MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #22

LAST TIME

- INTERNAL END ALGEBRAS
- OSTRIK'S THEOREM

END OF VOLUME I

TOPICS:

I. NOTIONS OF SAMENESS (RECAP)

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$ (§4.9)

III. BIMODULES AND BEYOND (§4.10.1)

IV. CATEGORICAL MORITA EQUIVALENCE (§4.10.2)

I. NOTIONS OF SAMENESS

FOR $\Bbbk$-ALGEBRAS

$A = B$

EQUALITY OF ALGEBRAS

WEAKEN

$A \cong B$

ISOMORPHISM OF ALGEBRAS

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$$A \underset{\text{MORITA}}{\sim} B$$

"EQUIVALENCE"
OF ALGEBRAS

$\Vert$ DEF

$$A\text{-Mod}_{\Bbbk} \simeq B\text{-Mod}_{\Bbbk}$$

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FOR CATEGORIES

$$\mathcal{C} = \mathcal{D}$$

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EQUIVALENCE OF CATEGORIES

WEAKEN

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left] \end{tikzcd}$$

ADJUNCTION

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ADJUNCTION

FOR MONOIDAL CATEGORIES

$\mathcal{C} \overset{\otimes}{=} \mathcal{D}$

EQUALITY OF $\otimes$ CATEGORIES

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HAVE MODULES IN/OVER $\mathcal{C}$

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CAN GET A NOTION OF MORITA EQUIVALENCE

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MORITA'S THM $\Updownarrow$

$\exists$ BIMODULES$/\Bbbk$: ${}_AP_B$, ${}_BQ_A$

$$\text{s.t.} \begin{cases} P \otimes_B Q \cong A_{\text{reg}} \text{ IN } A\text{-Bimod}_{\Bbbk} \\ Q \otimes_A P \cong B_{\text{reg}} \text{ IN } B\text{-Bimod}_{\Bbbk} \end{cases}$$

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$\exists$ FIN. GEN. PROJ. $M \in A\text{-Mod}_{\Bbbk}$ WITH $\mathrm{Hom}_{A\text{-Mod}_{\Bbbk}}(M, -)$ FAITHFUL

s.t. $B^{op} \cong \mathrm{End}_{A\text{-Mod}}(M)$ AS $\Bbbk$-ALGEBRAS

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FOR MONOIDAL CATEGORIES

$\mathcal{C}\text{-Mod}$

OBJECTS : LEFT $\mathcal{C}$-MODULE CATEGS.

MORPHISMS : $\mathcal{C}$-MODULE FUNCTORS

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FOR MONOIDAL CATEGORIES

$\mathcal{C}$-Mod

OBJECTS : LEFT $\mathcal{C}$-MODULE CATEGS.

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THIS EQUIVALENCE IS BEST HANDLED IN A "2-CATEGORICAL" SETTING

... INCORPORATES MORPHISMS OF MORPHISMS

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FOR MONOIDAL CATEGORIES

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... CAN MIMIC VIA BIMODULE CATEGORIES
$\exists$ ${}_{\mathcal{C}}\mathcal{P}_{\mathcal{D}}$, ${}_{\mathcal{D}}\mathcal{Q}_{\mathcal{C}}$
+ CONDITIONS

... CAN MIMIC VIA CERTAIN ENDOFUNCTOR CATEGORIES
$\text{End}_{\mathcal{C}\text{-Mod}}(\mathcal{M})$

FOR MONOIDAL CATEGORIES

$\mathcal{C}$-Mod

OBJECTS : LEFT $\mathcal{C}$-MODULE CATEGS.

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... CAN MIMIC VIA CERTAIN ENDOFUNCTOR CATEGORIES $\text{End}_{\mathcal{C}\text{-Mod}}(\mathcal{M})$

VIA ALGEBRAS IN $\otimes$ CATS.

CAN USE OSTRIK'S THEOREM

ON MODULE CATEGORIES OVER FUSION CATEGS.

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VIA ALGEBRAS IN $\otimes$ CATS.

CAN USE OSTRIK'S THEOREM
(OR A GENERALIZATION)
ON MODULE CATEGORIES OVER FUSION CATEGS.
(OVER FINITE TENSOR CATEGS.)

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VIA ALGEBRAS IN $\otimes$ CATS.

CAN USE OSTRIK'S THEOREM (OR A GENERALIZATION) ON MODULE CATEGORIES OVER FUSION CATEGS. (OVER FINITE TENSOR CATEGS.)

