MATH 466/566
SPRING 2024

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RICE U.

LECTURE #21

LAST TIME

- ALG. & (BI)MOD. OPERATIONS
- GEN'D EILENBERG-WATTS THM
- GEN'D MORITA'S THM

TOPICS:

I. INTERNAL END ALGEBRAS (§4.8.1)

II. OSTRIK'S THEOREM (§4.8.2)

≡ RECALL ≡

TAKE A MONOIDAL CATEGORY $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$,
HAVE TWO TYPES OF MODULES:

≡ RECALL ≡

TAKE A MONOIDAL CATEGORY $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$,
HAVE TWO TYPES OF MODULES :

INTERNAL TO $\mathcal{C}$

FOR $A := (A, m, u) \in \mathrm{Alg}(\mathcal{C})$

Mod-A($\mathcal{C}$)

RIGHT A-MODULES
$M := (M, \triangleleft)$ IN $\mathcal{C}$

≡ RECALL ≡

TAKE A MONOIDAL CATEGORY $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$,
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INTERNAL TO $\mathcal{C}$

FOR $A := (A, m, u) \in \mathrm{Alg}(\mathcal{C})$

$\mathrm{Mod}\text{-}A(\mathcal{C})$

RIGHT A-MODULES
$M := (M, \triangleleft)$ IN $\mathcal{C}$

EXTERNAL TO $\mathcal{C}$

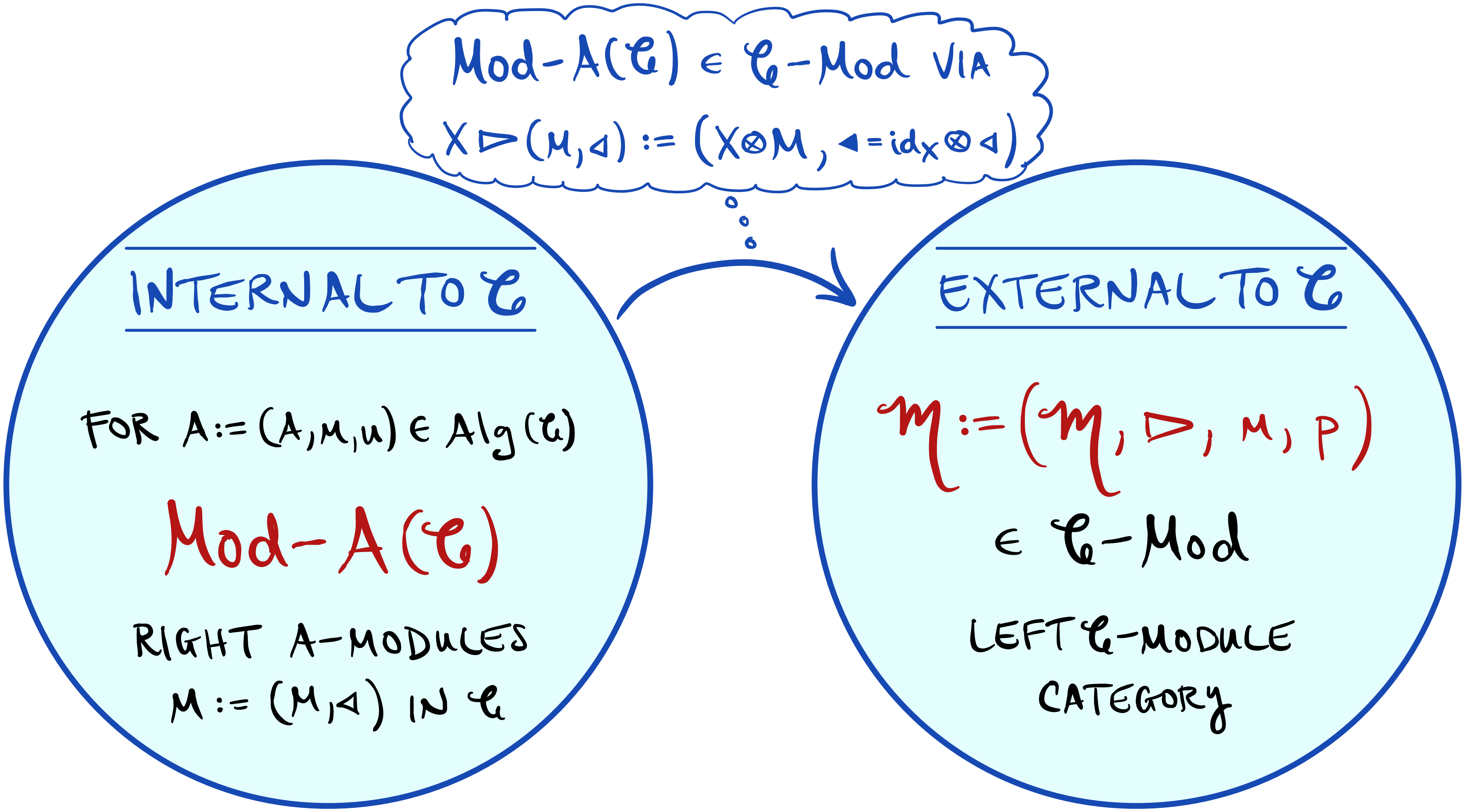
$\mathcal{M} := (\mathcal{M}, \triangleright, m, p)$

$\in \mathcal{C}\text{-}\mathrm{Mod}$

LEFT $\mathcal{C}$-MODULE
CATEGORY

≡ RECALL ≡

TAKE A MONOIDAL CATEGORY $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, a, \ell, r)$,
HAVE TWO TYPES OF MODULES:



$\text{Mod-}A(\mathcal{C}) \in \mathcal{C}\text{-Mod}$ VIA
$X \triangleright (M, \triangleleft) := (X \otimes M, \blacktriangleleft = \mathrm{id}_X \otimes \triangleleft)$

INTERNAL TO $\mathcal{C}$

FOR $A := (A, m, u) \in \mathsf{Alg}(\mathcal{C})$

$\text{Mod-}A(\mathcal{C})$

RIGHT A-MODULES
$M := (M, \triangleleft)$ IN $\mathcal{C}$

EXTERNAL TO $\mathcal{C}$

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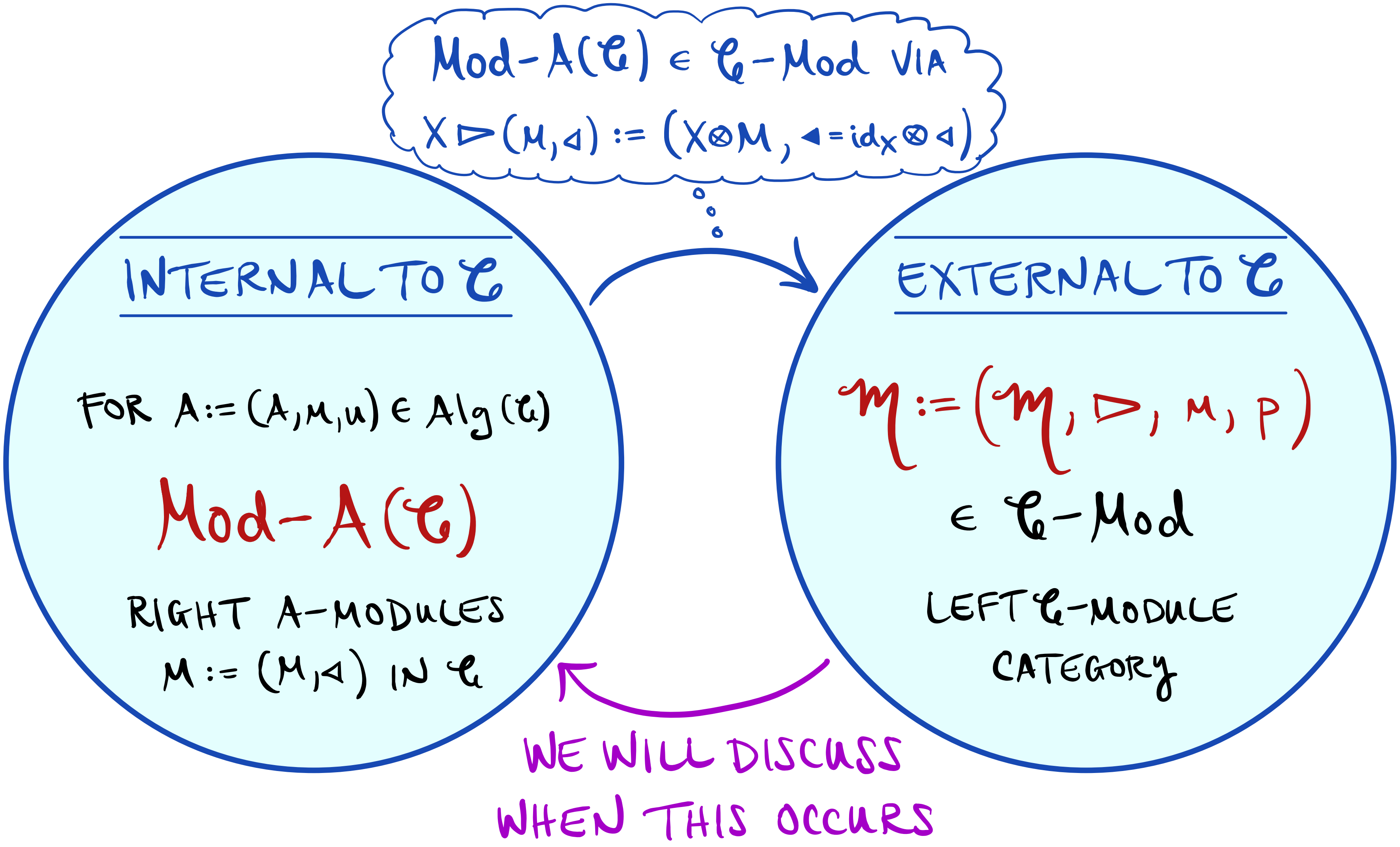

$\mathrm{Mod}\text{-}A(\mathcal{C}) \in \mathcal{C}\text{-}\mathrm{Mod}$ VIA

$X \triangleright (M, \triangleleft) := (X \otimes M, \blacktriangleleft = \mathrm{id}_X \otimes \triangleleft)$

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FOR $A := (A, m, u) \in \mathrm{Alg}(\mathcal{C})$

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RIGHT A-MODULES
$M := (M, \triangleleft)$ IN $\mathcal{C}$

EXTERNAL TO $\mathcal{C}$

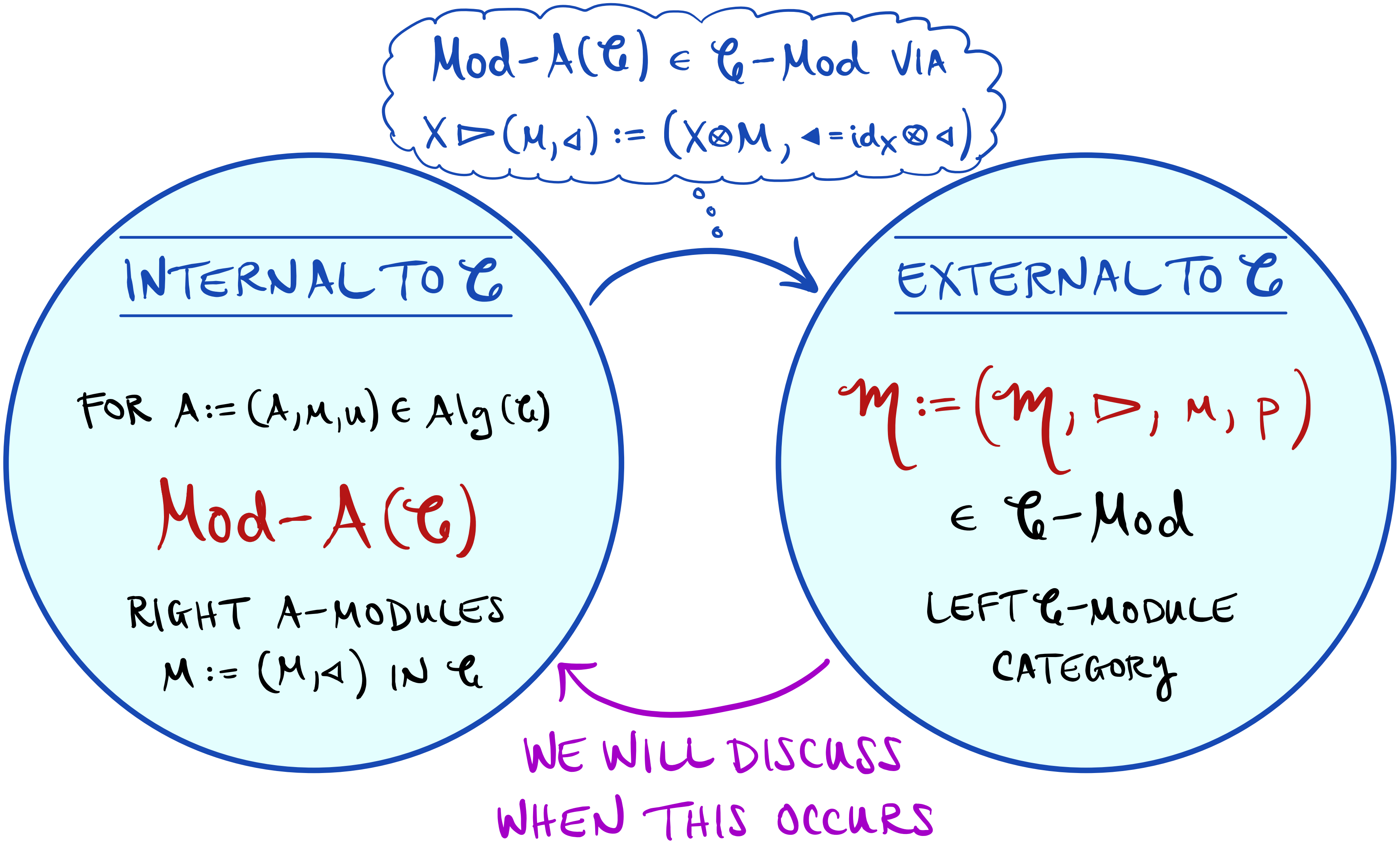
$\mathcal{M} := (\mathcal{M}, \triangleright, m, p)$

$\in \mathcal{C}\text{-}\mathrm{Mod}$

LEFT $\mathcal{C}$-MODULE
CATEGORY

WE WILL DISCUSS
WHEN THIS OCCURS

$\mathcal{M} \in \mathcal{C}$-Mod IS REPRESENTED BY $A \in \mathsf{Alg}(\mathcal{C})$ IF $\mathcal{M} \simeq$ Mod-$A(\mathcal{C})$ AS LEFT $\mathcal{C}$-MODULE CATEGORIES.



Mod-$A(\mathcal{C}) \in \mathcal{C}$-Mod VIA

$X \triangleright (M, \triangleleft) := (X \otimes M, \blacktriangleleft = \mathrm{id}_X \otimes \triangleleft)$

INTERNAL TO $\mathcal{C}$

FOR $A := (A, m, u) \in \mathsf{Alg}(\mathcal{C})$

Mod-$A(\mathcal{C})$

RIGHT A-MODULES
$M := (M, \triangleleft)$ IN $\mathcal{C}$

EXTERNAL TO $\mathcal{C}$

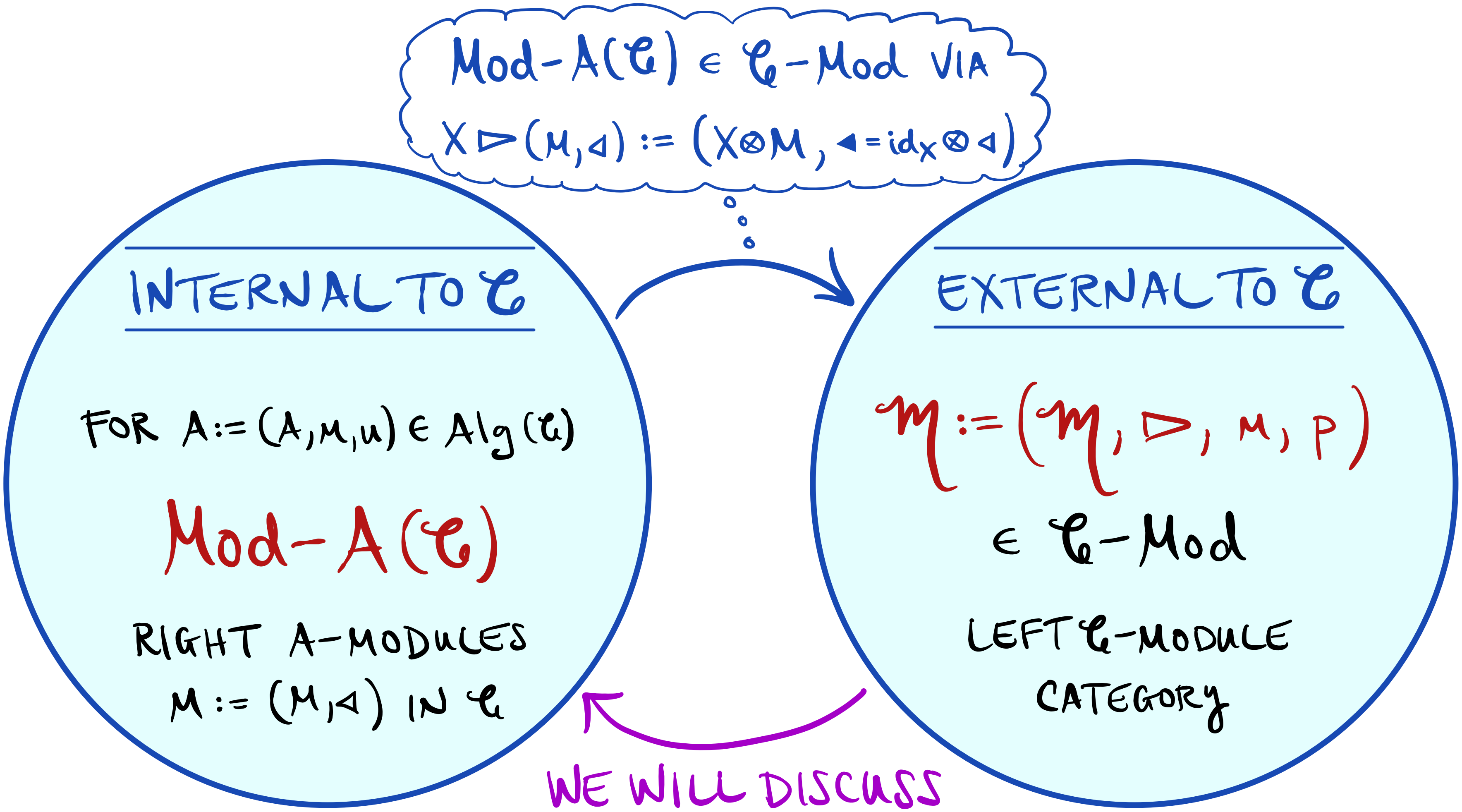
$\mathcal{M} := (\mathcal{M}, \triangleright, m, p)$

$\in \mathcal{C}$-Mod

LEFT $\mathcal{C}$-MODULE
CATEGORY

WE WILL DISCUSS
WHEN THIS OCCURS

$\mathcal{M} \in \text{Mod-}\mathcal{C}$ IS REPRESENTED BY $A \in \text{Alg}(\mathcal{C})$ IF $\mathcal{M} \simeq A\text{-Mod}(\mathcal{C})$ AS RIGHT $\mathcal{C}$-MODULE CATEGORIES.



