZED MORITA'S THEOREM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$. THEN:

$$A\text{-Mod}(\mathcal{C}) \simeq B\text{-Mod}(\mathcal{C})$$

AS RIGHT $\mathcal{C}$-MODULE CATEGORIES

$$\Updownarrow$$

$\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES IN $\mathcal{C}$

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES IN $\mathcal{C}$.

GENERALIZED EW THEOREM

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$.

THEN, WE GET AN EQUIV. OF CATS:

$$\mathrm{Rex}_{\text{Mod-}\mathcal{C}}\big(A\text{-Mod}(\mathcal{C}), B\text{-Mod}(\mathcal{C})\big) \simeq (B, A)\text{-Bimod}(\mathcal{C}).$$

RIGHT EXACT
MOD. CAT. FUNCTORS

MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

NEXT TIME

OSTRIK'S THEOREM

LECTURE #20

TOPICS:

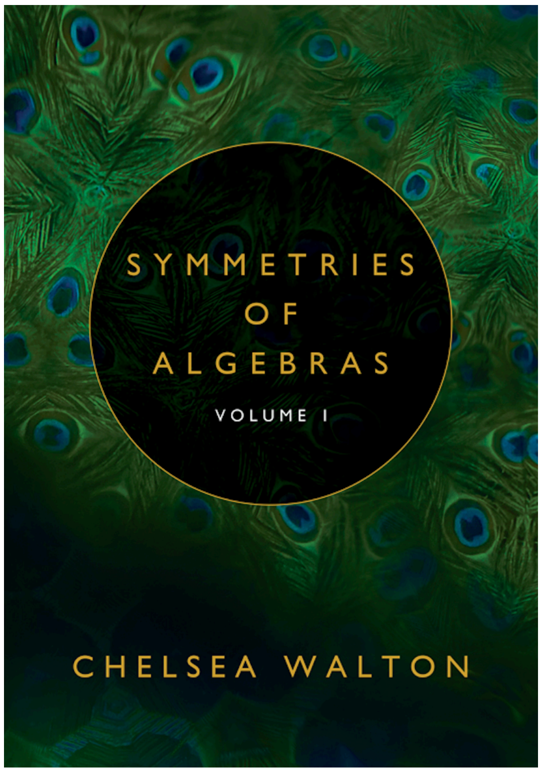
I. OPERATIONS ON ALGEBRAS & (BI)MODULES (§4.5)

II. GENERALIZED EILENBERG-WATTS THEOREM (§4.7.1)

III. GENERALIZED MORITA'S THEOREM (§4.7.2)

Enjoy this lecture?
You'll enjoy the textbook!

C. Walton's "Symmetries of Algebras, Volume 1" (2024)



Available for purchase at :

619 Wreath (at a discount)

https://www.619wreath.com/

Also on Amazon
&
Google Play

Lecture #20 keywords: algebraic operations in monoidal categories, Generalized Eilenberg-Watts Theorem, Generalized Morita's Theorem