$\vdots$

$\mathcal{M} \simeq \text{Mod-}A(\mathcal{C})$

FOR SOME $A \in \mathrm{Alg}(\mathcal{C})$

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+ CONDITIONS

... SHOOT FOR TENSOR EQUIVALENCE

$\mathcal{D}^{\otimes op} \simeq \mathrm{Rex}_{\mathcal{C}\text{-Mod}}(\mathcal{M}, \mathcal{M})$

VIA ALGEBRAS IN $\otimes$ CATS.

CAN USE OSTRIK'S THEOREM (OR A GENERALIZATION)

ON MODULE CATEGORIES OVER FUSION CATEGS. (OVER FINITE TENSOR CATEGS.)

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$\mathcal{D}^{\otimes op} \simeq \mathrm{Rex}_{\mathcal{C}\text{-Mod}}(\mathcal{M}, \mathcal{M}) \simeq (A\text{-Bimod}(\mathcal{C}))^{\otimes op}$

VIA ALGEBRAS IN $\otimes$ CATS.

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FOR SOME $A \in \mathrm{Alg}(\mathcal{C})$

[GENERALIZED EILENBERG–WATTS THEOREM]

I. NOTIONS OF SAMENESS

WANT

$$\boxed{\begin{gathered}\mathcal{C} \underset{\text{MORITA}}{\sim} \mathcal{D} \\ \text{|||}\ \text{DEF} \\ \mathcal{C}\text{-Mod} \simeq \mathcal{D}\text{-Mod}\end{gathered}}$$

VIA ALGEBRAS IN $\otimes$ CATS.

CAN USE

OSTRIK'S THEOREM

(OR A GENERALIZATION)

ON MODULE CATEGORIES OVER FUSION CATEGS.

(OVER FINITE TENSOR CATEGS.)

⋮

$$\mathcal{M} \simeq \text{Mod-}A(\mathcal{C})$$

FOR SOME $A \in \mathrm{Alg}(\mathcal{C})$

SHOOT FOR TENSOR EQUIVALENCE

GENERALIZED EILENBERG–WATTS THEOREM

$$\mathcal{D}^{\otimes\mathrm{op}} \simeq \mathrm{Rex}_{\mathcal{C}\text{-Mod}}(\mathcal{M},\mathcal{M}) \simeq \left(A\text{-Bimod}(\mathcal{C})\right)^{\otimes\mathrm{op}}$$

I. NOTIONS OF SAMENESS

FOR THIS STUDY OF SAMENESS, $\mathcal{C}$ AND $\mathcal{D}$ SHOULD LIE IN THE SAME SETTING

E.G., BOTH FUSION/ FINITE TENSOR

WANT

$$\mathcal{C} \underset{\text{MORITA}}{\sim} \mathcal{D}$$

$$\overset{\text{DEF}}{\Updownarrow}$$

$$\mathcal{C}\text{-Mod} \simeq \mathcal{D}\text{-Mod}$$

VIA ALGEBRAS IN $\otimes$ CATS.

CAN USE OSTRIK'S THEOREM (OR A GENERALIZATION) ON MODULE CATEGORIES OVER FUSION CATEGS. (OVER FINITE TENSOR CATEGS.)

$\vdots$

$$\mathcal{M} \simeq \text{Mod-}A(\mathcal{C})$$

FOR SOME $A \in \text{Alg}(\mathcal{C})$

GENERALIZED EILENBERG–WATTS THEOREM

SHOOT FOR TENSOR EQUIVALENCE

$$\mathcal{D}^{\otimes\text{op}} \simeq \text{Rex}_{\mathcal{C}\text{-Mod}}(\mathcal{M},\mathcal{M}) \simeq \left(A\text{-Bimod}(\mathcal{C})\right)^{\otimes\text{op}}$$

I. NOTIONS OF SAMENESS

FOR THIS STUDY OF SAMENESS, $\mathcal{C}$ AND $\mathcal{D}$ SHOULD LIE IN THE SAME SETTING

E.G., BOTH FUSION/ FINITE TENSOR

NEED TO STUDY WHEN A-BIMOD($\mathcal{C}$) IS FUSION/FINITE TENSOR

WANT

$$\boxed{\begin{array}{c}\mathcal{C} \underset{\text{MORITA}}{\sim} \mathcal{D} \\ \text{|||}\ \text{DEF} \\ \mathcal{C}\text{-Mod} \simeq \mathcal{D}\text{-Mod}\end{array}}$$

VIA ALGEBRAS IN $\otimes$ CATS.