$\text{Mod-}A(\mathcal{C}) \in \mathcal{C}\text{-Mod}$ VIA

$X \triangleright (M, \triangleleft) := (X \otimes M, \blacktriangleleft = \mathrm{id}_X \otimes \triangleleft)$

INTERNAL TO $\mathcal{C}$

FOR $A := (A, m, u) \in \text{Alg}(\mathcal{C})$

$\text{Mod-}A(\mathcal{C})$

RIGHT A-MODULES
$M := (M, \triangleleft)$ IN $\mathcal{C}$

EXTERNAL TO $\mathcal{C}$

$\mathcal{M} := (\mathcal{M}, \triangleright, m, p)$

$\in \mathcal{C}\text{-Mod}$

LEFT $\mathcal{C}$-MODULE
CATEGORY

WE WILL DISCUSS
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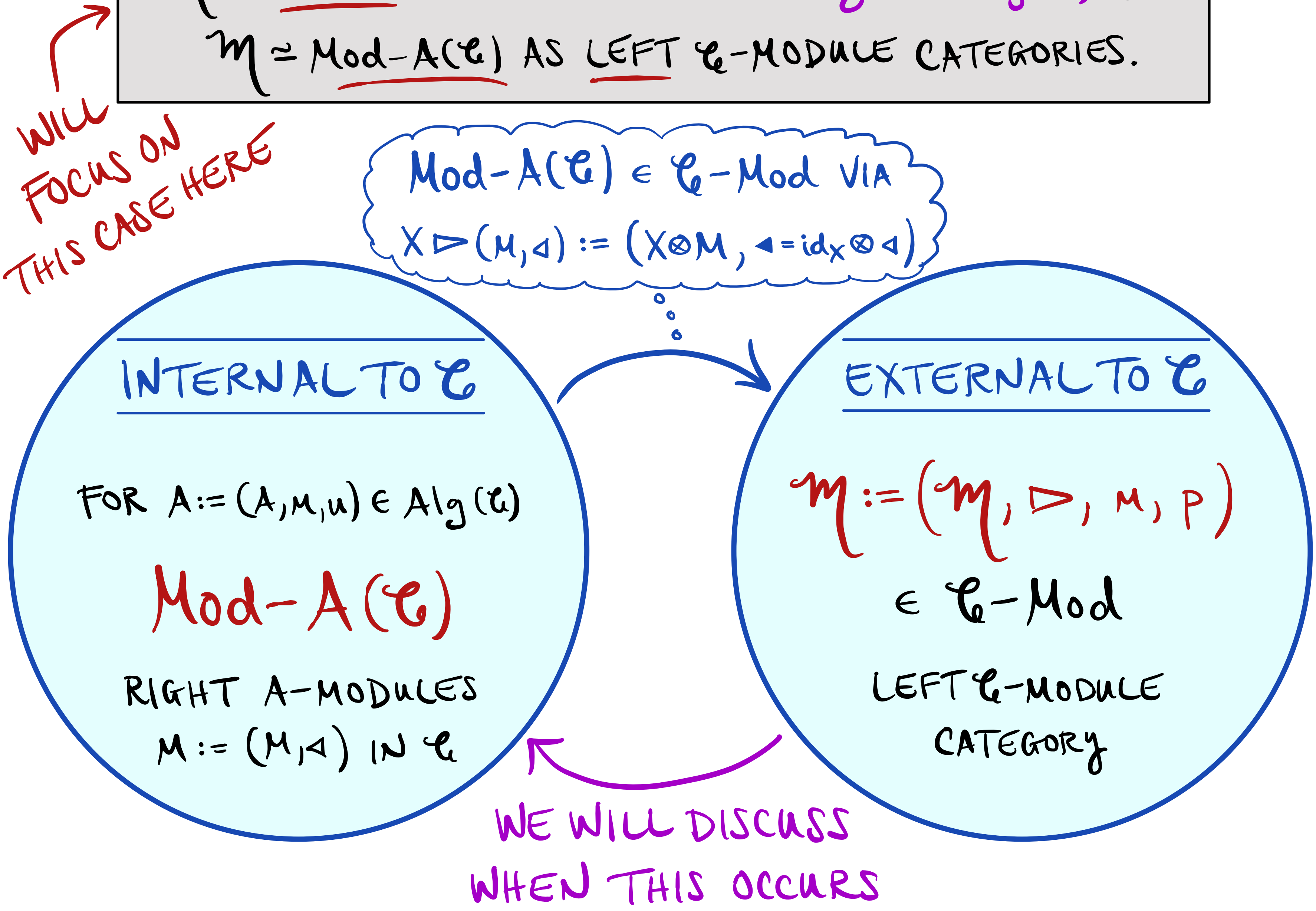
$\mathcal{M} \in \mathcal{C}$-Mod IS REPRESENTED BY $A \in \mathsf{Alg}(\mathcal{C})$ IF $\mathcal{M} \simeq$ Mod-$A(\mathcal{C})$ AS LEFT $\mathcal{C}$-MODULE CATEGORIES.



WILL FOCUS ON THIS CASE HERE

Mod-$A(\mathcal{C}) \in \mathcal{C}$-Mod VIA

$X \triangleright (M, \triangleleft) := (X \otimes M, \blacktriangleleft = \mathrm{id}_X \otimes \triangleleft)$

INTERNAL TO $\mathcal{C}$

FOR $A := (A, m, u) \in \mathsf{Alg}(\mathcal{C})$

Mod-$A(\mathcal{C})$

RIGHT A-MODULES $M := (M, \triangleleft)$ IN $\mathcal{C}$

EXTERNAL TO $\mathcal{C}$

$\mathcal{M} := (\mathcal{M}, \triangleright, m, p)$

$\in \mathcal{C}$-Mod

LEFT $\mathcal{C}$-MODULE CATEGORY

WE WILL DISCUSS WHEN THIS OCCURS

I. INTERNAL END ALGEBRAS

≡ RECALL ≡

A LEFT $\mathcal{C}$-MODULE CATEGORY

$$\begin{tikzcd}[column sep=huge]
\Bigg(\mathcal{M},\ \triangleright : \mathcal{C}\times\mathcal{M}\to\mathcal{M},\ \mathcal{C}\times\mathcal{C}\times\mathcal{M} \arrow[r, bend left=50, "\triangleright(\otimes\times\mathrm{Id}_{\mathcal{M}})"{name=U1}] \arrow[r, bend right=50, "\triangleright(\mathrm{Id}_{\mathcal{C}}\times\triangleright)"'{name=D1}] & \mathcal{M},\ \mathcal{M} \arrow[r, bend left=50, "\triangleright(\mathbb{1}\times\mathrm{Id}_{\mathcal{M}})"{name=U2}] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{M}}"'{name=D2}] & \mathcal{M} \Bigg)
\arrow[Rightarrow, from=U1, to=D1, "m", "\sim"']
\arrow[Rightarrow, from=U2, to=D2, "p", "\sim"']
\end{tikzcd}$$

IS CLOSED IF ...

I. INTERNAL END ALGEBRAS

≡ RECALL ≡

A LEFT $\mathcal{C}$-MODULE CATEGORY

$$\begin{tikzcd}[column sep=huge]
\Big(\mathcal{M},\ \triangleright:\mathcal{C}\times\mathcal{M}\to\mathcal{M},\ \mathcal{C}\times\mathcal{C}\times\mathcal{M} \arrow[r, bend left=50, "\triangleright(\otimes\times\mathrm{Id}_{\mathcal{M}})"{name=U}] \arrow[r, bend right=50, "\triangleright(\mathrm{Id}_{\mathcal{C}}\times\triangleright)"'{name=D}] & \mathcal{M},\ \mathcal{M} \arrow[r, bend left=50, "\triangleright(\mathbb{1}\times\mathrm{Id}_{\mathcal{M}})"{name=U2}] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{M}}"'{name=D2}] & \mathcal{M}\Big)
\arrow[Rightarrow, from=U, to=D, "m", "\sim"']
\arrow[Rightarrow, from=U2, to=D2, "p", "\sim"']
\end{tikzcd}$$

IS **CLOSED** IF $(-\triangleright M):\mathcal{C}\longrightarrow\mathcal{M}$ HAS A RIGHT ADJOINT $\underline{\mathrm{Hom}}(M,-):\mathcal{M}\longrightarrow\mathcal{C}$, $\forall M\in\mathcal{M}$.

I. INTERNAL END ALGEBRAS

≡ RECALL ≡

A LEFT $\mathcal{C}$-MODULE CATEGORY

$$\left(\mathcal{M},\ \triangleright : \mathcal{C}\times\mathcal{M}\to\mathcal{M},\ \begin{tikzcd} \mathcal{C}\times\mathcal{C}\times\mathcal{M} \arrow[r, bend left=60, "\triangleright(\otimes\times\mathrm{Id}_{\mathcal{M}})"{name=U}] \arrow[r, bend right=60, "\triangleright(\mathrm{Id}_{\mathcal{C}}\times\triangleright)"'{name=D}] & \mathcal{M} \arrow[from=U, to=D, Rightarrow, "m", "\sim"'] \end{tikzcd},\ \begin{tikzcd} \mathcal{M} \arrow[r, bend left=60, "\triangleright(\mathbb{1}\times\mathrm{Id}_{\mathcal{M}})"{name=U}] \arrow[r, bend right=60, "\mathrm{Id}_{\mathcal{M}}"'{name=D}] & \mathcal{M} \arrow[from=U, to=D, Rightarrow, "p", "\sim"'] \end{tikzcd} \right)$$

IS CLOSED IF $(- \triangleright M) : \mathcal{C} \to \mathcal{M}$ HAS A RIGHT ADJOINT $\underline{\mathrm{Hom}}(M, -) : \mathcal{M} \to \mathcal{C}$, $\forall M \in \mathcal{M}$.

$\underline{\mathrm{Hom}}(M,N) \in \mathcal{C}$ ≡ INTERNAL HOM OF $M, N \in \mathcal{M}$

$\underline{\mathrm{End}}(M) \in \mathcal{C}$ ≡ INTERNAL END OF $M \in \mathcal{M}$

I. INTERNAL END ALGEBRAS

≡ RECALL ≡

A LEFT $\mathcal{C}$-MODULE CATEGORY

$$\begin{tikzcd} \Big(\mathcal{M},\ \triangleright:\mathcal{C}\times\mathcal{M}\to\mathcal{M},\ \mathcal{C}\times\mathcal{C}\times\mathcal{M} \arrow[r, bend left, "\triangleright(\otimes\times\mathrm{Id}_{\mathcal{M}})"] \arrow[r, bend right, "\triangleright(\mathrm{Id}_{\mathcal{C}}\times\triangleright)"'] \arrow[r, phantom, "\sim\Downarrow M"] & \mathcal{M},\ \mathcal{M} \arrow[r, bend left, "\triangleright(\mathbb{1}\times\mathrm{Id}_{\mathcal{M}})"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{M}}"'] \arrow[r, phantom, "\sim\Downarrow P"] & \mathcal{M}\Big) \end{tikzcd}$$

IS CLOSED IF $(-\triangleright M):\mathcal{C}\to\mathcal{M}$ HAS A RIGHT ADJOINT $\underline{\mathrm{Hom}}(M,-):\mathcal{M}\to\mathcal{C}$, $\forall M\in\mathcal{M}$.

$\underline{\mathrm{Hom}}(M,N)\in\mathcal{C}$ ≡ INTERNAL HOM OF $M,N\in\mathcal{M}$

$\underline{\mathrm{End}}(M)\in\mathcal{C}$ ≡ INTERNAL END OF $M\in\mathcal{M}$

WILL GIVE THESE ALGEBRAIC STRUCTURE IN $\mathcal{C}$

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) : \mathcal{C} \to \mathcal{M}$ HAS A RIGHT ADJOINT $\underline{\mathrm{Hom}}(M, -) : \mathcal{M} \to \mathcal{C}$, $\forall M \in \mathcal{M}$.

$$\mathfrak{J} := \mathfrak{J}_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M, N))$$

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS **CLOSED** IF $(- \triangleright M) : \mathcal{C} \to \mathcal{M}$ HAS A RIGHT ADJOINT $\underline{\mathrm{Hom}}(M, -) : \mathcal{M} \to \mathcal{C}$, $\forall M \in \mathcal{M}$.

$$\mathcal{J} := \mathcal{J}_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$\uparrow$

$\mathrm{id}_{\underline{\mathrm{Hom}}(M,N)}$

TAKE
$Z = \underline{\mathrm{Hom}}(M,N)$

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) : \mathcal{C} \to \mathcal{M}$ HAS A RIGHT ADJOINT $\underline{\mathrm{Hom}}(M, -) : \mathcal{M} \to \mathcal{C}$, $\forall M \in \mathcal{M}$.

$\varsigma := \varsigma_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$$\left[\underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N\right] \overset{\varsigma^{-1}}{\longleftarrow\!\shortmid} \mathrm{id}_{\underline{\mathrm{Hom}}(M,N)}$$

TAKE $Z = \underline{\mathrm{Hom}}(M,N)$

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) : \mathcal{C} \to \mathcal{M}$ HAS A RIGHT ADJOINT $\underline{\mathrm{Hom}}(M, -) : \mathcal{M} \to \mathcal{C}$, $\forall M \in \mathcal{M}$.

$$\mathfrak{J} := \mathfrak{J}_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$$\left[\mathrm{ev}_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N\right] \overset{\mathfrak{J}^{-1}}{\longleftarrow\!\shortmid} \mathrm{id}_{\underline{\mathrm{Hom}}(M,N)}$$

MORPHISM IN $\mathcal{M}$

TAKE $Z = \underline{\mathrm{Hom}}(M,N)$

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) : \mathcal{C} \to \mathcal{M}$ HAS A RIGHT ADJOINT $\underline{\mathrm{Hom}}(M, -) : \mathcal{M} \to \mathcal{C}$, $\forall M \in \mathcal{M}$.

$$\zeta := \zeta_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$$\left[ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \right] \overset{\zeta^{-1}}{\longleftarrow\!\!\shortmid} id_{\underline{\mathrm{Hom}}(M,N)}$$

MORPHISM IN $\mathcal{M}$

DEFINE:

$$\left(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \right) \triangleright M \overset{ev_{M,N,P}}{\dashrightarrow} P$$

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) : \mathcal{C} \to \mathcal{M}$ HAS A RIGHT ADJOINT $\underline{\mathrm{Hom}}(M, -) : \mathcal{M} \to \mathcal{C}$, $\forall M \in \mathcal{M}$.

$$\mathfrak{J} := \mathfrak{J}_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$$\underbrace{\left[ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \right]}_{\text{MORPHISM IN } \mathcal{M}} \overset{\mathfrak{J}^{-1}}{\longmapsfrom} \mathrm{id}_{\underline{\mathrm{Hom}}(M,N)}$$

DEFINE:

$$\begin{tikzcd}
\left(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)\right) \triangleright M \arrow[r, dashed, "ev_{M,N,P}"] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P), \underline{\mathrm{Hom}}(M,N), M}"'] & P \\
\underline{\mathrm{Hom}}(N,P) \triangleright \left(\underline{\mathrm{Hom}}(M,N) \triangleright M\right) &
\end{tikzcd}$$

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) : \mathcal{C} \to \mathcal{M}$ HAS A RIGHT ADJOINT $\underline{\mathrm{Hom}}(M, -) : \mathcal{M} \to \mathcal{C}$, $\forall M \in \mathcal{M}$.

$\mathfrak{J} := \mathfrak{J}_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$$\left[ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \right] \xleftarrow{\mathfrak{J}^{-1}} id_{\underline{\mathrm{Hom}}(M,N)}$$

MORPHISM IN $\mathcal{M}$

DEFINE:

$$\begin{tikzcd}
\left(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)\right) \triangleright M \arrow[r, dashed, "ev_{M,N,P}"] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P), \underline{\mathrm{Hom}}(M,N), M}"'] & P \\
\underline{\mathrm{Hom}}(N,P) \triangleright \left(\underline{\mathrm{Hom}}(M,N) \triangleright M\right) \arrow[r, "id \triangleright ev_{M,N}"] & \underline{\mathrm{Hom}}(N,P) \triangleright N
\end{tikzcd}$$

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) : \mathcal{C} \to \mathcal{M}$ HAS A RIGHT ADJOINT $\underline{\mathrm{Hom}}(M, -) : \mathcal{M} \to \mathcal{C}$, $\forall M \in \mathcal{M}$.

$$\varsigma := \varsigma_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$$\Big[\underbrace{\mathrm{ev}_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N}_{\text{MORPHISM IN } \mathcal{M}}\Big] \xleftarrow{\varsigma^{-1}} \mathrm{id}_{\underline{\mathrm{Hom}}(M,N)}$$

DEFINE:

$$\begin{tikzcd}
\big(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)\big) \triangleright M \arrow[r, dashed, "\mathrm{ev}_{M,N,P}"] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P), \underline{\mathrm{Hom}}(M,N), M}"'] \arrow[dr, phantom, "\underline{\mathrm{DEF}}"] & P \\
\underline{\mathrm{Hom}}(N,P) \triangleright \big(\underline{\mathrm{Hom}}(M,N) \triangleright M\big) \arrow[r, "\mathrm{id} \triangleright \mathrm{ev}_{M,N}"] & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[u, "\mathrm{ev}_{N,P}"']
\end{tikzcd}$$

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) : \mathcal{C} \to \mathcal{M}$ HAS A RIGHT ADJOINT $\underline{\mathrm{Hom}}(M, -) : \mathcal{M} \to \mathcal{C}$, $\forall M \in \mathcal{M}$.

$\mathfrak{J} := \mathfrak{J}_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$$\left[ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \right] \overset{\mathfrak{J}^{-1}}{\longleftarrow\!\!\shortmid} id_{\underline{\mathrm{Hom}}(M,N)}$$

MORPHISM IN $\mathcal{M}$

DEFINE:

$$\begin{tikzcd}
\left(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)\right) \triangleright M \arrow[r, dashed, "ev_{M,N,P}", "\text{DEF}"'] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P), \underline{\mathrm{Hom}}(M,N), M}"'] & P \\
\underline{\mathrm{Hom}}(N,P) \triangleright \left(\underline{\mathrm{Hom}}(M,N) \triangleright M\right) \arrow[r, "id \triangleright ev_{M,N}"] & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[u, "ev_{N,P}"']
\end{tikzcd}$$

TAKE
$Z = \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)$

$$\mathfrak{J}(ev_{M,N,P}) : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P)$$

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) : \mathcal{C} \to \mathcal{M}$ HAS A RIGHT ADJOINT $\underline{\mathrm{Hom}}(M, -) : \mathcal{M} \to \mathcal{C}$, $\forall M \in \mathcal{M}$.

$\mathfrak{J} := \mathfrak{J}_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$$\left[ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \right] \overset{\mathfrak{J}^{-1}}{\longleftarrow\!\!\shortmid} id_{\underline{\mathrm{Hom}}(M,N)}$$

MORPHISM IN $\mathcal{M}$

DEFINE:

$$\begin{tikzcd}
\left(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)\right) \triangleright M \arrow[rr, dashed, "ev_{M,N,P}"] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P), \underline{\mathrm{Hom}}(M,N), M}"'] \arrow[drr, phantom, "\underline{\mathrm{DEF}}"] & & P \\
\underline{\mathrm{Hom}}(N,P) \triangleright \left(\underline{\mathrm{Hom}}(M,N) \triangleright M\right) \arrow[rr, "id \triangleright ev_{M,N}"] & & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[u, "ev_{N,P}"']
\end{tikzcd}$$

OBTAIN:

$$\mathrm{COMP}_{M,N,P} := \mathfrak{J}(ev_{M,N,P}) : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P)$$

MORPHISM IN $\mathcal{C}$

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) \dashv \underline{\mathrm{Hom}}(M, -)$ $\forall M$

$$\mathfrak{J} := \mathfrak{J}_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$\mathrm{ev}_{M,N} := \mathfrak{J}^{-1}(\mathrm{id}_{\underline{\mathrm{Hom}}(M,N)})$:

$$\underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$$

$$\begin{tikzcd}
(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)) \triangleright M \arrow[rr, dashed, "\mathrm{ev}_{M,N,P}"] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P), \underline{\mathrm{Hom}}(M,N), M}"'] \arrow[drr, phantom, "\underline{\mathrm{DEF}}"] & & P \in \mathcal{M} \\
\underline{\mathrm{Hom}}(N,P) \triangleright (\underline{\mathrm{Hom}}(M,N) \triangleright M) \arrow[rr, "\mathrm{id} \triangleright \mathrm{ev}_{M,N}"] & & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[u, "\mathrm{ev}_{N,P}"']
\end{tikzcd}$$

$$\mathrm{comp}_{M,N,P} := \mathfrak{J}(\mathrm{ev}_{M,N,P}) : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$$

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) \dashv \underline{\mathrm{Hom}}(M, -)$ $\forall M$

$$\varsigma := \varsigma_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$$\mathrm{ev}_{M,N} := \varsigma^{-1}\left(\mathrm{id}_{\underline{\mathrm{Hom}}(M,N)}\right) :$$

$$\underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$$

$$\begin{tikzcd}
\left(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)\right) \triangleright M \arrow[r, dashed, "\mathrm{ev}_{M,N,P}"] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P), \underline{\mathrm{Hom}}(M,N), M}"'] \arrow[dr, phantom, "\underline{\mathrm{DEF}}"] & P \in \mathcal{M} \\
\underline{\mathrm{Hom}}(N,P) \triangleright \left(\underline{\mathrm{Hom}}(M,N) \triangleright M\right) \arrow[r, "\mathrm{id} \triangleright \mathrm{ev}_{M,N}"] & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[u, "\mathrm{ev}_{N,P}"']
\end{tikzcd}$$

$$\mathrm{comp}_{M,N,P} := \varsigma(\mathrm{ev}_{M,N,P}) : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$$

<u>PROP</u>: LET $(\mathcal{M}, \triangleright, m, p)$ BE A CLOSED LEFT $\mathcal{C}$-MODULE CATEGORY.

THEN (a) $\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$ WITH

$\forall M \in \mathcal{M}$

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) \dashv \underline{\mathrm{Hom}}(M, -)$ $\forall M$

$$\mathfrak{J} := \mathfrak{J}_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$\mathrm{ev}_{M,N} := \mathfrak{J}^{-1}(\mathrm{id}_{\underline{\mathrm{Hom}}(M,N)})$:

$$\underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$$

$$\begin{tikzcd}
(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)) \triangleright M \arrow[rr, dashed, "\mathrm{ev}_{M,N,P}"] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P), \underline{\mathrm{Hom}}(M,N), M}"'] & \underline{\mathrm{DEF}} & P \in \mathcal{M} \\
\underline{\mathrm{Hom}}(N,P) \triangleright (\underline{\mathrm{Hom}}(M,N) \triangleright M) \arrow[rr, "\mathrm{id} \triangleright \mathrm{ev}_{M,N}"] & & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[u, "\mathrm{ev}_{N,P}"']
\end{tikzcd}$$

$$\mathrm{comp}_{M,N,P} := \mathfrak{J}(\mathrm{ev}_{M,N,P}) : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$$

PROP: LET $(\mathcal{M}, \triangleright, m, p)$ BE A CLOSED LEFT $\mathcal{C}$-MODULE CATEGORY.

THEN (a) $\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$ WITH

$\forall M \in \mathcal{M}$ [MULTIPLICATION] ??

[UNIT] ??

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) \dashv \underline{\mathrm{Hom}}(M, -)$ $\forall M$

$$\varphi := \varphi_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$\mathrm{ev}_{M,N} := \varphi^{-1}(\mathrm{id}_{\underline{\mathrm{Hom}}(M,N)})$:

$$\underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$$

$$\begin{tikzcd}
(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)) \triangleright M \arrow[r, dashed, "\mathrm{ev}_{M,N,P}"] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P), \underline{\mathrm{Hom}}(M,N), M}"'] \arrow[dr, phantom, "\underline{\mathrm{DEF}}"] & P \in \mathcal{M} \\
\underline{\mathrm{Hom}}(N,P) \triangleright (\underline{\mathrm{Hom}}(M,N) \triangleright M) \arrow[r, "\mathrm{id} \triangleright \mathrm{ev}_{M,N}"] & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[u, "\mathrm{ev}_{N,P}"']
\end{tikzcd}$$

$$\mathrm{comp}_{M,N,P} := \varphi(\mathrm{ev}_{M,N,P}) : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$$

<u>PROP</u>: LET $(\mathcal{M}, \triangleright, m, p)$ BE A CLOSED LEFT $\mathcal{C}$-MODULE CATEGORY.

THEN (a) $\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$ WITH

$\forall M \in \mathcal{M}$

[MULTIPLICATION] $\mathrm{comp}_{M,M,M} : \underline{\mathrm{End}}(M) \otimes \underline{\mathrm{End}}(M) \longrightarrow \underline{\mathrm{End}}(M)$,

[UNIT] $\varphi(p_M) : \mathbb{1} \longrightarrow \underline{\mathrm{End}}(M)$.

$p_M : \mathbb{1} \triangleright M \xrightarrow{\sim} M$

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) \dashv \underline{\mathrm{Hom}}(M,-)$ $\forall M$

$$\varphi := \varphi_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$\mathrm{ev}_{M,N} := \varphi^{-1}(\mathrm{id}_{\underline{\mathrm{Hom}}(M,N)})$:

$$\underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$$

$$\begin{tikzcd}
(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)) \triangleright M \arrow[rr, dashed, "\mathrm{ev}_{M,N,P}"] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P),\underline{\mathrm{Hom}}(M,N),M}"'] & \mathrm{DEF} & P \in \mathcal{M} \\
\underline{\mathrm{Hom}}(N,P) \triangleright (\underline{\mathrm{Hom}}(M,N) \triangleright M) \arrow[rr, "\mathrm{id} \triangleright \mathrm{ev}_{M,N}"] & & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[u, "\mathrm{ev}_{N,P}"']
\end{tikzcd}$$

$$\mathrm{comp}_{M,N,P} := \varphi(\mathrm{ev}_{M,N,P}) : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$$

PROP: LET $(\mathcal{M}, \triangleright, m, p)$ BE A CLOSED LEFT $\mathcal{C}$-MODULE CATEGORY.

THEN (a) $\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$ WITH

$\forall M \in \mathcal{M}$

[MULTIPLICATION] $\mathrm{comp}_{M,M,M} : \underline{\mathrm{End}}(M) \otimes \underline{\mathrm{End}}(M) \longrightarrow \underline{\mathrm{End}}(M)$,

[UNIT] $\varphi(p_M) : \mathbb{1} \longrightarrow \underline{\mathrm{End}}(M)$.

(b) $\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$ WITH

$\forall M, N \in \mathcal{M}$

[RIGHT ACTION] ??

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) \dashv \underline{\mathrm{Hom}}(M,-)$ $\forall M$

$$\mathcal{J} := \mathcal{J}_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$ev_{M,N} := \mathcal{J}^{-1}(\mathrm{id}_{\underline{\mathrm{Hom}}(M,N)})$:

$$\underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$$

$$\begin{tikzcd}
(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)) \triangleright M \arrow[rr, dashed, "ev_{M,N,P}"] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P), \underline{\mathrm{Hom}}(M,N), M}"'] & \underline{\mathrm{DEF}} & P \in \mathcal{M} \\
\underline{\mathrm{Hom}}(N,P) \triangleright (\underline{\mathrm{Hom}}(M,N) \triangleright M) \arrow[rr, "\mathrm{id} \triangleright ev_{M,N}"] & & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[u, "ev_{N,P}"']
\end{tikzcd}$$

$$comp_{M,N,P} := \mathcal{J}(ev_{M,N,P}) : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$$

PROP: LET $(\mathcal{M}, \triangleright, m, p)$ BE A CLOSED LEFT $\mathcal{C}$-MODULE CATEGORY.

THEN (a) $\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$ WITH

$\forall M \in \mathcal{M}$

[MULTIPLICATION] $comp_{M,M,M} : \underline{\mathrm{End}}(M) \otimes \underline{\mathrm{End}}(M) \longrightarrow \underline{\mathrm{End}}(M)$,

[UNIT] $\mathcal{J}(p_M) : \mathbb{1} \longrightarrow \underline{\mathrm{End}}(M)$.

(b) $\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$ WITH

$\forall M, N \in \mathcal{M}$

[RIGHT ACTION] $comp_{M,M,N} : \underline{\mathrm{Hom}}(M,N) \otimes \underline{\mathrm{End}}(M) \longrightarrow \underline{\mathrm{Hom}}(M,N)$.

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) \dashv \underline{\mathrm{Hom}}(M, -)$ $\forall M$

$$\varphi := \varphi_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$\mathrm{ev}_{M,N} := \varphi^{-1}(\mathrm{id}_{\underline{\mathrm{Hom}}(M,N)})$:

$$\underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$$

$$\begin{tikzcd}
\left(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)\right) \triangleright M \arrow[r, dashed, "\mathrm{ev}_{M,N,P}"] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P), \underline{\mathrm{Hom}}(M,N), M}"'] \arrow[dr, phantom, "\underline{\mathrm{DEF}}"] & P \in \mathcal{M} \\
\underline{\mathrm{Hom}}(N,P) \triangleright \left(\underline{\mathrm{Hom}}(M,N) \triangleright M\right) \arrow[r, "\mathrm{id} \triangleright \mathrm{ev}_{M,N}"] & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[u, "\mathrm{ev}_{N,P}"']
\end{tikzcd}$$

$$\mathrm{comp}_{M,N,P} := \varphi(\mathrm{ev}_{M,N,P}) : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$$

PROP: LET $(\mathcal{M}, \triangleright, m, p)$ BE A CLOSED LEFT $\mathcal{C}$-MODULE CATEGORY.

THEN (a) $\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$ WITH

$\forall M \in \mathcal{M}$

[MULTIPLICATION] $\mathrm{comp}_{M,M,M} : \underline{\mathrm{End}}(M) \otimes \underline{\mathrm{End}}(M) \longrightarrow \underline{\mathrm{End}}(M)$,

[UNIT] $\varphi(p_M) : \mathbb{1} \longrightarrow \underline{\mathrm{End}}(M)$.

⋮

(b)

⋮

(c) $\underline{\mathrm{Hom}}(N, M) \in \underline{\mathrm{End}}(M)\text{-}\mathrm{Mod}(\mathcal{C})$ WITH

$\forall M, N \in \mathcal{M}$

[LEFT ACTION] ??

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) \dashv \underline{\mathrm{Hom}}(M, -)$ $\forall M$

$$\varsigma := \varsigma_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$ev_{M,N} := \varsigma^{-1}(\mathrm{id}_{\underline{\mathrm{Hom}}(M,N)})$:

$$\underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$$

$$\begin{tikzcd}
(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)) \triangleright M \arrow[rr, dashed, "ev_{M,N,P}"] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P), \underline{\mathrm{Hom}}(M,N), M}"'] & \underline{\mathrm{DEF}} & P \in \mathcal{M} \\
\underline{\mathrm{Hom}}(N,P) \triangleright (\underline{\mathrm{Hom}}(M,N) \triangleright M) \arrow[rr, "\mathrm{id} \triangleright ev_{M,N}"] & & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[u, "ev_{N,P}"']
\end{tikzcd}$$

$$comp_{M,N,P} := \varsigma(ev_{M,N,P}) : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$$

<u>PROP</u>: LET $(\mathcal{M}, \triangleright, m, p)$ BE A CLOSED LEFT $\mathcal{C}$-MODULE CATEGORY.

THEN (a) $\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$ WITH

$\forall M \in \mathcal{M}$ [MULTIPLICATION] $comp_{M,M,M} : \underline{\mathrm{End}}(M) \otimes \underline{\mathrm{End}}(M) \longrightarrow \underline{\mathrm{End}}(M)$,

⋮

[UNIT] $\varsigma(p_M) : \mathbb{1} \longrightarrow \underline{\mathrm{End}}(M)$.

(b)

⋮

(c) $\underline{\mathrm{Hom}}(N,M) \in \underline{\mathrm{End}}(M)\text{-}\mathrm{Mod}(\mathcal{C})$ WITH

$\forall M, N \in \mathcal{M}$ [LEFT ACTION] $comp_{N,M,M} : \underline{\mathrm{End}}(M) \otimes \underline{\mathrm{Hom}}(N,M) \longrightarrow \underline{\mathrm{Hom}}(N,M)$.

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) \dashv \underline{\mathrm{Hom}}(M, -)$ $\forall M$

$$\varsigma := \varsigma_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$ev_{M,N} := \varsigma^{-1}\left(id_{\underline{\mathrm{Hom}}(M,N)}\right)$:

$$\underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$$

$$\begin{tikzcd}
\left(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)\right) \triangleright M \arrow[r, dashed, "ev_{M,N,P}"] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P), \underline{\mathrm{Hom}}(M,N), M}"'] \arrow[dr, phantom, "\text{DEF}"] & P \in \mathcal{M} \\
\underline{\mathrm{Hom}}(N,P) \triangleright \left(\underline{\mathrm{Hom}}(M,N) \triangleright M\right) \arrow[r, "id \triangleright ev_{M,N}"] & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[u, "ev_{N,P}"']
\end{tikzcd}$$

$$comp_{M,N,P} := \varsigma(ev_{M,N,P}) : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$$

PROP: LET $(\mathcal{M}, \triangleright, m, p)$ BE A CLOSED LEFT $\mathcal{C}$-MODULE CATEGORY.

THEN (a) $\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$ WITH "INTERNAL END ALGEBRA"

$\forall M \in \mathcal{M}$

[MULTIPLICATION] $comp_{M,M,M} : \underline{\mathrm{End}}(M) \otimes \underline{\mathrm{End}}(M) \longrightarrow \underline{\mathrm{End}}(M)$,

[UNIT] $\varsigma(p_M) : \mathbb{1} \longrightarrow \underline{\mathrm{End}}(M)$.

(b) $\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$ WITH "INTERNAL HOM MODULE"

$\forall M, N \in \mathcal{M}$ $\vdots$ (c) [RIGHT ACTION] $comp_{M,M,N} : \underline{\mathrm{Hom}}(M,N) \otimes \underline{\mathrm{End}}(M) \longrightarrow \underline{\mathrm{Hom}}(M,N)$.

I. INTERNAL END ALGEBRAS

LEFT $\mathcal{C}$-MODULE CATEGORY $(\mathcal{M}, \triangleright, m, p)$ IS CLOSED IF $(- \triangleright M) \dashv \underline{\mathrm{Hom}}(M,-)$ $\forall M$

$$\varsigma := \varsigma_{Z,N} : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$ev_{M,N} := \varsigma^{-1}(id_{\underline{\mathrm{Hom}}(M,N)})$:

$$\underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$$

$$\begin{tikzcd}
(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)) \triangleright M \arrow[r, dashed, "ev_{M,N,P}"] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P), \underline{\mathrm{Hom}}(M,N), M}"'] \arrow[dr, phantom, "\text{DEF}"] & P \in \mathcal{M} \\
\underline{\mathrm{Hom}}(N,P) \triangleright (\underline{\mathrm{Hom}}(M,N) \triangleright M) \arrow[r, "id \triangleright ev_{M,N}"] & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[u, "ev_{N,P}"']
\end{tikzcd}$$

$$comp_{M,N,P} := \varsigma(ev_{M,N,P}) : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$$

PROP: LET $(\mathcal{M}, \triangleright, m, p)$ BE A CLOSED LEFT $\mathcal{C}$-MODULE CATEGORY.

PROOF ≡ EXER. 4.51

THEN (a) $\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$ WITH "INTERNAL END ALGEBRA"

$\forall M \in \mathcal{M}$

[MULTIPLICATION] $comp_{M,M,M} : \underline{\mathrm{End}}(M) \otimes \underline{\mathrm{End}}(M) \longrightarrow \underline{\mathrm{End}}(M)$,

[UNIT] $\varsigma(p_M) : \mathbb{1} \longrightarrow \underline{\mathrm{End}}(M)$.

(b) $\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$ WITH "INTERNAL HOM MODULE"

$\forall M, N \in \mathcal{M}$ ⋮

(c) [RIGHT ACTION] $comp_{M,M,N} : \underline{\mathrm{Hom}}(M,N) \otimes \underline{\mathrm{End}}(M) \longrightarrow \underline{\mathrm{Hom}}(M,N)$.

I. INTERNAL END ALGEBRAS

TAKE $(\mathcal{M}, \triangleright, m, p) \in \mathcal{C}\text{-Mod}$ CLOSED

$\varphi : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$

$ev_{M,N,P} : \big(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)\big) \triangleright M \longrightarrow P \in \mathcal{M}$

$comp_{M,N,P} : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$

$\big(\underline{\mathrm{End}}(M), comp_{M,M,M}, \varphi(p_M)\big) \in \mathrm{Alg}(\mathcal{C})$

"INTERNAL END ALGEBRA"

$\big(\underline{\mathrm{Hom}}(M,N), comp_{M,M,N}\big) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

"INTERNAL HOM MODULE"

EXAMPLE

I. INTERNAL END ALGEBRAS

TAKE $(\mathcal{M}, \triangleright, m, p) \in \mathcal{C}\text{-}\mathrm{Mod}$ CLOSED

$$\varphi : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$$\mathrm{ev}_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \ \in \mathcal{M}$$

$$\mathrm{ev}_{M,N,P} : \big(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)\big) \triangleright M \longrightarrow P \ \in \mathcal{M}$$

$$\mathrm{comp}_{M,N,P} : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \ \in \mathcal{C}$$

$$\big(\underline{\mathrm{End}}(M), \mathrm{comp}_{M,M,M}, \varphi(p_M)\big) \in \mathrm{Alg}(\mathcal{C})$$

"INTERNAL END ALGEBRA"

$$\big(\underline{\mathrm{Hom}}(M,N), \mathrm{comp}_{M,M,N}\big) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

"INTERNAL HOM MODULE"

EXAMPLE $\mathcal{C}$ RIGID $\Longrightarrow$

$(\mathcal{C}_{\mathrm{reg}}, \triangleright := \otimes) \in \mathcal{C}\text{-}\mathrm{Mod}$ IS CLOSED:

$$\underline{\mathrm{Hom}}(Y,Z) := Z \otimes Y^*.$$

I. INTERNAL END ALGEBRAS

TAKE $(\mathcal{M}, \triangleright, M, p) \in \mathcal{C}\text{-Mod}$ CLOSED

$\varphi : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$

$ev_{M,N,P} : (\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)) \triangleright M \longrightarrow P \in \mathcal{M}$

$comp_{M,N,P} : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$

$(\underline{\mathrm{End}}(M), comp_{M,M,M}, \varphi(p_M)) \in \mathrm{Alg}(\mathcal{C})$

"INTERNAL END ALGEBRA"

$(\underline{\mathrm{Hom}}(M,N), comp_{M,M,N}) \in \text{Mod-}\underline{\mathrm{End}}(M)(\mathcal{C})$

"INTERNAL HOM MODULE"

EXAMPLE $\mathcal{C}$ RIGID $\Longrightarrow$

$(\mathcal{C}_{reg}, \triangleright := \otimes) \in \mathcal{C}\text{-Mod}$ IS CLOSED:

$$\underline{\mathrm{Hom}}(Y, Z) := Z \otimes Y^*.$$

INDEED, $(- \otimes Y) \dashv (- \otimes Y^*)$, i.e.

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y^*).$$

I. INTERNAL END ALGEBRAS

TAKE $(\mathcal{M}, \triangleright, m, p) \in \mathcal{C}\text{-Mod}$ CLOSED

$\varphi : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$

$ev_{M,N,P} : (\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)) \triangleright M \longrightarrow P \in \mathcal{M}$

$comp_{M,N,P} : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$

$(\underline{\mathrm{End}}(M), comp_{M,M,M}, \varphi(p_M)) \in \mathrm{Alg}(\mathcal{C})$

"INTERNAL END ALGEBRA"

$(\underline{\mathrm{Hom}}(M,N), comp_{M,M,N}) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

"INTERNAL HOM MODULE"

EXAMPLE $\mathcal{C}$ RIGID $\Longrightarrow$

$(\mathcal{C}_{reg}, \triangleright := \otimes) \in \mathcal{C}\text{-Mod}$ IS CLOSED:

$$\underline{\mathrm{Hom}}(Y, Z) := Z \otimes Y^*.$$

INDEED, $(- \otimes Y) \dashv (- \otimes Y^*)$, i.e.

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y^*).$$

$$f \longmapsto [X \xrightarrow{id \otimes coev_Y^L} X \otimes Y \otimes Y^* \xrightarrow{f \otimes id} Z \otimes Y^*]$$

I. INTERNAL END ALGEBRAS

TAKE $(\mathcal{M}, \triangleright, M, P) \in \mathcal{C}\text{-Mod}$ CLOSED

$$\varphi : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$$ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \quad \in \mathcal{M}$$

$$ev_{M,N,P} : (\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)) \triangleright M \longrightarrow P \quad \in \mathcal{M}$$

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$$(\underline{\mathrm{End}}(M), comp_{M,M,M}, \varphi(p_M)) \in \mathrm{Alg}(\mathcal{C})$$

"INTERNAL END ALGEBRA"

$$(\underline{\mathrm{Hom}}(M,N), comp_{M,M,N}) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

"INTERNAL HOM MODULE"

EXAMPLE $\mathcal{C}$ RIGID $\Longrightarrow$

$(\mathcal{C}_{reg}, \triangleright := \otimes) \in \mathcal{C}\text{-Mod}$ IS CLOSED:

$$\underline{\mathrm{Hom}}(Y, Z) := Z \otimes Y^*.$$

INDEED, $(- \otimes Y) \dashv (- \otimes Y^*)$, i.e.