CAN USE OSTRIK'S THEOREM (OR A GENERALIZATION) ON MODULE CATEGORIES OVER FUSION CATEGS. (OVER FINITE TENSOR CATEGS.)

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$\mathcal{M} \simeq \text{Mod-}A(\mathcal{C})$

FOR SOME $A \in \text{Alg}(\mathcal{C})$

GENERALIZED EILENBERG–WATTS THEOREM

SHOOT FOR TENSOR EQUIVALENCE

$\mathcal{D}^{\otimes\text{op}} \simeq \text{Rex}_{\mathcal{C}\text{-Mod}}(\mathcal{M},\mathcal{M}) \simeq \left(A\text{-Bimod}(\mathcal{C})\right)^{\otimes\text{op}}$

I. NOTIONS OF SAMENESS

FOR THIS STUDY OF SAMENESS, $\mathcal{C}$ AND $\mathcal{D}$ SHOULD LIE IN THE SAME SETTING

E.G., BOTH FUSION/ FINITE TENSOR

NEED TO STUDY WHEN A-$\mathrm{Bimod}(\mathcal{C})$ IS FUSION / FINITE TENSOR

WANT

$$\mathcal{C} \underset{\text{MORITA}}{\sim} \mathcal{D}$$

$$\overset{\text{DEF}}{\iff}$$

$$\mathcal{C}\text{-Mod} \simeq \mathcal{D}\text{-Mod}$$

PROPERTIES OF $A \in \mathrm{Alg}(\mathcal{C})$ ARE NEEDED

SHOOT FOR TENSOR EQUIVALENCE

$$\mathcal{D}^{\otimes op} \simeq \mathrm{Rex}_{\mathcal{C}\text{-Mod}}(\mathcal{M}, \mathcal{M}) \cong (A\text{-}\mathrm{Bimod}(\mathcal{C}))^{\otimes op}$$

VIA ALGEBRAS IN $\otimes$ CATS.

CAN USE OSTRIK'S THEOREM (OR A GENERALIZATION) ON MODULE CATEGORIES OVER FUSION CATEGS. (OVER FINITE TENSOR CATEGS.)

⋮

$$\mathcal{M} \simeq \mathrm{Mod}\text{-}A(\mathcal{C})$$

FOR SOME $A \in \mathrm{Alg}(\mathcal{C})$

[GENERALIZED EILENBERG–WATTS THEOREM]

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

CONNECTEDNESS

INDECOMPOSABILITY

SIMPLICITY

EXACTNESS

SEMISIMPLICITY

SEPARABILITY

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

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CONNECTEDNESS

DEF. A IS CONNECTED IF

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1.$$

EX.

PROP.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

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CONNECTEDNESS

DEF. A IS CONNECTED IF

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1.$$

EX. $\mathcal{C} = \mathrm{Vec} \rightsquigarrow$ A IS A $\Bbbk$-ALGEBRA.

A CONNECTED

$\Leftrightarrow \dim_{\Bbbk} A = \dim_{\Bbbk} \mathrm{Hom}_{\Bbbk}(\Bbbk, A) = 1$

$\Leftrightarrow A \cong \Bbbk.$

PROP.

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(STRONG CONDITION)

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$\Leftrightarrow A \cong \Bbbk.$

PROP. (a) $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$ IS A $\Bbbk$-ALGEBRA WITH:

$$m(f \otimes f') : \mathbb{1} \xrightarrow{l_{\mathbb{1}}^{-1}} \mathbb{1} \otimes \mathbb{1} \xrightarrow{f \otimes f'} A \otimes A \xrightarrow{m_A} A \quad \& \quad u(f) := u_A : \mathbb{1} \longrightarrow A$$

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

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(b) $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) \cong \mathrm{End}_{A\text{-Mod}(\mathcal{C})}({}_A A_{\mathrm{reg}})$ AS $\Bbbk$-ALGS

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

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$A := (A,\ m_A : A\otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathsf{Alg}(\mathcal{C})$

CONNECTEDNESS

(STRONG CONDITION)

DEF. A IS CONNECTED IF

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1.$$

EX. $\mathcal{C} = \mathsf{Vec} \rightsquigarrow A$ IS A $\Bbbk$-ALGEBRA.