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathcal{C}}(X, Z \otimes Y^*).$$

$$f \longmapsto [X \xrightarrow{id \otimes coev_Y^L} X \otimes Y \otimes Y^* \xrightarrow{f \otimes id} Z \otimes Y^*]$$

$$[X \otimes Y \xrightarrow{g \otimes id} Z \otimes Y^* \otimes Y \xrightarrow{id \otimes ev_Y^L} Z] \longmapsfrom g$$

I. INTERNAL END ALGEBRAS

TAKE $(\mathcal{M}, \triangleright, m, p) \in \mathcal{C}\text{-Mod}$ CLOSED

$$\varphi : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$$ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$$

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"INTERNAL END ALGEBRA"

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I. INTERNAL END ALGEBRAS

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$\varsigma : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

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"INTERNAL END ALGEBRA"

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ς USES coev $\quad \varsigma^{-1}$ USES ev

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= STRICT FOR EASE =

I. INTERNAL END ALGEBRAS

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"INTERNAL END ALGEBRA"

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GET $\underline{\mathrm{End}}(Y) := Y \otimes Y^* \in \mathrm{Alg}(\mathcal{C})$ VIA:

= STRICT FOR EASE =

I. INTERNAL END ALGEBRAS

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"INTERNAL END ALGEBRA"

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$$\mathrm{ev}_{Y,Y} : Y \otimes Y^* \otimes Y \xrightarrow{\mathrm{id}_Y \otimes \mathrm{ev}_Y^{L}} Y$$

$$\begin{tikzcd}
(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)) \triangleright M \arrow[r, dashed, "\mathrm{ev}_{M,N,P}"] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P), \underline{\mathrm{Hom}}(M,N), M}"'] \arrow[dr, phantom, "\underline{\mathrm{DEF}}"] & P \\
\underline{\mathrm{Hom}}(N,P) \triangleright (\underline{\mathrm{Hom}}(M,N) \triangleright M) \arrow[r, "\mathrm{id} \triangleright \mathrm{ev}_{M,N}"] & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[u, "\mathrm{ev}_{N,P}"']
\end{tikzcd}$$

<u>EXAMPLE</u> $\quad \mathcal{C}$ RIGID $\Longrightarrow$

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$$\underline{\mathrm{Hom}}(Y, Z) := Z \otimes Y^*.$$

GET $\underline{\mathrm{End}}(Y) := Y \otimes Y^* \in \mathrm{Alg}(\mathcal{C})$ VIA:

$$m_{\underline{\mathrm{End}}(Y)} := \mathrm{comp}_{Y,Y,Y} = \varphi(\mathrm{ev}_{Y,Y,Y}):$$

$$\begin{tikzcd}
Y \otimes Y^* \otimes Y \otimes Y^* \arrow[r] & Y \otimes Y^*
\end{tikzcd}$$

= STRICT FOR EASE =

I. INTERNAL END ALGEBRAS

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"INTERNAL END ALGEBRA"

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\underline{\mathrm{Hom}}(N,P) \triangleright (\underline{\mathrm{Hom}}(M,N) \triangleright M) \arrow[rr, "\mathrm{id} \triangleright \mathrm{ev}_{M,N}"] & & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[u, "\mathrm{ev}_{N,P}"']
\end{tikzcd}$$

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$m_{\underline{\mathrm{End}}(Y)} := \mathrm{comp}_{Y,Y,Y} = \varrho(\mathrm{ev}_{Y,Y,Y})$:

$$Y \otimes Y^* \otimes Y \otimes Y^* \longrightarrow Y \otimes Y^*$$

$$\begin{tikzcd}
((Y \otimes Y^*) \otimes (Y \otimes Y^*)) \otimes Y \arrow[r, dashed, "\mathrm{ev}_{Y,Y,Y}"] \arrow[d, "a_{Y \otimes Y^*, Y \otimes Y^*, Y}"'] & Y \\
(Y \otimes Y^*) \otimes ((Y \otimes Y^*) \otimes Y) &
\end{tikzcd}$$

≡ STRICT FOR EASE ≡

I. INTERNAL END ALGEBRAS

TAKE $(\mathcal{M}, \triangleright, m, p) \in \mathcal{C}\text{-Mod}$ CLOSED

$$\varsigma : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

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"INTERNAL END ALGEBRA"

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$$\mathrm{ev}_{Y,Y} : Y \otimes Y^* \otimes Y \xrightarrow{\mathrm{id}_Y \otimes \mathrm{ev}_Y^{\llcorner}} Y$$

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\underline{\mathrm{Hom}}(N,P) \triangleright (\underline{\mathrm{Hom}}(M,N) \triangleright M) \arrow[rr, "\mathrm{id} \triangleright \mathrm{ev}_{M,N}"] & & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[u, "\mathrm{ev}_{N,P}"']
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$$Y \otimes Y^* \otimes Y \otimes Y^* \longrightarrow Y \otimes Y^*$$

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((Y \otimes Y^*) \otimes (Y \otimes Y^*)) \otimes Y \arrow[r, dashed, "\mathrm{ev}_{Y,Y,Y}"] \arrow[d, "a_{Y \otimes Y^*, Y \otimes Y^*, Y}"'] & Y \\
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\end{tikzcd}$$

≡ STRICT FOR EASE ≡

I. INTERNAL END ALGEBRAS

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$\varsigma : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

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"INTERNAL END ALGEBRA"

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"INTERNAL HOM MODULE"

ς USES COEV $\qquad$ ς^{-1} USES EV

$$\mathrm{ev}_{Y,Y} : Y \otimes Y^* \otimes Y \xrightarrow{\mathrm{id}_Y \otimes \mathrm{ev}^{\smile}_Y} Y$$

$$\begin{tikzcd}
(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)) \triangleright M \arrow[rr, dashed, "\mathrm{ev}_{M,N,P}"] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P), \underline{\mathrm{Hom}}(M,N), M}"'] & \underline{\mathrm{DEF}} & P \\
\underline{\mathrm{Hom}}(N,P) \triangleright (\underline{\mathrm{Hom}}(M,N) \triangleright M) \arrow[rr, "\mathrm{id} \triangleright \mathrm{ev}_{M,N}"] & & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[u, "\mathrm{ev}_{N,P}"']
\end{tikzcd}$$

EXAMPLE

$\mathcal{C}$ RIGID $\Longrightarrow$

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GET $\underline{\mathrm{End}}(Y) := Y \otimes Y^* \in \mathrm{Alg}(\mathcal{C})$ VIA:

$m_{\underline{\mathrm{End}}(Y)} := \mathrm{comp}_{Y,Y,Y} = \varsigma(\mathrm{ev}_{Y,Y,Y})$:

$$Y \otimes Y^* \otimes Y \otimes Y^* \longrightarrow Y \otimes Y^*$$

$$\begin{tikzcd}
((Y \otimes Y^*) \otimes (Y \otimes Y^*)) \otimes Y \arrow[r, dashed, "\mathrm{ev}_{Y,Y,Y}"] \arrow[d, bend right, "a_{Y \otimes Y^*, Y \otimes Y^*, Y}"'] & Y \\
(Y \otimes Y^*) \otimes ((Y \otimes Y^*) \otimes Y) \arrow[r, "\mathrm{id}_{Y \otimes Y^* \otimes Y} \otimes \mathrm{ev}^{\smile}_Y"] & Y \otimes Y^* \otimes Y \arrow[u, bend right, "\mathrm{id}_Y \otimes \mathrm{ev}^{\smile}_Y"']
\end{tikzcd}$$

≡ STRICT FOR EASE ≡

I. INTERNAL END ALGEBRAS

TAKE $(\mathcal{M}, \triangleright, m, p) \in \mathcal{C}\text{-Mod}$ CLOSED

$$\phi : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

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"INTERNAL END ALGEBRA"

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(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)) \triangleright M \arrow[rr, dashed, "\mathrm{ev}_{M,N,P}"] \arrow[d, "m_{\underline{\mathrm{Hom}}(N,P), \underline{\mathrm{Hom}}(M,N), M}"'] \arrow[drr, phantom, "\underline{\mathrm{DEF}}"] & & P \\
\underline{\mathrm{Hom}}(N,P) \triangleright (\underline{\mathrm{Hom}}(M,N) \triangleright M) \arrow[r, "\mathrm{id} \triangleright \mathrm{ev}_{M,N}"] & \underline{\mathrm{Hom}}(N,P) \triangleright N \arrow[ur, "\mathrm{ev}_{N,P}"'] &
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$$Y \otimes Y^* \otimes Y \otimes Y^* \xrightarrow{\mathrm{id}_Y \otimes \mathrm{ev}_Y^{L} \otimes \mathrm{id}_{Y^*}} Y \otimes Y^*$$

$$\rotatebox{90}{$\rightsquigarrow$}$$

$$\begin{tikzcd}
((Y \otimes Y^*) \otimes (Y \otimes Y^*)) \otimes Y \arrow[r, dashed, "\mathrm{ev}_{Y,Y,Y}"] \arrow[d, "a_{Y \otimes Y^*, Y \otimes Y^*, Y}"'] & Y \\
(Y \otimes Y^*) \otimes ((Y \otimes Y^*) \otimes Y) \arrow[r, "\mathrm{id}_{Y \otimes Y^* \otimes Y} \otimes \mathrm{ev}_Y^{L}"] & Y \otimes Y^* \otimes Y \arrow[u, "\mathrm{id}_Y \otimes \mathrm{ev}_Y^{L}"']
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= STRICT FOR EASE =

I. INTERNAL END ALGEBRAS

TAKE $(\mathcal{M}, \triangleright, M, p) \in \mathcal{C}\text{-Mod}$ CLOSED

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"INTERNAL END ALGEBRA"

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"INTERNAL HOM MODULE"

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$m_{\underline{\mathrm{End}}(Y)} := \mathrm{comp}_{Y,Y,Y} = \varrho(\mathrm{ev}_{Y,Y,Y})$:

$$Y \otimes Y^* \otimes Y \otimes Y^* \xrightarrow{\mathrm{id}_Y \otimes \mathrm{ev}^L_Y \otimes \mathrm{id}_{Y^*}} Y \otimes Y^*$$

$u_{\underline{\mathrm{End}}(Y)} := \varrho(p_Y) : \mathbb{1} \longrightarrow Y \otimes Y^*$

= STRICT FOR EASE =

I. INTERNAL END ALGEBRAS

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$$p_Y : \mathbb{1} \otimes Y \xrightarrow{l_Y} Y$$

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$$m_{\underline{\mathrm{End}}(Y)} := \mathrm{comp}_{Y,Y,Y} = \varsigma(\mathrm{ev}_{Y,Y,Y}) :$$

$$Y \otimes Y^* \otimes Y \otimes Y^* \xrightarrow{\mathrm{id}_Y \otimes \mathrm{ev}_Y^L \otimes \mathrm{id}_{Y^*}} Y \otimes Y^*$$

$$u_{\underline{\mathrm{End}}(Y)} := \varsigma(p_Y) : \mathbb{1} \longrightarrow Y \otimes Y^*$$

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I. INTERNAL END ALGEBRAS

TAKE $(\mathcal{M}, \triangleright, M, p) \in \mathcal{C}\text{-Mod}$ CLOSED

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$$ev_{M,N,P} : (\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)) \triangleright M \longrightarrow P \in \mathcal{M}$$

$$comp_{M,N,P} : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$$

$$(\underline{\mathrm{End}}(M), comp_{M,M,M}, \varsigma(p_M)) \in \mathrm{Alg}(\mathcal{C})$$

"INTERNAL END ALGEBRA"

$$(\underline{\mathrm{Hom}}(M,N), comp_{M,M,N}) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

"INTERNAL HOM MODULE"

ς USES coev $\quad$ ς^{-1} USES ev

$$p_Y : \mathbb{1} \otimes Y \xrightarrow{l_Y} Y$$

EXAMPLE $\mathcal{C}$ RIGID $\Longrightarrow$

$(\mathcal{C}_{reg}, \triangleright := \otimes) \in \mathcal{C}\text{-Mod}$ IS CLOSED:

$$\underline{\mathrm{Hom}}(Y, Z) := Z \otimes Y^*.$$

GET $\underline{\mathrm{End}}(Y) := Y \otimes Y^* \in \mathrm{Alg}(\mathcal{C})$ VIA:

$$m_{\underline{\mathrm{End}}(Y)} := comp_{Y,Y,Y} = \varsigma(ev_{Y,Y,Y}) :$$

$$Y \otimes Y^* \otimes Y \otimes Y^* \xrightarrow{id_Y \otimes ev_Y^L \otimes id_{Y^*}} Y \otimes Y^*$$

$$u_{\underline{\mathrm{End}}(Y)} := \varsigma(p_Y) : \mathbb{1} \xrightarrow{coev_Y^L} Y \otimes Y^*$$

= STRICT FOR EASE =

I. INTERNAL END ALGEBRAS

TAKE $(\mathcal{M}, \triangleright, M, p) \in \mathcal{C}\text{-Mod}$ CLOSED

$\varsigma : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$

$ev_{M,N,P} : (\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)) \triangleright M \longrightarrow P \in \mathcal{M}$

$comp_{M,N,P} : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$

$(\underline{\mathrm{End}}(M), comp_{M,M,M}, \varsigma(p_M)) \in \mathrm{Alg}(\mathcal{C})$

"INTERNAL END ALGEBRA"

$(\underline{\mathrm{Hom}}(M,N), comp_{M,M,N}) \in \text{Mod-}\underline{\mathrm{End}}(M)(\mathcal{C})$

"INTERNAL HOM MODULE"

EXAMPLE $\mathcal{C}$ RIGID $\Longrightarrow$

$(\mathcal{C}_{reg}, \triangleright := \otimes) \in \mathcal{C}\text{-Mod}$ IS CLOSED:

$\underline{\mathrm{Hom}}(Y,Z) := Z \otimes Y^*$.

GET $\underline{\mathrm{End}}(Y) := Y \otimes Y^* \in \mathrm{Alg}(\mathcal{C})$ VIA:

$m_{\underline{\mathrm{End}}(Y)} := comp_{Y,Y,Y} = \varsigma(ev_{Y,Y,Y})$:

$$Y \otimes Y^* \otimes Y \otimes Y^* \xrightarrow{id_Y \otimes ev^L_Y \otimes id_{Y^*}} Y \otimes Y^*$$

$$u_{\underline{\mathrm{End}}(Y)} := \varsigma(p_Y) : \mathbb{1} \xrightarrow{coev^L_Y} Y \otimes Y^*$$

EXERCISE 4.52

CHECK THE DETAILS

I. INTERNAL END ALGEBRAS

TAKE $(\mathcal{M}, \triangleright, m, p) \in \mathcal{C}\text{-Mod}$ CLOSED

$$\varsigma : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$$ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$$

$$ev_{M,N,P} : \big(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)\big) \triangleright M \longrightarrow P \in \mathcal{M}$$

$$comp_{M,N,P} : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$$

$$\big(\underline{\mathrm{End}}(M), comp_{M,M,M}, \varsigma(p_M)\big) \in \mathrm{Alg}(\mathcal{C})$$

"INTERNAL END ALGEBRA"

$$\big(\underline{\mathrm{Hom}}(M,N), comp_{M,M,N}\big) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

"INTERNAL HOM MODULE"

EXERCISE 4.54: $G \equiv$ GROUP

$\mathrm{Forg} : G\text{-Mod} \to \mathrm{Vec}$ IS STRONG MONOIDAL

$\Rightarrow \mathrm{Vec} \in (G\text{-Mod})\text{-Mod}$

EXAMPLE $\mathcal{C}$ RIGID $\Longrightarrow$

$(\mathcal{C}_{reg}, \triangleright := \otimes) \in \mathcal{C}\text{-Mod}$ IS CLOSED:

$$\underline{\mathrm{Hom}}(Y, Z) := Z \otimes Y^*.$$

GET $\underline{\mathrm{End}}(Y) := Y \otimes Y^* \in \mathrm{Alg}(\mathcal{C})$ VIA:

$$m_{\underline{\mathrm{End}}(Y)} := comp_{Y,Y,Y} = \varsigma(ev_{Y,Y,Y}) :$$

$$Y \otimes Y^* \otimes Y \otimes Y^* \xrightarrow{id_Y \otimes ev^L_Y \otimes id_{Y^*}} Y \otimes Y^*$$

$$u_{\underline{\mathrm{End}}(Y)} := \varsigma(p_Y) : \mathbb{1} \xrightarrow{coev^L_Y} Y \otimes Y^*$$

EXERCISE 4.52

CHECK THE DETAILS

I. INTERNAL END ALGEBRAS

TAKE $(\mathcal{M}, \triangleright, m, p) \in \mathcal{C}\text{-Mod}$ CLOSED

$$\varphi : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$$ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$$

$$ev_{M,N,P} : (\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)) \triangleright M \longrightarrow P \in \mathcal{M}$$

$$comp_{M,N,P} : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$$

$$(\underline{\mathrm{End}}(M), comp_{M,M,M}, \varphi(p_M)) \in \mathrm{Alg}(\mathcal{C})$$

"INTERNAL END ALGEBRA"

$$(\underline{\mathrm{Hom}}(M,N), comp_{M,M,N}) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

"INTERNAL HOM MODULE"

EXERCISE 4.54: $G \equiv$ GROUP

$\mathrm{Forg} : G\text{-Mod} \to \mathrm{Vec}$ IS STRONG MONOIDAL

$\Rightarrow \mathrm{Vec} \in (G\text{-Mod})\text{-Mod}$

For $\Bbbk \in \mathrm{Vec}$, GET:

$$\underline{\mathrm{End}}(\Bbbk) \in \mathrm{Alg}(G\text{-Mod}).$$

EXAMPLE $\mathcal{C}$ RIGID $\Longrightarrow$

$(\mathcal{C}_{reg}, \triangleright := \otimes) \in \mathcal{C}\text{-Mod}$ IS CLOSED:

$$\underline{\mathrm{Hom}}(Y, Z) := Z \otimes Y^*.$$

GET $\underline{\mathrm{End}}(Y) := Y \otimes Y^* \in \mathrm{Alg}(\mathcal{C})$ VIA:

$$m_{\underline{\mathrm{End}}(Y)} := comp_{Y,Y,Y} = \varphi(ev_{Y,Y,Y}) :$$

$$Y \otimes Y^* \otimes Y \otimes Y^* \xrightarrow{id_Y \otimes ev^L_Y \otimes id_{Y^*}} Y \otimes Y^*$$

$$u_{\underline{\mathrm{End}}(Y)} := \varphi(p_Y) : \mathbb{1} \xrightarrow{coev^L_Y} Y \otimes Y^*$$

EXERCISE 4.52

CHECK THE DETAILS

I. INTERNAL END ALGEBRAS

TAKE $(\mathcal{M}, \triangleright, m, p) \in \mathcal{C}\text{-Mod}$ CLOSED

$$\varsigma : \mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$$ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N \in \mathcal{M}$$

$$ev_{M,N,P} : \left(\underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N)\right) \triangleright M \longrightarrow P \in \mathcal{M}$$

$$comp_{M,N,P} : \underline{\mathrm{Hom}}(N,P) \otimes \underline{\mathrm{Hom}}(M,N) \longrightarrow \underline{\mathrm{Hom}}(M,P) \in \mathcal{C}$$

$$\left(\underline{\mathrm{End}}(M),\, comp_{M,M,M},\, \varsigma(p_M)\right) \in \mathrm{Alg}(\mathcal{C})$$

"INTERNAL END ALGEBRA"

$$\left(\underline{\mathrm{Hom}}(M,N),\, comp_{M,M,N}\right) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

"INTERNAL HOM MODULE"

EXERCISE 4.54: $G \equiv$ GROUP

$\mathrm{Forg} : G\text{-Mod} \to \mathrm{Vec}$ IS $\overset{\text{STRONG}}{\text{MONOIDAL}}$

$\Rightarrow \mathrm{Vec} \in (G\text{-Mod})\text{-Mod}$

For $\Bbbk \in \mathrm{Vec}$, GET:

$$\underline{\mathrm{End}}(\Bbbk) \cong (\Bbbk G)^*$$
$$\in \mathrm{Alg}(G\text{-Mod}).$$

EXAMPLE $\mathcal{C}$ RIGID $\Longrightarrow$

$(\mathcal{C}_{reg}, \triangleright := \otimes) \in \mathcal{C}\text{-Mod}$ IS CLOSED:

$$\underline{\mathrm{Hom}}(Y, Z) := Z \otimes Y^*.$$

GET $\underline{\mathrm{End}}(Y) := Y \otimes Y^* \in \mathrm{Alg}(\mathcal{C})$ VIA:

$$m_{\underline{\mathrm{End}}(Y)} := comp_{Y,Y,Y} = \varsigma(ev_{Y,Y,Y}) :$$

$$Y \otimes Y^* \otimes Y \otimes Y^* \xrightarrow{id_Y \otimes ev_Y^L \otimes id_{Y^*}} Y \otimes Y^*$$

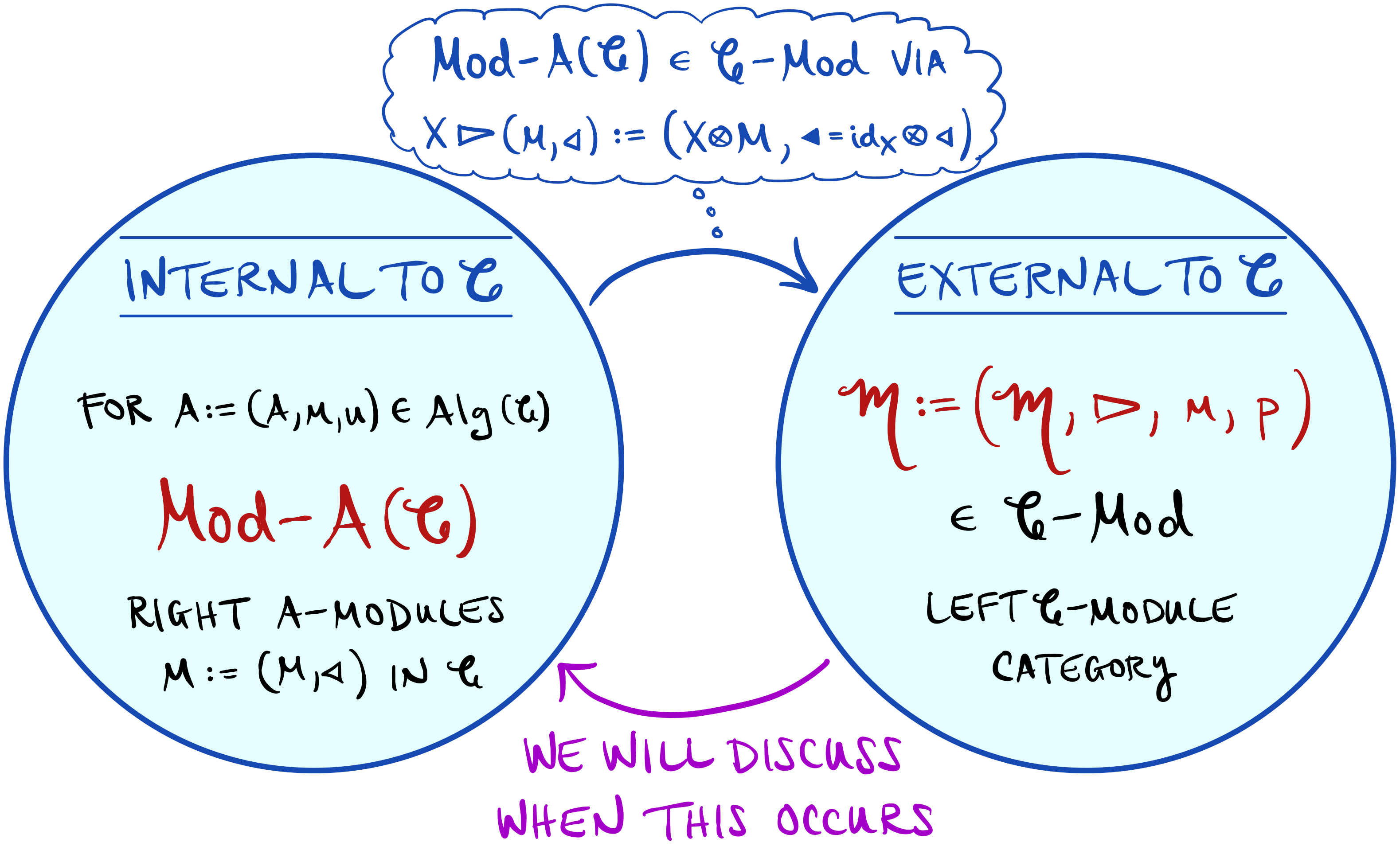
$$u_{\underline{\mathrm{End}}(Y)} := \varsigma(p_Y) : \mathbb{1} \xrightarrow{coev_Y^L} Y \otimes Y^*$$

EXERCISE 4.52

CHECK THE DETAILS

II. OSTRIK'S THEOREM

$\mathcal{M} \in \mathcal{C}$-Mod IS **REPRESENTED BY** $A \in \mathrm{Alg}(\mathcal{C})$ IF $\mathcal{M} \simeq \text{Mod-}A(\mathcal{C})$ AS LEFT $\mathcal{C}$-MODULE CATEGORIES.



Mod-$A(\mathcal{C}) \in \mathcal{C}$-Mod VIA

$X \triangleright (M, \triangleleft) := (X \otimes M, \blacktriangleleft = \mathrm{id}_X \otimes \triangleleft)$

INTERNAL TO $\mathcal{C}$

FOR $A := (A, m, u) \in \mathrm{Alg}(\mathcal{C})$

Mod-$A(\mathcal{C})$

RIGHT A-MODULES

$M := (M, \triangleleft)$ IN $\mathcal{C}$

EXTERNAL TO $\mathcal{C}$

$\mathcal{M} := (\mathcal{M}, \triangleright, m, p)$

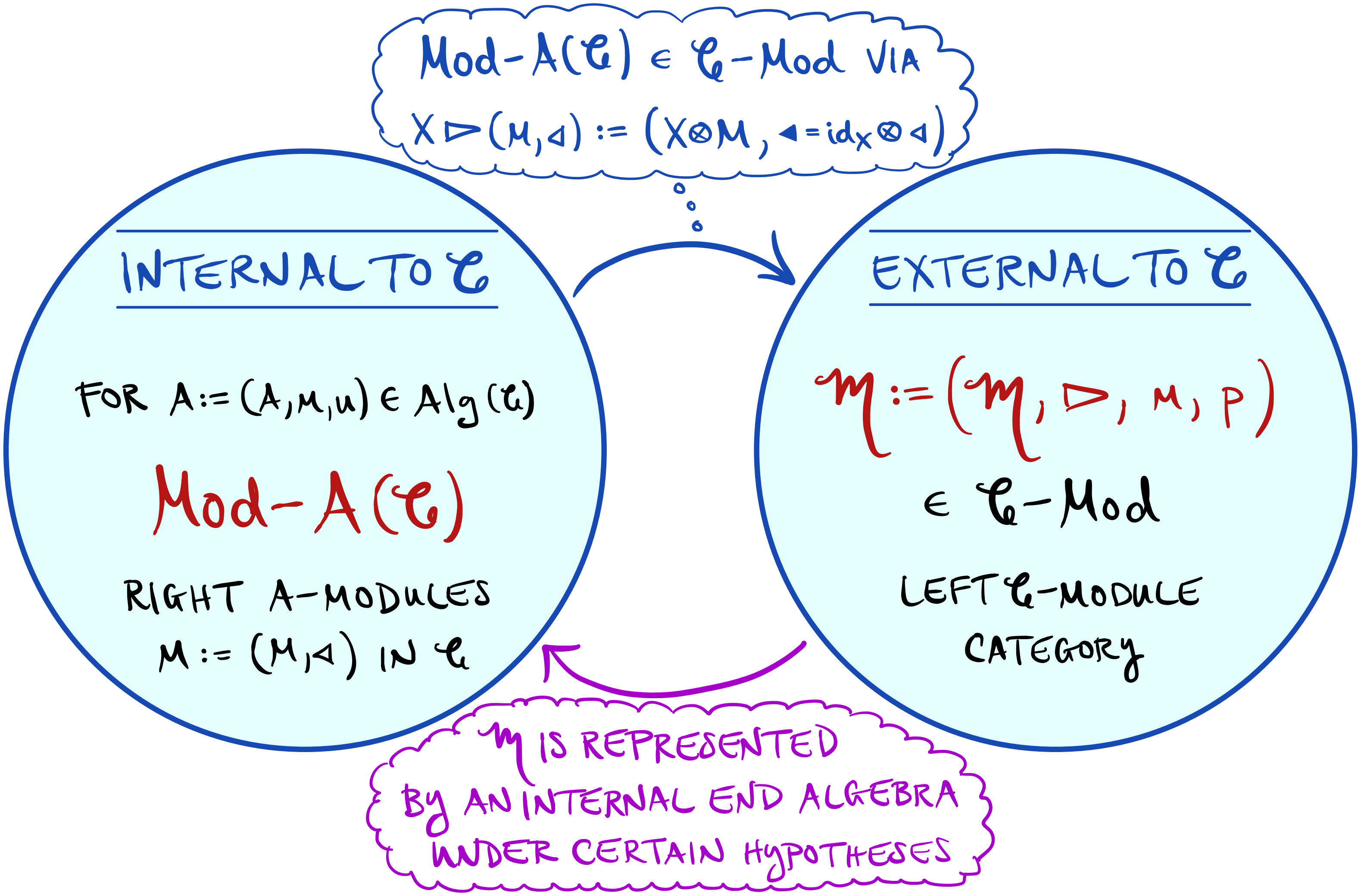
$\in \mathcal{C}$-Mod

LEFT $\mathcal{C}$-MODULE CATEGORY

WE WILL DISCUSS WHEN THIS OCCURS

II. OSTRIK'S THEOREM

$\mathcal{M} \in \mathcal{C}$-Mod IS REPRESENTED BY $A \in \mathrm{Alg}(\mathcal{C})$ IF $\mathcal{M} \simeq \text{Mod-}A(\mathcal{C})$ AS LEFT $\mathcal{C}$-MODULE CATEGORIES.



Mod-$A(\mathcal{C}) \in \mathcal{C}$-Mod VIA

$X \triangleright (M, \triangleleft) := (X \otimes M, \blacktriangleleft = \mathrm{id}_X \otimes \triangleleft)$

INTERNAL TO $\mathcal{C}$

FOR $A := (A, m, u) \in \mathrm{Alg}(\mathcal{C})$

Mod-$A(\mathcal{C})$

RIGHT A-MODULES

$M := (M, \triangleleft)$ IN $\mathcal{C}$

EXTERNAL TO $\mathcal{C}$

$\mathcal{M} := (\mathcal{M}, \triangleright, m, p)$

$\in \mathcal{C}$-Mod

LEFT $\mathcal{C}$-MODULE CATEGORY

$\mathcal{M}$ IS REPRESENTED BY AN INTERNAL END ALGEBRA UNDER CERTAIN HYPOTHESES

II. OSTRIK'S THEOREM

$(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ CLOSED $\Leftrightarrow$

$$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

LET'S COLLECT SOME FACTS —

II. OSTRIK'S THEOREM

$(\mathcal{M}, \triangleright) \in \mathcal{C}$-Mod CLOSED $\Leftrightarrow$

$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N)$

$\cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

LET'S COLLECT SOME FACTS —

LEMMA: WHEN $(\mathcal{M}, \triangleright) \in \mathcal{C}$-Mod IS CLOSED, GET FOR ALL $X \in \mathcal{C}$ AND $M, N \in \mathcal{M}$:

$$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N).$$

II. OSTRIK'S THEOREM

$(\mathcal{M},\triangleright) \in \mathcal{C}\text{-Mod}$ CLOSED $\Leftrightarrow$

$\mathrm{Hom}_{\mathcal{M}}(Z\triangleright M, N)$

$\cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

LET'S COLLECT SOME FACTS –

<u>LEMMA</u>: WHEN $(\mathcal{M},\triangleright) \in \mathcal{C}\text{-Mod}$ IS CLOSED, GET FOR ALL $X \in \mathcal{C}$ AND $M, N \in \mathcal{M}$:

$$\underline{\mathrm{Hom}}(M, X\triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N).$$

PF/ $\mathrm{Hom}_{\mathcal{C}}(Y, \underline{\mathrm{Hom}}(M, X\triangleright N))$

$\cong \mathrm{Hom}_{\mathcal{M}}(Y\triangleright M, X\triangleright N)$

II. OSTRIK'S THEOREM

$(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ CLOSED $\Leftrightarrow$

$$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

LET'S COLLECT SOME FACTS —

LEMMA: WHEN $(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ IS CLOSED, GET FOR ALL $X \in \mathcal{C}$ AND $M, N \in \mathcal{M}$:

$$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N).$$

PF/ $\mathrm{Hom}_{\mathcal{C}}(Y, \underline{\mathrm{Hom}}(M, X \triangleright N))$

$$\cong \mathrm{Hom}_{\mathcal{M}}(Y \triangleright M, X \triangleright N)$$

$$\cong \mathrm{Hom}_{\mathcal{M}}(X^* \triangleright (Y \triangleright M), N)$$

$(X^* \triangleright -) \dashv (X \triangleright -)$

II. OSTRIK'S THEOREM

$(\mathcal{M}, \triangleright) \in \mathcal{C}$-Mod CLOSED $\Leftrightarrow$

$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

LET'S COLLECT SOME FACTS —

LEMMA: WHEN $(\mathcal{M}, \triangleright) \in \mathcal{C}$-Mod IS CLOSED, GET FOR ALL $X \in \mathcal{C}$ AND $M, N \in \mathcal{M}$:

$$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N).$$

PF/ $\mathrm{Hom}_{\mathcal{C}}(Y, \underline{\mathrm{Hom}}(M, X \triangleright N))$

$$\begin{aligned}
&\cong \mathrm{Hom}_{\mathcal{M}}(Y \triangleright M, X \triangleright N) \\
&\cong \mathrm{Hom}_{\mathcal{M}}(X^* \triangleright (Y \triangleright M), N) \\
&\overset{\text{MOD CAT ASSOC.}}{\cong} \mathrm{Hom}_{\mathcal{M}}((X^* \otimes Y) \triangleright M, N)
\end{aligned}$$

$(X^* \triangleright -) \dashv (X \triangleright -)$

II. OSTRIK'S THEOREM

$(\mathcal{M},\triangleright)\in\mathcal{C}\text{-Mod}$ CLOSED $\Leftrightarrow$

$$\mathrm{Hom}_{\mathcal{M}}(Z\triangleright M,N)\cong \mathrm{Hom}_{\mathcal{C}}(Z,\underline{\mathrm{Hom}}(M,N))$$

$\underline{\mathrm{End}}(M)\in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N)\in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

LET'S COLLECT SOME FACTS –

LEMMA: WHEN $(\mathcal{M},\triangleright)\in\mathcal{C}\text{-Mod}$ IS CLOSED, GET FOR ALL $X\in\mathcal{C}$ AND $M,N\in\mathcal{M}$:

$$\underline{\mathrm{Hom}}(M,X\triangleright N)\cong X\otimes\underline{\mathrm{Hom}}(M,N).$$

PF/
$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{C}}(Y,\underline{\mathrm{Hom}}(M,X\triangleright N))\\
&\cong \mathrm{Hom}_{\mathcal{M}}(Y\triangleright M,X\triangleright N)\\
&\cong \mathrm{Hom}_{\mathcal{M}}(X^*\triangleright(Y\triangleright M),N)\\
&\cong \mathrm{Hom}_{\mathcal{M}}((X^*\otimes Y)\triangleright M,N)\\
&\cong \mathrm{Hom}_{\mathcal{C}}(X^*\otimes Y,\underline{\mathrm{Hom}}(M,N))
\end{aligned}$$

MOD CAT ASSOC.

$$(X^*\triangleright -)\dashv(X\triangleright -)$$

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II. OSTRIK'S THEOREM

$(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ CLOSED $\Leftrightarrow$

$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

LET'S COLLECT SOME FACTS –

<u>LEMMA</u>: WHEN $(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ IS CLOSED, GET FOR ALL $X \in \mathcal{C}$ AND $M, N \in \mathcal{M}$:

$$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N).$$

PF/

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{C}}(Y, \underline{\mathrm{Hom}}(M, X \triangleright N)) \\
&\cong \mathrm{Hom}_{\mathcal{M}}(Y \triangleright M, X \triangleright N) \\
&\cong \mathrm{Hom}_{\mathcal{M}}(X^* \triangleright (Y \triangleright M), N) \\
&\cong \mathrm{Hom}_{\mathcal{M}}((X^* \otimes Y) \triangleright M, N) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(X^* \otimes Y, \underline{\mathrm{Hom}}(M,N)) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(Y, X \otimes \underline{\mathrm{Hom}}(M,N)).
\end{aligned}$$

MOD CAT ASSOC.

$(X^* \triangleright -) \dashv (X \triangleright -)$

$(X^* \otimes -) \dashv (X \otimes -)$

II. OSTRIK'S THEOREM

$(\mathcal{M}, \triangleright) \in \mathcal{C}$-Mod CLOSED $\Leftrightarrow$

$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N)$
$\cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

LET'S COLLECT SOME FACTS –

LEMMA: WHEN $(\mathcal{M}, \triangleright) \in \mathcal{C}$-Mod IS CLOSED, GET FOR ALL $X \in \mathcal{C}$ AND $M, N \in \mathcal{M}$:

$$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N).$$

PF/ $\mathrm{Hom}_{\mathcal{C}}(Y, \underline{\mathrm{Hom}}(M, X \triangleright N))$

$$\begin{aligned}
&\cong \mathrm{Hom}_{\mathcal{M}}(Y \triangleright M, X \triangleright N)\\
&\cong \mathrm{Hom}_{\mathcal{M}}(X^* \triangleright (Y \triangleright M), N)\\
&\overset{\text{MOD CAT ASSOC.}}{\cong} \mathrm{Hom}_{\mathcal{M}}((X^* \otimes Y) \triangleright M, N)\\
&\cong \mathrm{Hom}_{\mathcal{C}}(X^* \otimes Y, \underline{\mathrm{Hom}}(M,N))\\
&\cong \mathrm{Hom}_{\mathcal{C}}(Y, X \otimes \underline{\mathrm{Hom}}(M,N)).
\end{aligned}$$

$(X^* \triangleright -) \dashv (X \triangleright -)$

$(X^* \otimes -) \dashv (X \otimes -)$

THEN APPLY YONEDA'S LEMMA. ///

II. OSTRIK'S THEOREM

$(\mathcal{M}, \triangleright) \in \mathcal{C}$-Mod CLOSED $\Leftrightarrow$

$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N)$

$\forall X \in \mathcal{C};\ M, N \in \mathcal{M}$

LET'S COLLECT SOME FACTS —

LEMMA: WHEN $(\mathcal{M}, \triangleright) \in \mathcal{C}$-Mod IS CLOSED, GET FOR ALL $X \in \mathcal{C}$ AND $M, N \in \mathcal{M}$:

$$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N).$$

PF/ $\mathrm{Hom}_{\mathcal{C}}(Y, \underline{\mathrm{Hom}}(M, X \triangleright N))$

$$\begin{aligned}
&\cong \mathrm{Hom}_{\mathcal{M}}(Y \triangleright M, X \triangleright N)\\
&\cong \mathrm{Hom}_{\mathcal{M}}(X^* \triangleright (Y \triangleright M), N)\\
&\overset{\text{MOD CAT ASSOC.}}{\cong} \mathrm{Hom}_{\mathcal{M}}((X^* \otimes Y) \triangleright M, N)\\
&\cong \mathrm{Hom}_{\mathcal{C}}(X^* \otimes Y, \underline{\mathrm{Hom}}(M,N))\\
&\cong \mathrm{Hom}_{\mathcal{C}}(Y, X \otimes \underline{\mathrm{Hom}}(M,N)) \quad \forall Y \in \mathcal{C}.
\end{aligned}$$

$(X^* \triangleright -) \dashv (X \triangleright -)$

$(X^* \otimes -) \dashv (X \otimes -)$

THEN APPLY YONEDA'S LEMMA. ///

II. OSTRIK'S THEOREM

$(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ CLOSED $\Leftrightarrow$

$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N)$
$\cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N)$
$\forall X \in \mathcal{C};\ M, N \in \mathcal{M}$

COOL FACTS: TAKE $\mathcal{C}$ MULTIFUSION CATEGORY.

II. OSTRIK'S THEOREM

$(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ CLOSED $\Leftrightarrow$

$$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

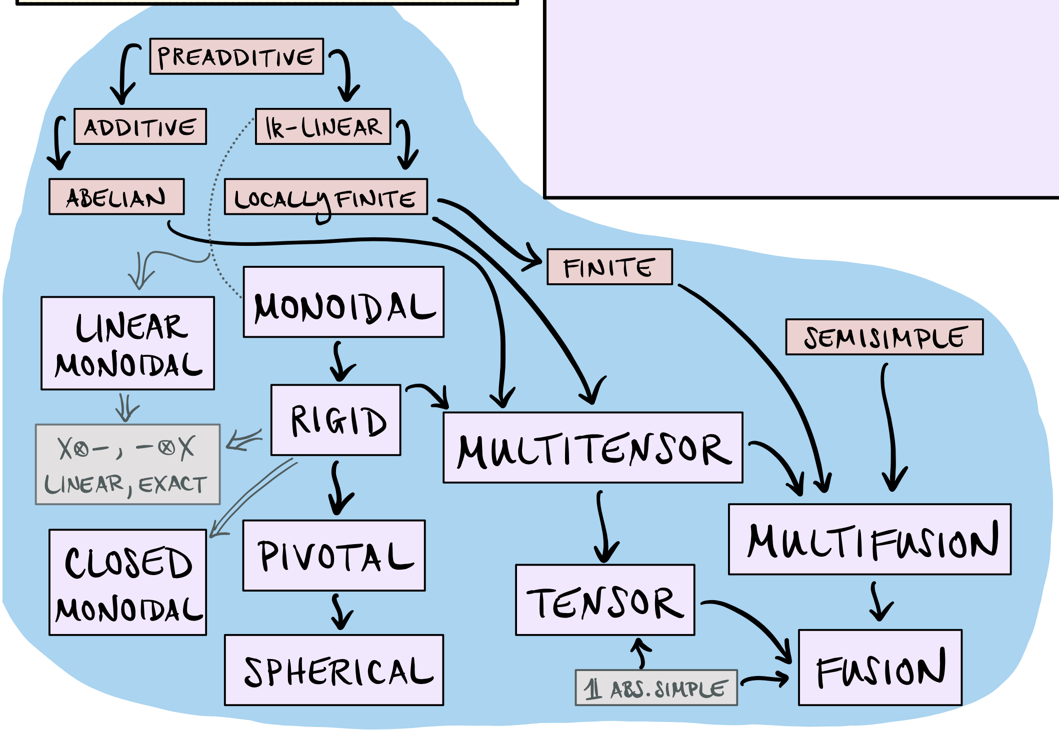
$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

$$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N)$$

$\forall X \in \mathcal{C};\ M, N \in \mathcal{M}$

COOL FACTS: TAKE $\mathcal{C}$ MULTIFUSION CATEGORY.



II. OSTRIK'S THEOREM

$(\mathcal{M},\triangleright)\in\mathcal{C}$-Mod CLOSED $\Leftrightarrow$

$$\mathrm{Hom}_{\mathcal{M}}(Z\triangleright M,N)\cong \mathrm{Hom}_{\mathcal{C}}(Z,\underline{\mathrm{Hom}}(M,N))$$

$\underline{\mathrm{End}}(M)\in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N)\in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

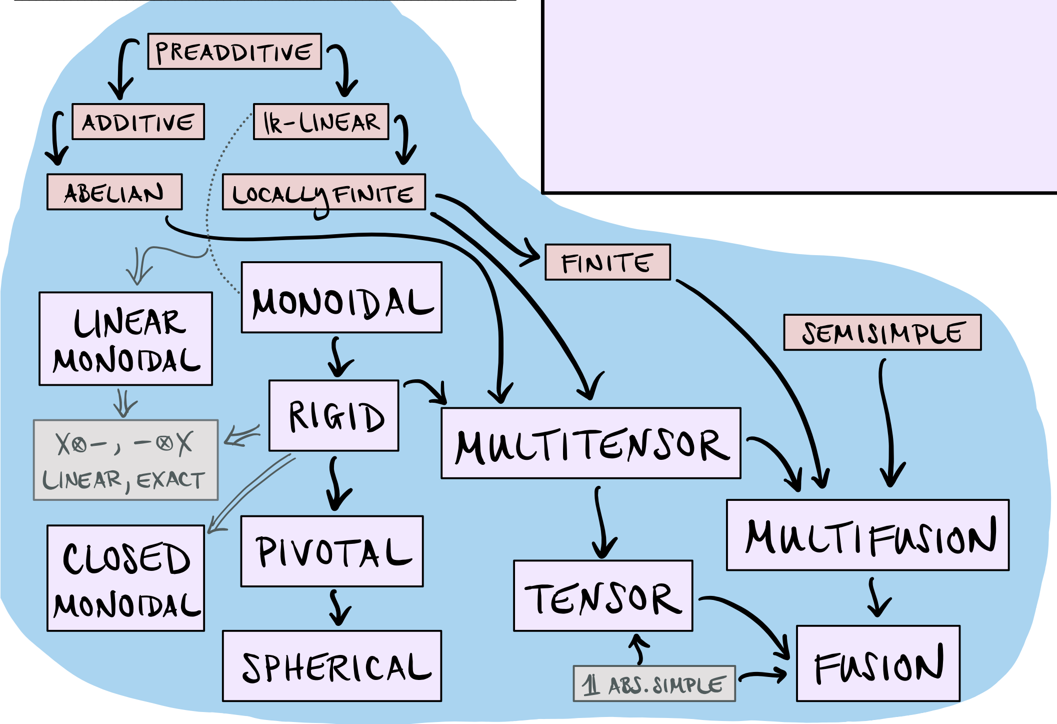
$$\underline{\mathrm{Hom}}(M,X\triangleright N)\cong X\otimes\underline{\mathrm{Hom}}(M,N)\quad \forall X\in\mathcal{C};\ M,N\in\mathcal{M}$$

COOL FACTS: TAKE $\mathcal{C}$ MULTIFUSION CATEGORY.