A CONNECTED

$\iff \dim_{\Bbbk} A = \dim_{\Bbbk} \mathrm{Hom}_{\Bbbk}(\Bbbk, A) = 1$

$\iff A \cong \Bbbk.$

PROP. (a) $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$ IS A $\Bbbk$-ALGEBRA WITH:

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(b) $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) \cong \mathrm{End}_{A\text{-}\mathsf{Mod}(\mathcal{C})}({}_A A_{reg})$ AS $\Bbbk$-ALGS

(c) A CONNECTED $\iff$ THESE ALGS ARE 1-DIMENSIONAL.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

CONNECTEDNESS

$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$

INDECOMPOSABILITY

SIMPLICITY

EXACTNESS

SEMISIMPLICITY

SEPARABILITY

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

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INDECOMPOSABILITY

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY
$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

INDECOMPOSABILITY

DEF. A IS **INDECOMPOSABLE** IF IT IS NOT $\cong A_1 \square A_2$ AS ALGS FOR $A_1^{\neq 0}, A_2^{\neq 0} \in \mathrm{Alg}(\mathcal{C})$.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

CONNECTEDNESS

$\dim_{\mathbb{k}} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$

$\Longrightarrow$

INDECOMPOSABILITY

DEF. A IS INDECOMPOSABLE IF IT IS NOT $\cong A_1 \square A_2$ AS ALGS FOR $A_1^{\neq 0}, A_2^{\neq 0} \in \mathrm{Alg}(\mathcal{C})$.

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CONNECTEDNESS

$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$

$\Longrightarrow$ **INDECOMPOSABILITY**

A IS DECOMPOSABLE

$\Rightarrow\ A \cong \overset{\neq 0}{A_1} \,\square\, \overset{\neq 0}{A_2}$ AS ALGS

$$\Rightarrow \quad \begin{tikzcd} & A_1 \arrow[dr, "\alpha_1"] & \\ \mathbb{1} \arrow[ur, "u_{A_1}"] \arrow[dr, "u_{A_2}"'] & & A_1 \square A_2 \\ & A_2 \arrow[ur, "\alpha_2"'] & \end{tikzcd}$$

ARE LINEARLY INDEP. ELTS
OF $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$

$\Rightarrow$ A IS NOT CONNECTED

DEF. A IS **INDECOMPOSABLE** IF IT IS NOT $\cong A_1 \square A_2$ AS ALGS FOR $A_1^{\neq 0}, A_2^{\neq 0} \in \mathrm{Alg}(\mathcal{C})$.

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CONNECTEDNESS

$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$

$\Longrightarrow$ **INDECOMPOSABILITY**

A IS DECOMPOSABLE

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EX. $\mathcal{C} = \mathsf{Vec}$:

$\mathrm{Mat}_n(\Bbbk)$ IS INDECOMP.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY
$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

CONNECTEDNESS $\Longrightarrow$ **INDECOMPOSABILITY** ($\not\Longleftarrow$)

$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$

A IS DECOMPOSABLE

$\Rightarrow\ A \cong \overset{\neq 0}{A_1} \,\square\, \overset{\neq 0}{A_2}$ AS ALGS

$$\Rightarrow \quad \begin{tikzcd} & A_1 \arrow[dr, "\alpha_1"] & \\ \mathbb{1} \arrow[ur, "u_{A_1}"] \arrow[dr, "u_{A_2}"'] & & A_1 \,\square\, A_2 \\ & A_2 \arrow[ur, "\alpha_2"'] & \end{tikzcd}$$

ARE LINEARLY INDEP. ELTS OF $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$

$\Rightarrow$ A IS NOT CONNECTED

DEF. A IS **INDECOMPOSABLE** IF IT IS NOT $\cong A_1 \,\square\, A_2$ AS ALGS FOR $A_1^{\neq 0}, A_2^{\neq 0} \in \mathrm{Alg}(\mathcal{C})$.

EX. $\mathcal{C} = \mathrm{Vec}$:

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II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY
$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

CONNECTEDNESS

$$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$$

$\Longrightarrow$ (and $\not\Longleftarrow$)

INDECOMPOSABILITY

A IS DECOMPOSABLE

$\Rightarrow$ $A \cong A_1^{\neq 0} \,\square\, A_2^{\neq 0}$ AS ALGS

$\Rightarrow$

$$\begin{tikzcd}
 & A_1 \arrow[dr, "\alpha_1"] & \\
\mathbb{1} \arrow[ur, "u_{A_1}"] \arrow[dr, "u_{A_2}"'] & & A_1 \square A_2 \\
 & A_2 \arrow[ur, "\alpha_2"'] &
\end{tikzcd}$$

ARE LINEARLY INDEP. ELTS OF $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$

$\Rightarrow$ A IS NOT CONNECTED

DEF. A IS **INDECOMPOSABLE** IF IT IS NOT $\cong A_1 \square A_2$ AS ALGS FOR $A_1^{\neq 0}, A_2^{\neq 0} \in \mathrm{Alg}(\mathcal{C})$.

EX. $\mathcal{C} = \mathrm{Vec}$:

$\mathrm{Mat}_n(\Bbbk)$ IS INDECOMP.

$\mathcal{C}$ IN GENERAL:

$\mathbb{1}$ AND $X \otimes X^* \in \mathrm{Alg}(\mathcal{C})$ ARE INDECOMP.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathsf{Alg}(\mathcal{C})$

CONNECTEDNESS $\;\overset{\not\Leftarrow}{\Longrightarrow}\;$ INDECOMPOSABILITY

$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$

DEF. A IS INDECOMPOSABLE IF IT IS NOT $\cong A_1 \,\square\, A_2$ AS ALGS FOR $A_1^{\neq 0}, A_2^{\neq 0} \in \mathsf{Alg}(\mathcal{C})$.

EX. $\mathcal{C} = \mathsf{Vec}$:

$\mathrm{Mat}_n(\Bbbk)$ IS INDECOMP.

$\mathcal{C}$ IN GENERAL:

$\mathbb{1}$ AND $X \otimes X^* \in \mathsf{Alg}(\mathcal{C})$ ARE INDECOMP.

PROP.

A-$\mathsf{Mod}(\mathcal{C}) \in \mathsf{Mod}$-$\mathcal{C}$ IS INDECOMP. $\Longrightarrow$ $A \in \mathsf{Alg}(\mathcal{C})$ IS INDECOMP.

Mod-$A(\mathcal{C}) \in \mathcal{C}$-$\mathsf{Mod}$ IS INDECOMP. $\Longrightarrow$ $A \in \mathsf{Alg}(\mathcal{C})$ IS INDECOMP.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

CONNECTEDNESS $\overset{\not\Leftarrow}{\Longrightarrow}$ **INDECOMPOSABILITY**

$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$

HAVE:

$(A_1 \square A_2)\text{-Mod}(\mathcal{C})$

$\simeq A_1\text{-Mod}(\mathcal{C}) \times A_2\text{-Mod}(\mathcal{C})$

<u>DEF.</u> A IS INDECOMPOSABLE IF IT IS NOT $\cong A_1 \square A_2$ AS ALGS FOR $A_1^{\neq 0}, A_2^{\neq 0} \in \mathrm{Alg}(\mathcal{C})$.

<u>EX.</u> $\mathcal{C} = \mathrm{Vec}$:

$\mathrm{Mat}_n(\Bbbk)$ IS INDECOMP.

$\mathcal{C}$ IN GENERAL:

$\mathbb{1}$ AND $X \otimes X^* \in \mathrm{Alg}(\mathcal{C})$ ARE INDECOMP.

<u>PROP.</u>

$A\text{-Mod}(\mathcal{C}) \in \mathrm{Mod}\text{-}\mathcal{C}$ IS INDECOMP. $\Longrightarrow$ $A \in \mathrm{Alg}(\mathcal{C})$ IS INDECOMP.

$\mathrm{Mod}\text{-}A(\mathcal{C}) \in \mathcal{C}\text{-}\mathrm{Mod}$ IS INDECOMP. $\Longrightarrow$ $A \in \mathrm{Alg}(\mathcal{C})$ IS INDECOMP.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

CONNECTEDNESS

$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$

$\Longrightarrow$

INDECOMPOSABILITY

$\ncong A_1 \square A_2$

AS ALGS IN $\mathcal{C}$

FOR $A_1 \neq 0,\ A_2 \neq 0 \in \mathrm{Alg}(\mathcal{C})$.

SIMPLICITY

EXACTNESS

SEMISIMPLICITY

SEPARABILITY

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

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$\boxed{\text{SIMPLICITY}}$

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

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$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

SIMPLICITY

DEF. A IS SIMPLE IF THE ONLY IDEALS OF A ARE THE ZERO IDEAL AND A ITSELF

EX.

PROP.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

SIMPLICITY

DEF. A IS SIMPLE IF THE ONLY IDEALS OF A ARE THE ZERO IDEAL AND A ITSELF

EX.

$\mathbb{1} \in \mathrm{Alg}(\mathcal{C})$ IS SIMPLE

($\mathbb{1}$ IS A SIMPLE OBJ.)