$\mathcal{M}^{\neq 0}\in\mathcal{C}$-Mod SEMISIMPLE & INDECOMP. $\Rightarrow$

(a) $\underline{\mathrm{Hom}}(M,-):\mathcal{M}\longrightarrow\mathcal{C}$ EXACT.

$\forall M^{\neq 0}\in\mathcal{M}$.



II. OSTRIK'S THEOREM

$(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ CLOSED $\Leftrightarrow$

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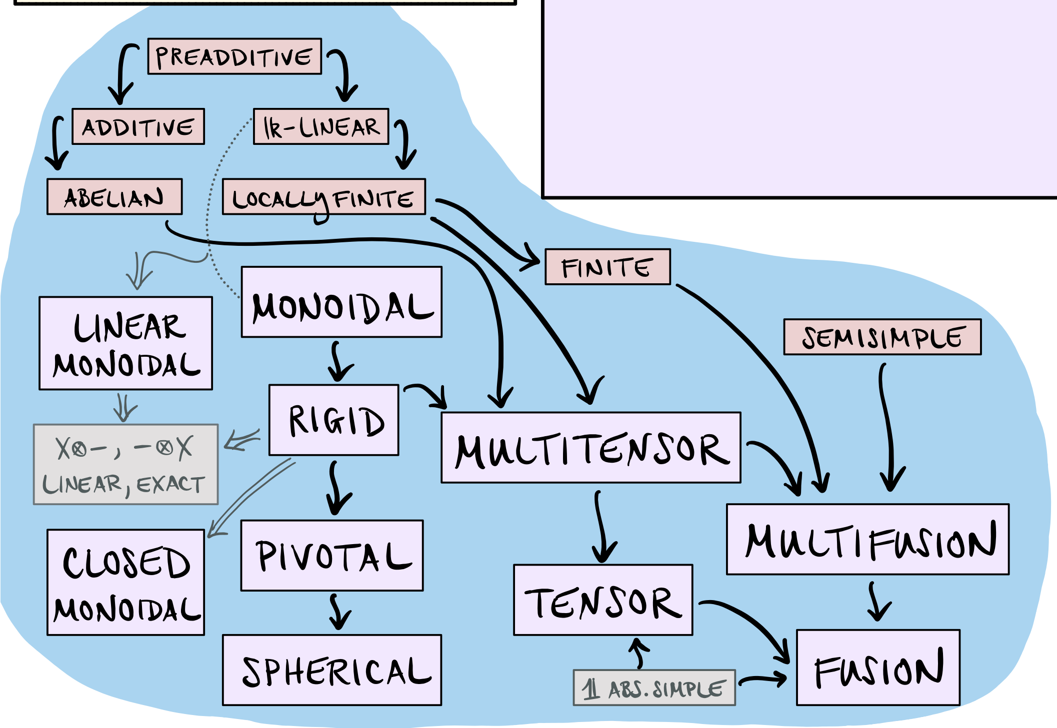

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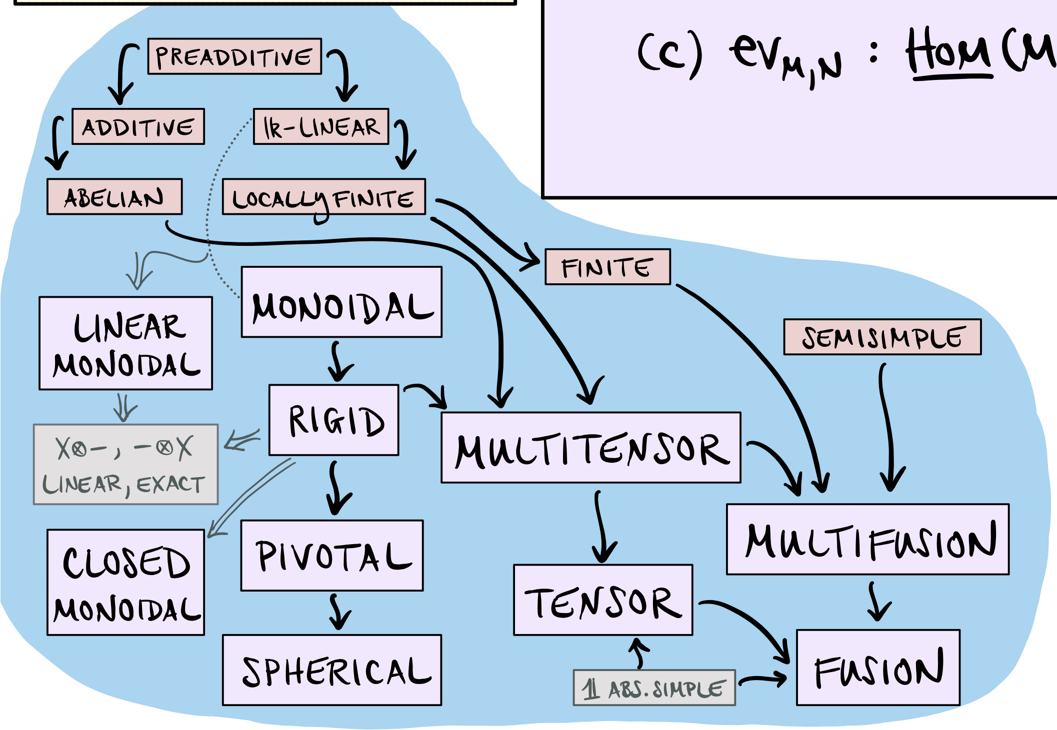

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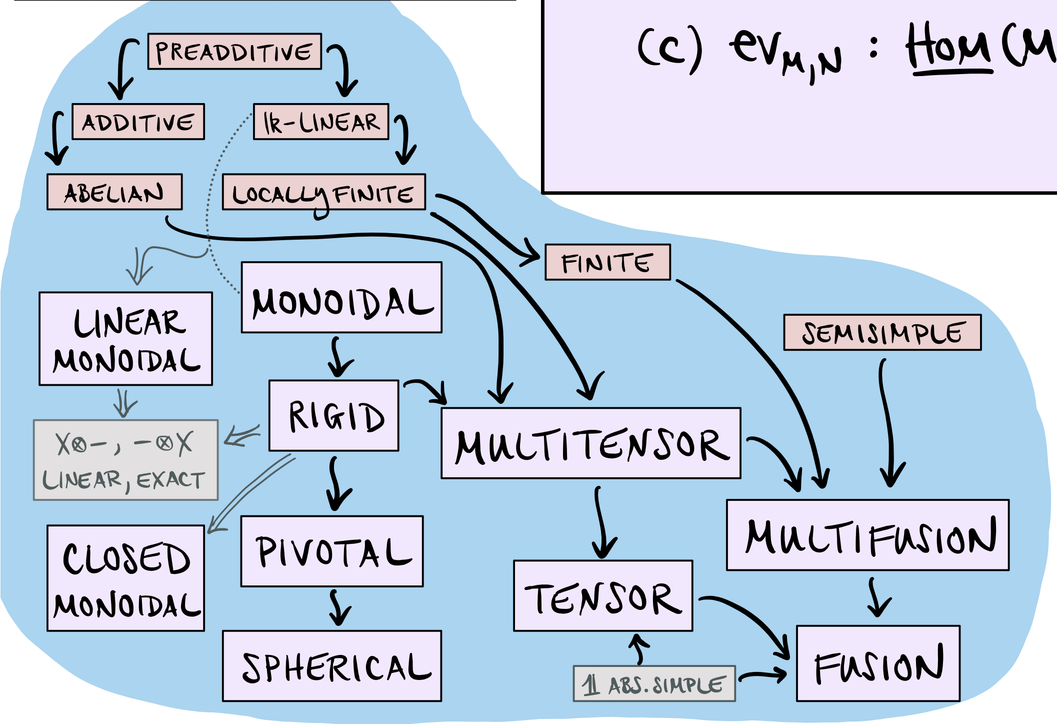

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⋮

ALSO GET THAT ALL (LEFT) $\mathcal{C}$-MODULE CATS ARE <u>CLOSED</u>

II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

$(\mathcal{M}, \triangleright) \in \mathcal{C}$-Mod CLOSED $\Leftrightarrow$

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$$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N) \quad \forall X \in \mathcal{C};\ M, N \in \mathcal{M}$$

$\mathcal{M} \neq 0$ SEMISIMPLE & INDECOMPOSABLE $\Rightarrow$

(a) $\underline{\mathrm{Hom}}(M,-) : \mathcal{M} \longrightarrow \mathcal{C}$ EXACT

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THEOREM [OSTRIK] $\mathcal{M}^{\neq 0} \in \mathcal{C}\text{-Mod}$. THEN

$$\mathcal{M} \simeq \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

AS LEFT $\mathcal{C}$-MODULE CATEGORIES $\forall M^{\neq 0} \in \mathcal{M}$.

II. OSTRIK'S THEOREM

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PF/

TAKE $A := \underline{\text{End}}(M)$,

$$F : \mathcal{M} \longrightarrow \text{Mod-}A(\mathcal{C})$$

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<u>WANT</u>: F IS AN EQUIV. OF LEFT $\mathcal{C}$-MOD. CATS.

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F IS A LEFT $\mathcal{C}$-MODULE FUNCTOR —

- FUNCTOR ✓

II. OSTRIK'S THEOREM

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$\boxed{\text{F IS A LEFT } \mathcal{C}\text{-MODULE FUNCTOR}}$ –

- FUNCTOR ✓
- … OF $\mathcal{C}$-MODULE CATEGORIES:

$$F(X \triangleright N) \overset{\text{DEF}}{=} \underline{\mathrm{Hom}}(M, X \triangleright N)$$

$\forall X \in \mathcal{C}, N \in \mathcal{M}$.

II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

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II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

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$$\overset{\mathrm{DEF}}{=} X \triangleright_A F(N)$$

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II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

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WANT: F IS AN EQUIV. $\overbrace{\text{OF LEFT } \mathcal{C}\text{-MOD.}}^{\checkmark}$ CATS.

$\boxed{F \text{ IS FULLY FAITHFUL}}$

II. OSTRIK'S THEOREM

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F IS FULLY FAITHFUL

IF $N = X \triangleright M$ FOR SOME $X \in \mathcal{C}$

II. OSTRIK'S THEOREM

$\{\mathcal{C}$ MULTIFUSION$\}$

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$\boxed{\text{F IS FULLY FAITHFUL}}$ $\{$IF $N = X \triangleright M$ FOR SOME $X \in \mathcal{C}\}$

$$\mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(F(N), F(N')) = \mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(F(X \triangleright M), F(N'))$$

$$\vdots$$

$$= \mathrm{Hom}_{\mathcal{M}}(N, N')$$

II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

$(\mathcal{M}, \triangleright) \in \mathcal{C}$-Mod CLOSED $\Leftrightarrow$

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$$\mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(F(N), F(N')) = \mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(F(X \triangleright M), F(N'))$$

$$\cong \mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(X \otimes A, F(N'))$$

$$\vdots$$

$$= \mathrm{Hom}_{\mathcal{M}}(N, N')$$

II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

$(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ CLOSED $\Leftrightarrow$

$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N)$ $\forall X \in \mathcal{C};\ M, N \in \mathcal{M}$

$\mathcal{M}^{\neq 0}$ SEMISIMPLE & INDECOMPOSABLE $\Rightarrow$

(a) $\underline{\mathrm{Hom}}(M,-): \mathcal{M} \longrightarrow \mathcal{C}$ EXACT

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THEOREM [OSTRIK] $\mathcal{M}^{\neq 0} \in \mathcal{C}\text{-Mod}$. THEN

$$\mathcal{M} \simeq \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

AS LEFT $\mathcal{C}$-MODULE CATEGORIES $\forall M^{\neq 0} \in \mathcal{M}$.

PF/

TAKE $A := \underline{\mathrm{End}}(M)$,

$F: \mathcal{M} \longrightarrow \mathrm{Mod}\text{-}A(\mathcal{C})$

$N \longmapsto \underline{\mathrm{Hom}}(M,N)$.

WANT: F IS AN EQUIV. $\overbrace{\text{OF LEFT } \mathcal{C}\text{-MOD.}}^{\checkmark}$ CATS.

F IS FULLY FAITHFUL | IF $N = X \triangleright M$ FOR SOME $X \in \mathcal{C}$

$$\mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(F(N), F(N')) = \mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(F(X \triangleright M), F(N'))$$

$$\cong \mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(X \otimes A, F(N'))$$

Free $\dashv$ Forg

$\mathrm{Free} = (- \otimes A)$

$$\cong \mathrm{Hom}_{\mathcal{C}}(X, \mathrm{Forg}(F(N')))$$

$$\vdots$$

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II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

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$\boxed{\text{F IS FULLY FAITHFUL}}$ IF $N = X \triangleright M$ FOR SOME $X \in \mathcal{C}$

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(F(N), F(N')) &= \mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(F(X \triangleright M), F(N')) \\
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&\overset{\mathrm{DEF}}{=} \mathrm{Hom}_{\mathcal{C}}(X, \underline{\mathrm{Hom}}(M,N')) \\
&\cong \mathrm{Hom}_{\mathcal{M}}(X \triangleright M, N') \\
&= \mathrm{Hom}_{\mathcal{M}}(N, N')
\end{aligned}$$

$\underset{\overset{\|}{(-\otimes A)}}{\mathrm{Free}} \dashv \mathrm{Forg}$

II. OSTRIK'S THEOREM

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II. OSTRIK'S THEOREM

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$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F", "{\shortparallel\ \underline{\mathrm{Hom}}(M,-)}"'] \arrow[rr, bend right] & \mathrm{Mod}\text{-}A(\mathcal{C}) \arrow[r, "\mathrm{Forg}"] & \mathcal{C}
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- FAITHFUL IN GENERAL:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F", "\|"'] \arrow[rr, bend right=50, "\underline{\mathrm{Hom}}(M{,}-)"'] & \text{Mod-}A(\mathcal{C}) \arrow[r, "\mathrm{Forg}"] & \mathcal{C} \end{tikzcd}$$

FAITHFUL

II. OSTRIK'S THEOREM

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FAITHFUL $\Longrightarrow$ F FAITHFUL

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$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[r, swap, "\shortparallel"] \arrow[rr, bend right, "\underline{\mathrm{Hom}}(M,-)"'] & \mathrm{Mod}\text{-}A(\mathcal{C}) \arrow[r, "\mathrm{Forg}"] & \mathcal{C}
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- FULL IN GENERAL:

GET S.E.S.: $0 \to \overset{\ker(ev_{M,N})}{\overset{\parallel}{K}} \longrightarrow \underline{\mathrm{Hom}}(M,N) \triangleright M \xrightarrow{ev_{M,N}} N \longrightarrow 0$

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$$\mathcal{M}\simeq \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

AS LEFT $\mathcal{C}$-MODULE CATEGORIES $\forall M^{\neq 0}\in\mathcal{M}$.

FAITHFUL✓

WANT: F IS AN EQUIV. $\overbrace{\text{OF LEFT } \mathcal{C}\text{-MOD. CATS.}}^{\checkmark}$

F IS FULLY FAITHFUL | IF $N=X\triangleright M$ FOR SOME $X\in\mathcal{C}$

- FULL IN GENERAL:

GET S.E.S.: $0\to \overset{\ker(ev_{M,N})}{\overset{\parallel}{K}}\longrightarrow \underline{\mathrm{Hom}}(M,N)\triangleright M\xrightarrow{ev_{M,N}} N\longrightarrow 0$

GET S.E.S.: $0\to F(K)\longrightarrow F(\underline{\mathrm{Hom}}(M,N)\triangleright M)\longrightarrow F(N)\longrightarrow 0$.

II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

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PF/

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GET S.E.S.: $0 \to F(K) \longrightarrow F(\underline{\mathrm{Hom}}(M,N) \triangleright M) \longrightarrow F(N) \longrightarrow 0$.

APPLY LEFT EXACT, CONTRAVARIANT FUNCTORS TO GET

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{M}}(N, N') \longrightarrow \mathrm{Hom}_{\mathcal{M}}(X \triangleright M, N') \longrightarrow \mathrm{Hom}_{\mathcal{M}}(K, N')$$

II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

$(\mathcal{M}, \triangleright) \in \mathcal{C}$-Mod CLOSED $\Leftrightarrow$

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$\cong$ $F_{0,0}$ ↻ ↻ ↻

"FOUR-LEMMA"

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\end{tikzcd}$$

"FOUR-LEMMA"

$\therefore$ $F_{N,N'}$ IS EPIC $\forall N,N'$

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II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

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$\boxed{\text{F IS ESSENTIALLY SURJECTIVE}}$

TAKE $Z \in \text{Mod-}A(\mathcal{C})$.

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$(\mathcal{M}, \triangleright) \in \mathcal{C}$-Mod CLOSED $\Leftrightarrow$

$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \text{Mod-}\underline{\mathrm{End}}(M)(\mathcal{C})$

$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N)$ $\forall X \in \mathcal{C};\ M,N \in \mathcal{M}$

$\mathcal{M}^{\neq 0}$ SEMISIMPLE & INDECOMPOSABLE $\Rightarrow$

(a) $\underline{\mathrm{Hom}}(M,-): \mathcal{M} \to \mathcal{C}$ EXACT

(b) $\underline{\mathrm{Hom}}(M,-): \mathcal{M} \to \mathcal{C}$ FAITHFUL

(c) $ev_{M,N}: \underline{\mathrm{Hom}}(M,N) \triangleright M \to N$ EPIC $\forall M^{\neq 0} \in \mathcal{M}$.

PF/

TAKE $A := \underline{\mathrm{End}}(M)$,

$F: \mathcal{M} \to \text{Mod-}A(\mathcal{C})$

$N \mapsto \underline{\mathrm{Hom}}(M,N)$.

THEOREM [OSTRIK] $\mathcal{M}^{\neq 0} \in \mathcal{C}$-Mod. THEN

$$\mathcal{M} \simeq \text{Mod-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

AS LEFT $\mathcal{C}$-MODULE CATEGORIES $\forall M^{\neq 0} \in \mathcal{M}$.

FULLY FAITHFUL ✓ ✓

WANT: F IS AN EQUIV. OF LEFT $\mathcal{C}$-MOD. CATS.

$\boxed{\text{F IS ESSENTIALLY SURJECTIVE}}$

TAKE $Z \in \text{Mod-}A(\mathcal{C})$.

CONSIDER ADJUNCTION:

$$\mathrm{Hom}_{\text{Mod-}A(\mathcal{C})}(Z \otimes A, Z) \xrightarrow{\wr} \mathrm{Hom}_{\mathcal{C}}(Z, \mathrm{Forg}(Z))$$

$$\Big[\underset{\text{(CHECK)}}{\text{EPI}}: Z \otimes A \longrightarrow Z\Big] \longmapsfrom \mathrm{id}_Z$$

II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

$(\mathcal{M},\triangleright) \in \mathcal{C}$-Mod CLOSED $\Leftrightarrow$

$$\mathrm{Hom}_{\mathcal{M}}(Z\triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

$\underline{\mathrm{Hom}}(M, X\triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N)$ $\forall X\in\mathcal{C};\ M,N\in\mathcal{M}$

$\mathcal{M}^{\neq 0}$ SEMISIMPLE & INDECOMPOSABLE $\Rightarrow$

(a) $\underline{\mathrm{Hom}}(M,-): \mathcal{M} \longrightarrow \mathcal{C}$ EXACT

(b) $\underline{\mathrm{Hom}}(M,-): \mathcal{M} \longrightarrow \mathcal{C}$ FAITHFUL

(c) $ev_{M,N}: \underline{\mathrm{Hom}}(M,N)\triangleright M \longrightarrow N$ EPIC $\forall M^{\neq 0}\in\mathcal{M}$.

THEOREM [OSTRIK] $\mathcal{M}^{\neq 0} \in \mathcal{C}$-Mod. THEN

$$\mathcal{M} \simeq \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

AS LEFT $\mathcal{C}$-MODULE CATEGORIES $\forall M^{\neq 0}\in\mathcal{M}$.