PROP.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY
$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

DEF. A IS SIMPLE IF THE ONLY IDEALS OF A ARE THE ZERO IDEAL AND A ITSELF

INDECOMPOSABILITY $\Longleftarrow$ SIMPLICITY

EX.
$\mathbb{1} \in \mathrm{Alg}(\mathcal{C})$ IS SIMPLE
($\mathbb{1}$ IS A SIMPLE OBJ.)

PROP.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathsf{Alg}(\mathcal{C})$

A IS DECOMPOSABLE $\Rightarrow$ $A \cong A_1^{\neq 0} \,\square\, A_2^{\neq 0}$ AS ALGS.

$\Rightarrow$ $(A_1)_{reg} \,\square\, 0$ IS A PROPER IDEAL OF A $\Rightarrow$ A IS NOT SIMPLE

INDECOMPOSABILITY $\Leftarrow$ **SIMPLICITY**

DEF. A IS SIMPLE IF THE ONLY IDEALS OF A ARE THE ZERO IDEAL AND A ITSELF

EX.

$\mathbb{1} \in \mathsf{Alg}(\mathcal{C})$ IS SIMPLE ($\mathbb{1}$ IS A SIMPLE OBJ.)

PROP.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A: A\otimes A \to A,\ u_A: \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

A IS DECOMPOSABLE $\Rightarrow$ $A \cong \overset{\neq 0}{A_1} \,\square\, \overset{\neq 0}{A_2}$ AS ALGS.

$\Rightarrow$ $(A_1)_{reg} \,\square\, 0$ IS A PROPER IDEAL OF A $\Rightarrow$ A IS NOT SIMPLE

DEF. A IS SIMPLE IF THE ONLY IDEALS OF A ARE THE ZERO IDEAL AND A ITSELF

INDECOMPOSABILITY $\Leftarrow$ SIMPLICITY

INDECOMPOSABILITY $\not\Rightarrow$ SIMPLICITY

(SAW THIS FOR $\mathcal{C} = \mathrm{Vec}$)

EX.

$\mathbb{1} \in \mathrm{Alg}(\mathcal{C})$ IS SIMPLE

($\mathbb{1}$ IS A SIMPLE OBJ.)

PROP.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY
$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

A IS DECOMPOSABLE $\Rightarrow$ $A \cong \overset{\neq 0}{A_1} \,\square\, \overset{\neq 0}{A_2}$ AS ALGS.

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INDECOMPOSABILITY $\Leftarrow$ SIMPLICITY

INDECOMPOSABILITY $\not\Rightarrow$ SIMPLICITY

(SAW THIS FOR $\mathcal{C} = \mathrm{Vec}$)

DEF. A IS SIMPLE IF THE ONLY IDEALS OF A ARE THE ZERO IDEAL AND A ITSELF

EX.
$\mathbb{1} \in \mathrm{Alg}(\mathcal{C})$ IS SIMPLE
($\mathbb{1}$ IS A SIMPLE OBJ.)

PROP. SIMPLICITY IS MORITA INVARIANT:

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$ WITH $A \underset{\text{MORITA}}{\simeq} B$.

THEN A IS SIMPLE AS AN ALG. IN $\mathcal{C}$

$\Leftrightarrow$ B IS SIMPLE AS AN ALG. IN $\mathcal{C}$.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

A IS DECOMPOSABLE $\Rightarrow$ $A \cong A_1^{\neq 0} \square A_2^{\neq 0}$ AS ALGS.

$\Rightarrow$ $(A_1)_{\mathrm{reg}} \square 0$ IS A PROPER IDEAL OF A $\Rightarrow$ A IS NOT SIMPLE

$\boxed{\text{INDECOMPOSABILITY}} \Leftarrow \boxed{\text{SIMPLICITY}}$

$\not\Rightarrow$

(SAW THIS FOR $\mathcal{C} = \mathrm{Vec}$)

DEF. A IS **SIMPLE** IF THE ONLY IDEALS OF A ARE THE ZERO IDEAL AND A ITSELF

EX.
$\mathbb{1} \in \mathrm{Alg}(\mathcal{C})$ IS SIMPLE
($\mathbb{1}$ IS A SIMPLE OBJ.)

PROP. SIMPLICITY IS **MORITA INVARIANT**:

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$ WITH $A \underset{\text{MORITA}}{\simeq} B$.

THEN A IS SIMPLE AS AN ALG. IN $\mathcal{C}$

$\Leftrightarrow$ B IS SIMPLE AS AN ALG. IN $\mathcal{C}$.

$\exists\ {}_A P_B$ AND ${}_B Q_A$ s.t. $P \otimes_B Q \cong A$ AND $Q \otimes_A P \cong B$.
FOR A PROPER IDEAL I OF A, GET
$Q \otimes_A I \otimes_A P$ IS $\cong$ TO A PROPER IDEAL OF B.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

A IS DECOMPOSABLE $\Rightarrow$ $A \cong A_1^{\neq 0} \,\square\, A_2^{\neq 0}$ AS ALGS.

$\Rightarrow$ $(A_1)_{reg} \,\square\, 0$ IS A PROPER IDEAL OF A $\Rightarrow$ A IS NOT SIMPLE

DEF. A IS *SIMPLE* IF THE ONLY IDEALS OF A ARE THE ZERO IDEAL AND A ITSELF

INDECOMPOSABILITY $\Leftarrow$ SIMPLICITY

INDECOMPOSABILITY $\not\Rightarrow$ SIMPLICITY

(SAW THIS FOR $\mathcal{C} = \mathrm{Vec}$)

PROP. SIMPLICITY IS *MORITA INVARIANT*:

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$ WITH $A \underset{\text{MORITA}}{\simeq} B$.

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$\exists\ {}_A P_B$ AND ${}_B Q_A$ $\ni$. $P \otimes_B Q \cong A$ AND $Q \otimes_A P \cong B$.

FOR A PROPER IDEAL I OF A, GET $Q \otimes_A I \otimes_A P$ IS $\cong$ TO A PROPER IDEAL OF B.

EX.

$\mathbb{1} \in \mathrm{Alg}(\mathcal{C})$ IS SIMPLE

($\mathbb{1}$ IS A SIMPLE OBJ.)

$\Downarrow$

$X \otimes X^* \in \mathrm{Alg}(\mathcal{C})$ IS SIMPLE FOR ANY $X \in \mathcal{C}$.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

CONNECTEDNESS

$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$

$\Longrightarrow$

INDECOMPOSABILITY

$\not\cong A_1 \square A_2$

AS ALGS IN $\mathcal{C}$

FOR $A_1 \neq 0,\ A_2 \neq 0 \in \mathrm{Alg}(\mathcal{C})$.

$\Longleftarrow$

SIMPLICITY

ONLY IDEALS ARE 0 AND A

EXACTNESS

SEMISIMPLICITY

SEPARABILITY

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY
$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

$\boxed{\text{SEMISIMPLICITY}}$

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

REM.

EX.

SEMISIMPLICITY

DEF. A IS SEMISIMPLE IF A-$\mathrm{Mod}(\mathcal{C})$ IS A SEMISIMPLE CATEGORY.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY
$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

REM.

SEMISIMPLICITY

DEF. A IS SEMISIMPLE IF $A\text{-}\mathrm{Mod}(\mathcal{C})$ IS A SEMISIMPLE CATEGORY.

EX.

$\mathbb{1} \in \mathrm{Alg}(\mathcal{C})$ IS SEMISIMPLE

$\Updownarrow$

$\mathcal{C}$ IS SEMISIMPLE

$[\mathbb{1}\text{-}\mathrm{Mod}(\mathcal{C}) \simeq \mathcal{C}]$

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

REM. SEMISIMPLICITY IS MORITA INVARIANT:

TAKE $A, B \in \mathrm{Alg}(\mathcal{C})$ WITH $A \underset{\text{MORITA}}{\simeq} B$.

THEN A IS SEMISIMPLE AS AN ALG. IN $\mathcal{C}$

$\Leftrightarrow$ B IS SEMISIMPLE AS AN ALG. IN $\mathcal{C}$.

EX.

$\mathbb{1} \in \mathrm{Alg}(\mathcal{C})$

IS SEMISIMPLE

$\Updownarrow$

$\mathcal{C}$ IS SEMISIMPLE

$[\mathbb{1}\text{-}\mathrm{Mod}(\mathcal{C}) \simeq \mathcal{C}]$

SEMISIMPLICITY

DEF. A IS SEMISIMPLE

IF $A\text{-}\mathrm{Mod}(\mathcal{C})$ IS A

SEMISIMPLE CATEGORY.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

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EX.

$\mathbb{1} \in \mathrm{Alg}(\mathcal{C})$

IS SEMISIMPLE

$\Updownarrow$

$\mathcal{C}$ IS SEMISIMPLE

$[\mathbb{1}\text{-Mod}(\mathcal{C}) \simeq \mathcal{C}]$

$X \otimes X^* \in \mathrm{Alg}(\mathcal{C})$

IS SEMISIMPLE

FOR ANY $X \in \mathcal{C}$.