PF/

TAKE $A := \underline{\mathrm{End}}(M)$,

$F: \mathcal{M} \longrightarrow \mathrm{Mod}\text{-}A(\mathcal{C})$

$N \longmapsto \underline{\mathrm{Hom}}(M,N)$.

WANT: $\overset{\text{FULLY FAITHFUL}\ \checkmark}{F \text{ IS AN EQUIV.}}\ \overbrace{\text{OF LEFT } \mathcal{C}\text{-MOD. CATS.}}^{\checkmark}$

$\boxed{\text{F IS ESSENTIALLY SURJECTIVE}}$

TAKE $Z \in \mathrm{Mod}\text{-}A(\mathcal{C})$.

CONSIDER ADJUNCTION:

$$\mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(Z\otimes A, Z) \xrightarrow{\xi} \mathrm{Hom}_{\mathcal{C}}(Z, \mathrm{Forg}(Z))$$

$$\big[\underset{\text{(CHECK)}}{\text{EPI}}: Z\otimes A \longrightarrow Z\big] \longleftarrow\!\shortmid\ id_Z$$

GET:

$$\begin{tikzcd} & 0 \arrow[d] & & \\ & X = \ker(\xi^{-1}(id_Z)) \arrow[d] & & \\ & Z\otimes A \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

$(\mathcal{M},\triangleright)\in\mathcal{C}$-Mod CLOSED $\Leftrightarrow$

$\mathrm{Hom}_{\mathcal{M}}(Z\triangleright M,N) \cong \mathrm{Hom}_{\mathcal{C}}(Z,\underline{\mathrm{Hom}}(M,N))$

$\underline{\mathrm{End}}(M)\in\mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N)\in\mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,X\triangleright N)\cong X\otimes\underline{\mathrm{Hom}}(M,N)$ $\forall X\in\mathcal{C};\ M,N\in\mathcal{M}$

$\mathcal{M}^{\neq 0}$ SEMISIMPLE & INDECOMPOSABLE $\Rightarrow$

(a) $\underline{\mathrm{Hom}}(M,-):\mathcal{M}\longrightarrow\mathcal{C}$ EXACT

(b) $\underline{\mathrm{Hom}}(M,-):\mathcal{M}\longrightarrow\mathcal{C}$ FAITHFUL

(c) $ev_{M,N}:\underline{\mathrm{Hom}}(M,N)\triangleright M\longrightarrow N$ EPIC $\forall M^{\neq 0}\in\mathcal{M}$.

THEOREM [OSTRIK] $\mathcal{M}^{\neq 0}\in\mathcal{C}$-Mod. THEN

$$\mathcal{M}\simeq\mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

AS LEFT $\mathcal{C}$-MODULE CATEGORIES $\forall M^{\neq 0}\in\mathcal{M}$.

PF/

TAKE $A:=\underline{\mathrm{End}}(M)$,

$F:\mathcal{M}\longrightarrow\mathrm{Mod}\text{-}A(\mathcal{C})$

$N\longmapsto\underline{\mathrm{Hom}}(M,N)$.

WANT: $\overset{\text{FULLY FAITHFUL ✓}}{\text{F IS}}$ AN EQUIV. $\overbrace{\text{OF LEFT }\mathcal{C}\text{-MOD.}}^{✓}$ CATS.

$\boxed{\text{F IS ESSENTIALLY SURJECTIVE}}$

TAKE $Z\in\mathrm{Mod}\text{-}A(\mathcal{C})$.

CONSIDER ADJUNCTION:

$$\mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(Z\otimes A, Z)\xrightarrow{\varphi}\mathrm{Hom}_{\mathcal{C}}(Z,\mathrm{Forg}(Z))$$

$$\Big[\underset{\text{(CHECK)}}{\text{EPI}}: Z\otimes A\longrightarrow Z\Big] \longleftarrow\!\shortmid\ id_Z$$

GET:

$$\begin{tikzcd}
& 0 \arrow[d] & & & \\
& X=\ker(\varphi^{-1}(id_Z)) \arrow[dr] & & & \\
X\otimes A \arrow[ur, "\varphi^{-1}(id_X)"] \arrow[rr] & & Z\otimes A \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

$(\mathcal{M}, \triangleright) \in \mathcal{C}$-Mod CLOSED $\Leftrightarrow$

$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N)$

$\cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N)$

$\forall X \in \mathcal{C};\ M, N \in \mathcal{M}$

$\mathcal{M}^{\neq 0}$ SEMISIMPLE & INDECOMPOSABLE $\Rightarrow$

(a) $\underline{\mathrm{Hom}}(M,-) : \mathcal{M} \longrightarrow \mathcal{C}$ EXACT

(b) $\underline{\mathrm{Hom}}(M,-) : \mathcal{M} \longrightarrow \mathcal{C}$ FAITHFUL

(c) $ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N$ EPIC $\forall M^{\neq 0} \in \mathcal{M}$.

PF/

TAKE $A := \underline{\mathrm{End}}(M)$,

$F : \mathcal{M} \longrightarrow \mathrm{Mod}\text{-}A(\mathcal{C})$

$N \longmapsto \underline{\mathrm{Hom}}(M,N)$.

THEOREM [OSTRIK] $\mathcal{M}^{\neq 0} \in \mathcal{C}$-Mod. THEN

$\mathcal{M} \simeq \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

AS LEFT $\mathcal{C}$-MODULE CATEGORIES $\forall M^{\neq 0} \in \mathcal{M}$.

WANT: $\overset{\text{FULLY FAITHFUL ✓}}{\text{F IS AN EQUIV.}}$ $\overbrace{\text{OF LEFT } \mathcal{C}\text{-MOD.}}^{✓}$ CATS.

$\boxed{\text{F IS ESSENTIALLY SURJECTIVE}}$

TAKE $Z \in \mathrm{Mod}\text{-}A(\mathcal{C})$.

CONSIDER ADJUNCTION:

$$\mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(Z \otimes A, Z) \xrightarrow{\varrho} \mathrm{Hom}_{\mathcal{C}}(Z, \mathrm{Forg}(Z))$$

$$\left[\underset{\text{(CHECK)}}{\text{EPI}} : Z \otimes A \longrightarrow Z\right] \longleftarrow\!\shortmid\ \mathrm{id}_Z$$

GET:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & X = \ker(\varrho^{-1}(\mathrm{id}_Z)) \arrow[dr] & & & \\
X \otimes A \arrow[ur, "\varrho^{-1}(\mathrm{id}_X)"] \arrow[rr, "\text{CALL THIS } \phi"'] & & Z \otimes A \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

$(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ CLOSED $\Leftrightarrow$

$$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \text{Mod-}\underline{\mathrm{End}}(M)(\mathcal{C})$

$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N)$ $\forall X \in \mathcal{C};\ M,N \in \mathcal{M}$

$\mathcal{M}^{\neq 0}$ SEMISIMPLE & INDECOMPOSABLE $\Rightarrow$

(a) $\underline{\mathrm{Hom}}(M,-): \mathcal{M} \longrightarrow \mathcal{C}$ EXACT

(b) $\underline{\mathrm{Hom}}(M,-): \mathcal{M} \longrightarrow \mathcal{C}$ FAITHFUL

(c) $ev_{M,N}: \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N$ EPIC $\forall M^{\neq 0} \in \mathcal{M}$.

PF/

TAKE $A := \underline{\mathrm{End}}(M)$,

$$F: \mathcal{M} \longrightarrow \text{Mod-}A(\mathcal{C})$$
$$N \longmapsto \underline{\mathrm{Hom}}(M,N).$$

$\underline{\text{THEOREM}}$ [OSTRIK] $\mathcal{M}^{\neq 0} \in \mathcal{C}\text{-Mod}$. THEN

$$\mathcal{M} \simeq \text{Mod-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

AS LEFT $\mathcal{C}$-MODULE CATEGORIES $\forall M^{\neq 0} \in \mathcal{M}$.

$\underline{\text{WANT}}$: F $\overset{\text{FULLY FAITHFUL ✓}}{\text{IS}}$ AN EQUIV. $\overbrace{\text{OF LEFT } \mathcal{C}\text{-MOD.}}^{✓}$ CATS.

$$\boxed{\text{F IS ESSENTIALLY SURJECTIVE}}$$

TAKE $Z \in \text{Mod-}A(\mathcal{C})$.

NOW:

$$\underset{\phi\,\in}{\mathrm{Hom}_{\text{Mod-}A(\mathcal{C})}}(X \otimes A, Z \otimes A)$$

GET:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & X = \ker(g^{-1}(id_Z)) \arrow[dr] & & & \\
X \otimes A \arrow[ur, "g^{-1}(id_X)"] \arrow[rr, "\text{CALL THIS } \phi"'] & & Z \otimes A \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

$(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ CLOSED $\Leftrightarrow$

$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N)$

$\forall X \in \mathcal{C};\ M, N \in \mathcal{M}$

$\mathcal{M}^{\neq 0}$ SEMISIMPLE & INDECOMPOSABLE $\Rightarrow$

(a) $\underline{\mathrm{Hom}}(M,-) : \mathcal{M} \to \mathcal{C}$ EXACT

(b) $\underline{\mathrm{Hom}}(M,-) : \mathcal{M} \to \mathcal{C}$ FAITHFUL

(c) $ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \to N$ EPIC $\forall M^{\neq 0} \in \mathcal{M}$.

THEOREM [OSTRIK] $\mathcal{M}^{\neq 0} \in \mathcal{C}\text{-Mod}$. THEN

$$\mathcal{M} \simeq \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

AS LEFT $\mathcal{C}$-MODULE CATEGORIES $\forall M^{\neq 0} \in \mathcal{M}$.

PF/

TAKE $A := \underline{\mathrm{End}}(M)$,

$$F : \mathcal{M} \longrightarrow \mathrm{Mod}\text{-}A(\mathcal{C})$$
$$N \longmapsto \underline{\mathrm{Hom}}(M,N).$$

FULLY FAITHFUL ✓ ✓

WANT: F IS AN EQUIV. OF LEFT $\mathcal{C}$-MOD. CATS.

F IS ESSENTIALLY SURJECTIVE

TAKE $Z \in \mathrm{Mod}\text{-}A(\mathcal{C})$.

NOW:

$$\underset{\phi^{\mathcal{C}}}{\mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(X \otimes A, Z \otimes A)} \overset{✓}{\cong} \mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(F(X \triangleright M), F(Z \triangleright M))$$

GET:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & X = \ker(g^{-1}(\mathrm{id}_Z)) \arrow[dr] & & & \\
X \otimes A \arrow[ur, "g^{-1}(\mathrm{id}_X)"] \arrow[rr, "\text{CALL THIS } \phi"'] & & Z \otimes A \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

$(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ CLOSED $\Leftrightarrow$

$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N)$

$\forall X \in \mathcal{C};\ M, N \in \mathcal{M}$

$\mathcal{M}^{\neq 0}$ SEMISIMPLE & INDECOMPOSABLE $\Rightarrow$

(a) $\underline{\mathrm{Hom}}(M,-) : \mathcal{M} \longrightarrow \mathcal{C}$ EXACT

(b) $\underline{\mathrm{Hom}}(M,-) : \mathcal{M} \longrightarrow \mathcal{C}$ FAITHFUL

(c) $ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N$ EPIC $\forall M^{\neq 0} \in \mathcal{M}$.

THEOREM [OSTRIK] $\mathcal{M}^{\neq 0} \in \mathcal{C}\text{-Mod}$. THEN

$$\mathcal{M} \simeq \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

AS LEFT $\mathcal{C}$-MODULE CATEGORIES $\forall M^{\neq 0} \in \mathcal{M}$.

PF/

TAKE $A := \underline{\mathrm{End}}(M)$,

$F : \mathcal{M} \longrightarrow \mathrm{Mod}\text{-}A(\mathcal{C})$

$N \longmapsto \underline{\mathrm{Hom}}(M,N)$.

FULLY FAITHFUL ✓

WANT: F IS AN EQUIV. $\overbrace{\text{OF LEFT } \mathcal{C}\text{-MOD.}}^{\checkmark}$ CATS.

$$\boxed{F \text{ IS ESSENTIALLY SURJECTIVE}}$$

TAKE $Z \in \mathrm{Mod}\text{-}A(\mathcal{C})$.

NOW:

$$\underset{\phi \in}{\mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}}(X \otimes A, Z \otimes A) \cong \mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(F(X \triangleright M), F(Z \triangleright M))$$

$$\cong \mathrm{Hom}_{\mathcal{M}}(X \triangleright M, Z \triangleright M)$$

[F FULLY FAITHFUL]

GET:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & X = \ker(g^{-1}(id_Z)) \arrow[dr] & & & \\
X \otimes A \arrow[ur, "g^{-1}(id_X)"] \arrow[rr, "\text{CALL THIS } \phi"'] & & Z \otimes A \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

$(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ CLOSED $\Leftrightarrow$

$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N)$

$\cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N)$

$\forall X \in \mathcal{C};\ M, N \in \mathcal{M}$

$\mathcal{M}^{\neq 0}$ SEMISIMPLE & INDECOMPOSABLE $\Rightarrow$

(a) $\underline{\mathrm{Hom}}(M,-): \mathcal{M} \longrightarrow \mathcal{C}$ EXACT

(b) $\underline{\mathrm{Hom}}(M,-): \mathcal{M} \longrightarrow \mathcal{C}$ FAITHFUL

(c) $ev_{M,N}: \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N$ EPIC

$\forall M^{\neq 0} \in \mathcal{M}$.

<u>THEOREM</u> [OSTRIK] $\mathcal{M}^{\neq 0} \in \mathcal{C}\text{-Mod}$. THEN

$$\mathcal{M} \simeq \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

AS LEFT $\mathcal{C}$-MODULE CATEGORIES $\forall M^{\neq 0} \in \mathcal{M}$.

PF/

TAKE $A := \underline{\mathrm{End}}(M)$,

$F: \mathcal{M} \longrightarrow \mathrm{Mod}\text{-}A(\mathcal{C})$

$N \longmapsto \underline{\mathrm{Hom}}(M,N)$.

<u>WANT</u>: $\overset{\text{FULLY FAITHFUL} \checkmark}{\text{F IS AN EQUIV.}}\ \overbrace{\text{OF LEFT } \mathcal{C}\text{-MOD. CATS}}^{\checkmark}.$

$\boxed{\text{F IS ESSENTIALLY SURJECTIVE}}$

TAKE $Z \in \mathrm{Mod}\text{-}A(\mathcal{C})$.

NOW:

$$\mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(X \otimes A, Z \otimes A) \cong \mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(F(X \triangleright M), F(Z \triangleright M))$$

$$\phi'' \longleftarrow\!\shortmid \qquad \cong \mathrm{Hom}_{\mathcal{M}}(X \triangleright M, Z \triangleright M) \qquad \longrightarrow \text{CALL THIS } \phi'$$

[F FULLY FAITHFUL]

GET:

$$\begin{tikzcd}
& 0 \arrow[d] & & & \\
& X = \ker(g^{-1}(id_Z)) \arrow[dr] & & & \\
X \otimes A \arrow[ur, "g^{-1}(id_X)"] \arrow[rr, "\text{CALL THIS } \phi"'] & & Z \otimes A \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

$(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ CLOSED $\Leftrightarrow$

$$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

$$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N) \quad \forall X \in \mathcal{C};\ M, N \in \mathcal{M}$$

$\mathcal{M}^{\neq 0}$ SEMISIMPLE & INDECOMPOSABLE $\Rightarrow$

(a) $\underline{\mathrm{Hom}}(M,-) : \mathcal{M} \longrightarrow \mathcal{C}$ EXACT

(b) $\underline{\mathrm{Hom}}(M,-) : \mathcal{M} \longrightarrow \mathcal{C}$ FAITHFUL

(c) $ev_{M,N} : \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N$ EPIC $\forall M^{\neq 0} \in \mathcal{M}$.

PF/

TAKE $A := \underline{\mathrm{End}}(M)$,

$F : \mathcal{M} \longrightarrow \mathrm{Mod}\text{-}A(\mathcal{C})$

$N \longmapsto \underline{\mathrm{Hom}}(M,N)$.

THEOREM [OSTRIK] $\mathcal{M}^{\neq 0} \in \mathcal{C}\text{-Mod}$. THEN

$$\mathcal{M} \simeq \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

AS LEFT $\mathcal{C}$-MODULE CATEGORIES $\forall M^{\neq 0} \in \mathcal{M}$.

FULLY FAITHFUL ✓ ✓

WANT: F IS AN EQUIV. OF LEFT $\mathcal{C}$-MOD. CATS.

F IS ESSENTIALLY SURJECTIVE

TAKE $Z \in \mathrm{Mod}\text{-}A(\mathcal{C})$.

NOW:

$$\underset{\phi}{\mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}}(X \otimes A, Z \otimes A) \cong \mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(F(X \triangleright M), F(Z \triangleright M)) \ni F(\phi')$$

$$\cong \mathrm{Hom}_{\mathcal{M}}(X \triangleright M, Z \triangleright M)$$

CALL THIS ϕ'

FINALLY:

$$F(\mathrm{coker}\,\phi') \cong \mathrm{coker}(F(\phi')) \cong Z.$$

II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

$(\mathcal{M}, \triangleright) \in \mathcal{C}$-Mod CLOSED $\Leftrightarrow$

$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N)$ $\forall X \in \mathcal{C};\ M, N \in \mathcal{M}$

$\mathcal{M}^{\neq 0}$ SEMISIMPLE & INDECOMPOSABLE $\Rightarrow$

(a) $\underline{\mathrm{Hom}}(M,-): \mathcal{M} \longrightarrow \mathcal{C}$ EXACT

(b) $\underline{\mathrm{Hom}}(M,-): \mathcal{M} \longrightarrow \mathcal{C}$ FAITHFUL

(c) $ev_{M,N}: \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N$ EPIC $\forall M^{\neq 0} \in \mathcal{M}$.

THEOREM [OSTRIK] $\mathcal{M}^{\neq 0} \in \mathcal{C}$-Mod. THEN

$$\mathcal{M} \simeq \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

AS LEFT $\mathcal{C}$-MODULE CATEGORIES $\forall M^{\neq 0} \in \mathcal{M}$.

PF/

TAKE $A := \underline{\mathrm{End}}(M)$,

$$F: \mathcal{M} \longrightarrow \mathrm{Mod}\text{-}A(\mathcal{C})$$
$$N \longmapsto \underline{\mathrm{Hom}}(M,N).$$

FULLY FAITHFUL ✓ ✓

WANT: F IS AN EQUIV. OF LEFT $\mathcal{C}$-MOD. CATS.

$\boxed{\text{F IS ESSENTIALLY SURJECTIVE}}$

TAKE $Z \in \mathrm{Mod}\text{-}A(\mathcal{C})$.

NOW:

$$\underset{\phi}{\mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(X \otimes A, Z \otimes A)} \cong \underset{F(\phi')}{\mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(F(X \triangleright M), F(Z \triangleright M))}$$
$$\cong \underset{\text{CALL THIS } \phi'}{\mathrm{Hom}_{\mathcal{M}}(X \triangleright M, Z \triangleright M)}$$

FINALLY:

$$F(\mathrm{coker}\, \phi') \cong \mathrm{coker}(F(\phi')) \cong Z.$$

II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

$(\mathcal{M}, \triangleright) \in \mathcal{C}\text{-Mod}$ CLOSED $\Leftrightarrow$

$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N)$ $\forall X \in \mathcal{C};\ M, N \in \mathcal{M}$

$\mathcal{M}^{\neq 0}$ SEMISIMPLE & INDECOMPOSABLE $\Rightarrow$

(a) $\underline{\mathrm{Hom}}(M,-): \mathcal{M} \longrightarrow \mathcal{C}$ EXACT

(b) $\underline{\mathrm{Hom}}(M,-): \mathcal{M} \longrightarrow \mathcal{C}$ FAITHFUL

(c) $ev_{M,N}: \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N$ EPIC $\forall M^{\neq 0} \in \mathcal{M}$.

THEOREM [OSTRIK] $\mathcal{M}^{\neq 0} \in \mathcal{C}\text{-Mod}$. THEN

$$\mathcal{M} \simeq \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

AS LEFT $\mathcal{C}$-MODULE CATEGORIES $\forall M^{\neq 0} \in \mathcal{M}$.

PF/

TAKE $A := \underline{\mathrm{End}}(M)$,

$F: \mathcal{M} \longrightarrow \mathrm{Mod}\text{-}A(\mathcal{C})$

$N \longmapsto \underline{\mathrm{Hom}}(M,N)$.

FULLY FAITHFUL ✓ ✓

WANT: F IS AN EQUIV. $\underbrace{\text{OF LEFT } \mathcal{C}\text{-MOD. CATS}}$.

$\boxed{\text{F IS ESSENTIALLY SURJECTIVE}}$

TAKE $Z \in \mathrm{Mod}\text{-}A(\mathcal{C})$.

NOW:

$$\underset{\phi}{\mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(X \otimes A, Z \otimes A)} \cong \overset{F(\phi')}{\mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(F(X \triangleright M), F(Z \triangleright M))}$$

$$\cong \mathrm{Hom}_{\mathcal{M}}(X \triangleright M, Z \triangleright M)$$

CALL THIS ϕ'

$$\begin{tikzcd}
0 \arrow[dr] & & 0 & & \\
 & X = \ker(\rho^{-1}(\mathrm{id}_Z)) \arrow[ur] \arrow[dr] & & & \\
X \otimes A \arrow[ur, "\rho^{-1}(\mathrm{id}_X)"] \arrow[rr, "\phi", "F(\phi')"'] & & Z \otimes A \arrow[r, "\rho^{-1}(\mathrm{id}_Z)"] & Z \arrow[r] & 0
\end{tikzcd}$$

FINALLY: $F(\mathrm{coker}\,\phi') \cong \mathrm{coker}(F(\phi')) \cong Z$.

II. OSTRIK'S THEOREM

$\mathcal{C}$ MULTIFUSION

$(\mathcal{M}, \triangleright) \in \mathcal{C}$-Mod CLOSED $\Leftrightarrow$

$\mathrm{Hom}_{\mathcal{M}}(Z \triangleright M, N) \cong \mathrm{Hom}_{\mathcal{C}}(Z, \underline{\mathrm{Hom}}(M,N))$

$\underline{\mathrm{End}}(M) \in \mathrm{Alg}(\mathcal{C})$

$\underline{\mathrm{Hom}}(M,N) \in \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$

$\underline{\mathrm{Hom}}(M, X \triangleright N) \cong X \otimes \underline{\mathrm{Hom}}(M,N)$ $\forall X \in \mathcal{C};\ M, N \in \mathcal{M}$

$\mathcal{M}^{\neq 0}$ SEMISIMPLE & INDECOMPOSABLE $\Rightarrow$

(a) $\underline{\mathrm{Hom}}(M,-): \mathcal{M} \longrightarrow \mathcal{C}$ EXACT

(b) $\underline{\mathrm{Hom}}(M,-): \mathcal{M} \longrightarrow \mathcal{C}$ FAITHFUL

(c) $ev_{M,N}: \underline{\mathrm{Hom}}(M,N) \triangleright M \longrightarrow N$ EPIC $\forall M^{\neq 0} \in \mathcal{M}$.

THEOREM [OSTRIK] $\mathcal{M}^{\neq 0} \in \mathcal{C}$-Mod. THEN

$$\mathcal{M} \simeq \mathrm{Mod}\text{-}\underline{\mathrm{End}}(M)(\mathcal{C})$$

AS LEFT $\mathcal{C}$-MODULE CATEGORIES $\forall M^{\neq 0} \in \mathcal{M}$.

PF/

TAKE $A := \underline{\mathrm{End}}(M)$,

$F: \mathcal{M} \longrightarrow \mathrm{Mod}\text{-}A(\mathcal{C})$

$N \longmapsto \underline{\mathrm{Hom}}(M,N)$.

FULLY FAITHFUL ✓ ✓

WANT: F IS AN EQUIV. $\overbrace{\text{OF LEFT } \mathcal{C}\text{-MOD. CATS.}}$

F IS ESSENTIALLY SURJECTIVE

TAKE $Z \in \mathrm{Mod}\text{-}A(\mathcal{C})$.

NOW:

$$\underset{\phi}{\mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(X \otimes A, Z \otimes A)} \cong \mathrm{Hom}_{\mathrm{Mod}\text{-}A(\mathcal{C})}(F(X \triangleright M), \underset{F(\phi')}{F(Z \triangleright M)})$$

$$\cong \mathrm{Hom}_{\mathcal{M}}(X \triangleright M, Z \triangleright M)$$

CALL THIS ϕ'

$$\begin{tikzcd}
0 \arrow[dr] & & 0 & & \\
& X = \ker(\varrho^{-1}(id_Z)) \arrow[ur] \arrow[dr] & & & \\
X \otimes A \arrow[ur, "\varrho^{-1}(id_X)"] \arrow[rr, "\phi", "F(\phi')"'] & & Z \otimes A \arrow[r, "\varrho^{-1}(id_Z)"] & Z \arrow[r] & 0
\end{tikzcd}$$

FINALLY:

$$F(\mathrm{coker}\,\phi') \cong \mathrm{coker}(F(\phi')) \cong Z.$$

$\therefore$ F IS ESSENTIALLY SURJECTIVE //

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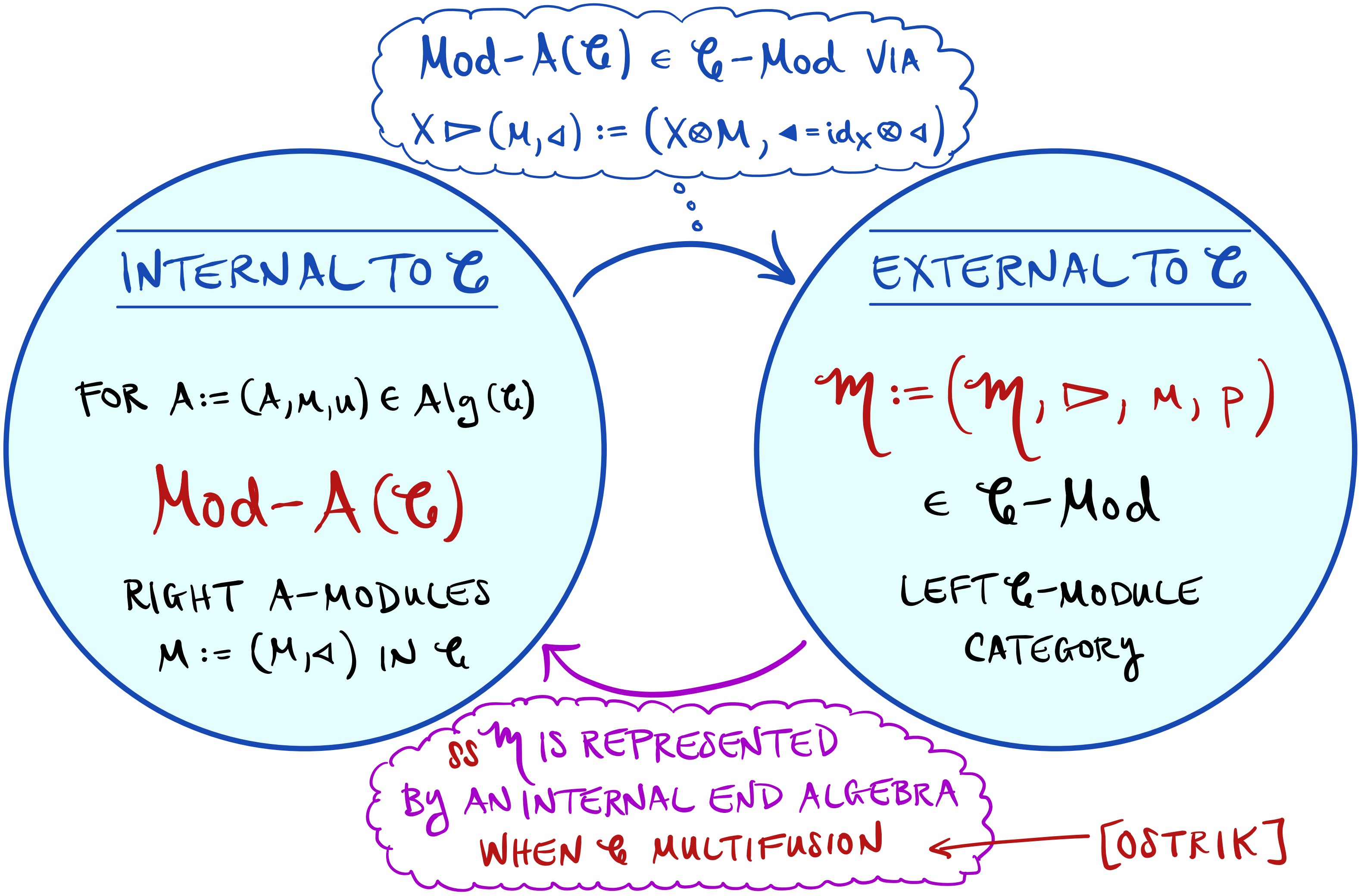
II. OSTRIK'S THEOREM

$\mathcal{M} \in \mathcal{C}$-Mod IS REPRESENTED BY $A \in \mathsf{Alg}(\mathcal{C})$ IF
$\mathcal{M} \simeq \text{Mod-}A(\mathcal{C})$ AS LEFT $\mathcal{C}$-MODULE CATEGORIES.



Mod-$A(\mathcal{C}) \in \mathcal{C}$-Mod VIA
$X \triangleright (M, \triangleleft) := (X \otimes M, \blacktriangleleft = \mathrm{id}_X \otimes \triangleleft)$

INTERNAL TO $\mathcal{C}$

FOR $A := (A, m, u) \in \mathsf{Alg}(\mathcal{C})$

Mod-$A(\mathcal{C})$

RIGHT A-MODULES
$M := (M, \triangleleft)$ IN $\mathcal{C}$

EXTERNAL TO $\mathcal{C}$

$\mathcal{M} := (\mathcal{M}, \triangleright, m, p)$

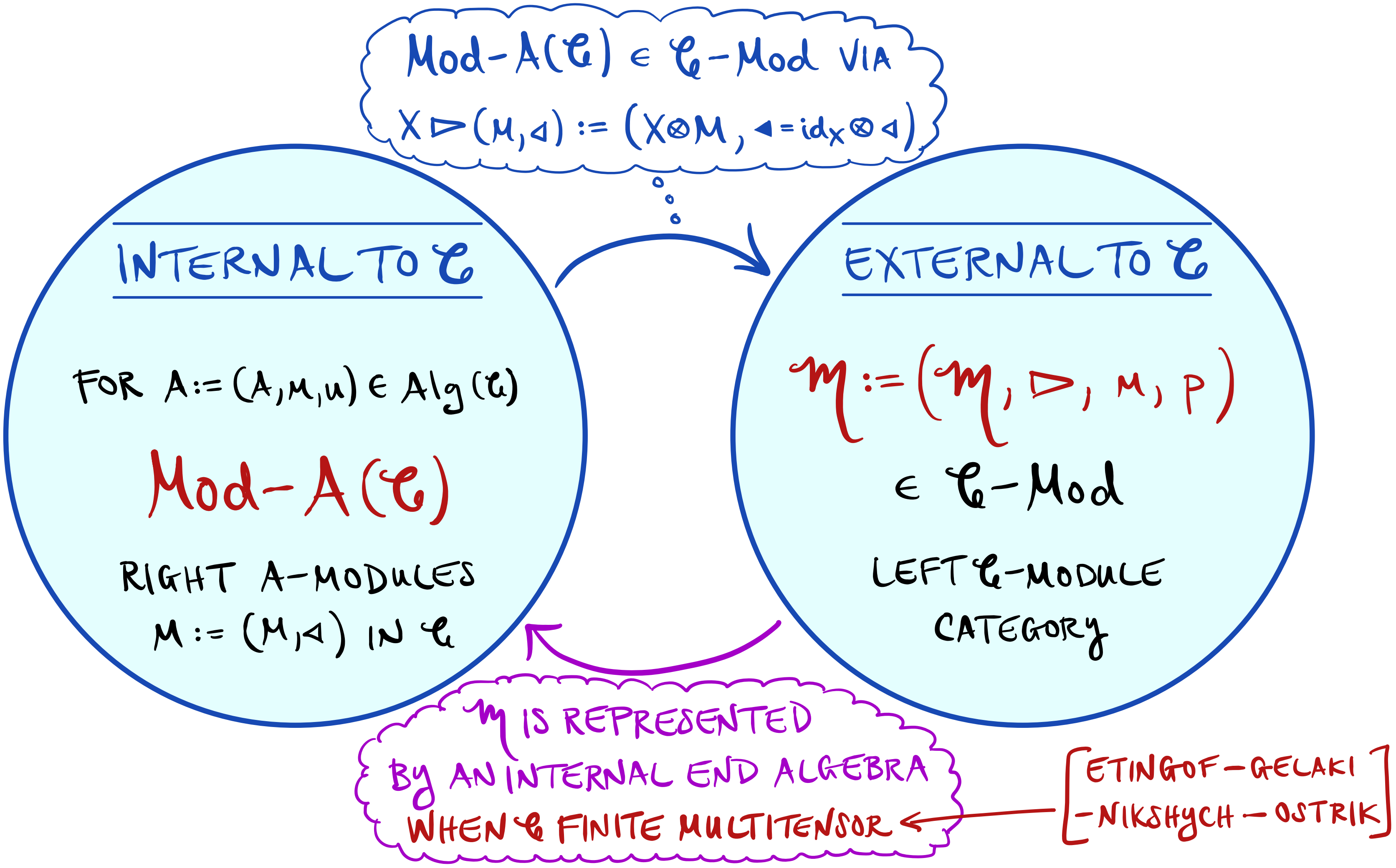
$\in \mathcal{C}$-Mod

LEFT $\mathcal{C}$-MODULE
CATEGORY

SS $\mathcal{M}$ IS REPRESENTED
BY AN INTERNAL END ALGEBRA
WHEN $\mathcal{C}$ MULTIFUSION ← [OSTRIK]

II. OSTRIK'S THEOREM

$\mathcal{M} \in \mathcal{C}$-Mod IS REPRESENTED BY $A \in \mathsf{Alg}(\mathcal{C})$ IF $\mathcal{M} \simeq \text{Mod-}A(\mathcal{C})$ AS LEFT $\mathcal{C}$-MODULE CATEGORIES.



$\text{Mod-}A(\mathcal{C}) \in \mathcal{C}\text{-Mod}$ VIA

$X \triangleright (M, \triangleleft) := (X \otimes M, \blacktriangleleft = \mathrm{id}_X \otimes \triangleleft)$

INTERNAL TO $\mathcal{C}$

FOR $A := (A, m, u) \in \mathsf{Alg}(\mathcal{C})$

$\text{Mod-}A(\mathcal{C})$

RIGHT A-MODULES $M := (M, \triangleleft)$ IN $\mathcal{C}$

EXTERNAL TO $\mathcal{C}$

$\mathcal{M} := (\mathcal{M}, \triangleright, m, p)$

$\in \mathcal{C}\text{-Mod}$

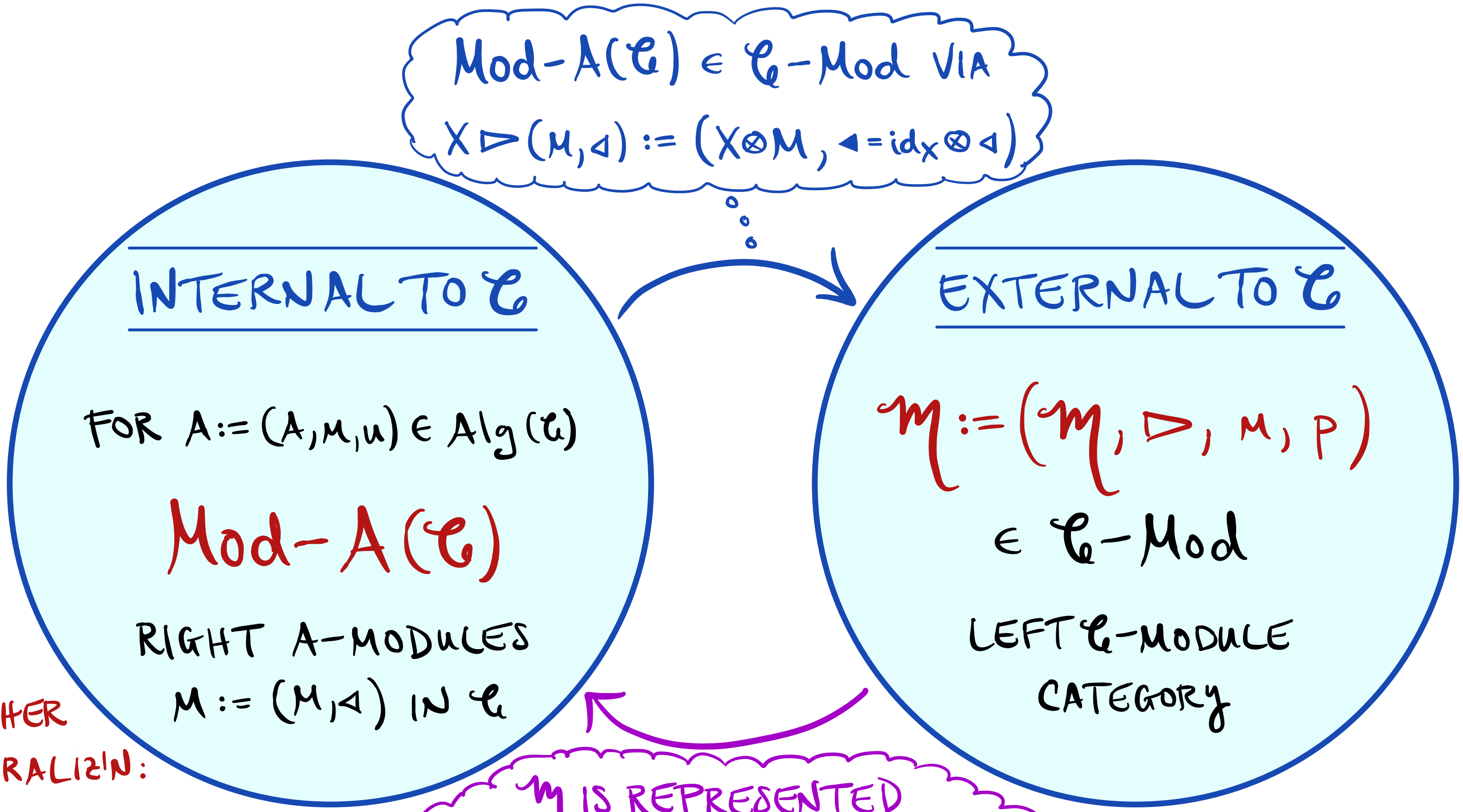
LEFT $\mathcal{C}$-MODULE CATEGORY

$\mathcal{M}$ IS REPRESENTED BY AN INTERNAL END ALGEBRA WHEN $\mathcal{C}$ FINITE MULTITENSOR

[ETINGOF–GELAKI–NIKSHYCH–OSTRIK]

II. OSTRIK'S THEOREM

$\mathcal{M} \in \mathcal{C}$-Mod IS REPRESENTED BY $A \in \mathrm{Alg}(\mathcal{C})$ IF
$\mathcal{M} \simeq$ Mod-$A(\mathcal{C})$ AS LEFT $\mathcal{C}$-MODULE CATEGORIES.



Mod-$A(\mathcal{C}) \in \mathcal{C}$-Mod VIA
$X \triangleright (M, \triangleleft) := (X \otimes M, \blacktriangleleft = \mathrm{id}_X \otimes \triangleleft)$

INTERNAL TO $\mathcal{C}$

FOR $A := (A, m, u) \in \mathrm{Alg}(\mathcal{C})$

Mod-$A(\mathcal{C})$

RIGHT A-MODULES
$M := (M, \triangleleft)$ IN $\mathcal{C}$

EXTERNAL TO $\mathcal{C}$

$\mathcal{M} := (\mathcal{M}, \triangleright, m, p)$
$\in \mathcal{C}$-Mod

LEFT $\mathcal{C}$-MODULE
CATEGORY

$\mathcal{M}$ IS REPRESENTED
BY AN INTERNAL END ALGEBRA
WHEN $\mathcal{C}$ FINITE MULTITENSOR ← [ETINGOF–GELAKI–NIKSHYCH–OSTRIK]

FURTHER
GENERALIZ'N:
[DOUGLAS
–SCHOMMER-PRIES
–SNYDER]

MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #21

NEXT TIME

NICE PROPERTIES
OF ALGEBRAS
IN MONOIDAL CATEGORIES

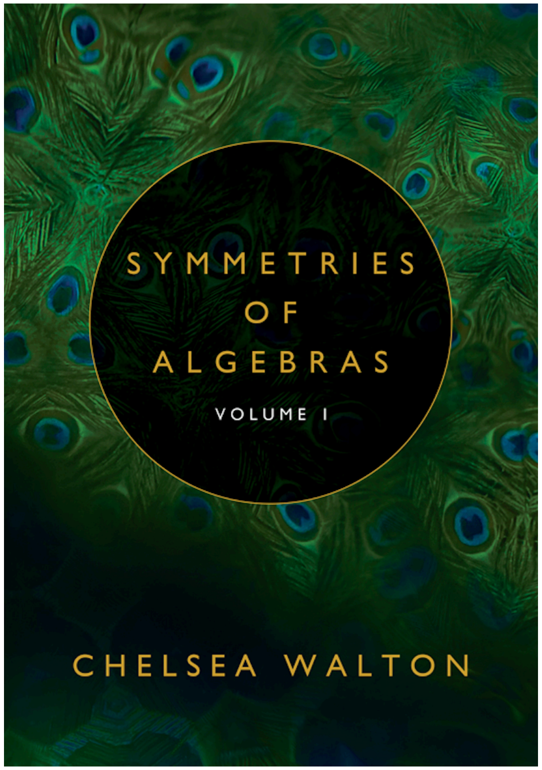
TOPICS:

✓ I. INTERNAL END ALGEBRAS (§4.8.1)

✓ II. OSTRIK'S THEOREM (§4.8.2)

Enjoy this lecture?
You'll enjoy the textbook!

C. Walton's "Symmetries of Algebras, Volume 1" (2024)



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Lecture #21 keywords: internal End algebra, internal Hom module, module category represented by an algebra, Ostrik's Theorem