$\Updownarrow$

$\mathcal{C}$ IS SEMISIMPLE

SEMISIMPLICITY

DEF. A IS SEMISIMPLE

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$\mathbb{1} \in \mathrm{Alg}(\mathcal{C})$

IS SEMISIMPLE

$\Updownarrow$

$\mathcal{C}$ IS SEMISIMPLE

$[\mathbb{1}\text{-Mod}(\mathcal{C}) \simeq \mathcal{C}]$

$X \otimes X^* \in \mathrm{Alg}(\mathcal{C})$

IS SEMISIMPLE

FOR ANY $X \in \mathcal{C}$.

$\Updownarrow$

$\mathcal{C}$ IS SEMISIMPLE

GOOD NOTION OF ARTIN-WEDDERBURN THEOREM

... UNRESOLVED

$\mathcal{C} = \mathrm{FdVec}$

A SEMISIMPLE

$\Longleftrightarrow A \cong \prod_{i=1}^{r} \mathrm{Mat}_{n_i}(\Bbbk)$

SEMISIMPLICITY

DEF. A IS SEMISIMPLE IF $A\text{-Mod}(\mathcal{C})$ IS A SEMISIMPLE CATEGORY.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

CONNECTEDNESS $\Longrightarrow$ **INDECOMPOSABILITY** $\Longleftarrow$ **SIMPLICITY**

CONNECTEDNESS

$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$

INDECOMPOSABILITY

$\ncong A_1 \boxplus A_2$

AS ALGS IN $\mathcal{C}$

FOR $A_1 \neq 0,\ A_2 \neq 0 \in \mathrm{Alg}(\mathcal{C})$.

SIMPLICITY

ONLY IDEALS

ARE 0 AND A

EXACTNESS

SEMISIMPLICITY

$A\text{-}\mathrm{Mod}(\mathcal{C})$ IS A

SEMISIMPLE CATEGORY

SEPARABILITY

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A\otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

PROP

EX.

DEF

A IS SEPARABLE IF

$\exists\ \phi : A \longrightarrow A \otimes A$

IN $A\text{-Bimod}(\mathcal{C})$

s.t. $m\,\phi = \mathrm{id}_A$

SEPARABILITY

EX.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY
$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

PROP

EX.

DEF

A IS SEPARABLE IF

$$\exists\, \phi : A \longrightarrow A \otimes A$$

IN A-Bimod($\mathcal{C}$)

$$\text{s.t.} \quad m\,\phi = \mathrm{id}_A$$

$$\left(\begin{array}{l} \text{HERE, } \triangleright_{A \otimes A} = (m_A \otimes \mathrm{id}_A)\, a^{-1}_{A,A,A} \\ \qquad\quad \triangleleft_{A \otimes A} = (\mathrm{id}_A \otimes m_A)\, a_{A,A,A} \end{array}\right)$$

SEPARABILITY

EX.

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY
$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

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$\ni$. $m\phi = \mathrm{id}_A$

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$\triangleleft_{A \otimes A} = (\mathrm{id}_A \otimes m_A)\, a_{A,A,A}$)

SEPARABILITY

EX.

$(\mathbb{1}, \ell_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}}) \in \mathrm{Alg}(\mathcal{C})$

IS SEPARABLE

WITH $\phi_{\mathbb{1}} := \ell_{\mathbb{1}}^{-1}$

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

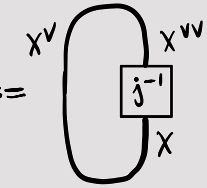
$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

PROP

EX. $\mathcal{C}$ PIVOTAL (STRICT FOR EASE)

TAKE $X \in \mathcal{C}$ $\ni$.

$\dim_j X :=$ 

IS AN ISO ($\Longleftrightarrow$ $\neq 0$)

DEF

A IS SEPARABLE IF

$\exists\ \phi : A \longrightarrow A \otimes A$

IN $A\text{-Bimod}(\mathcal{C})$

$\ni.\quad m\,\phi = \mathrm{id}_A$

(HERE, $\triangleright_{A\otimes A} = (m_A \otimes \mathrm{id}_A)\, a^{-1}_{A,A,A}$

$\triangleleft_{A\otimes A} = (\mathrm{id}_A \otimes m_A)\, a_{A,A,A}$)

SEPARABILITY

EX.

$(\mathbb{1}, l_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}}) \in \mathrm{Alg}(\mathcal{C})$

IS SEPARABLE

WITH $\phi_{\mathbb{1}} := l_{\mathbb{1}}^{-1}$

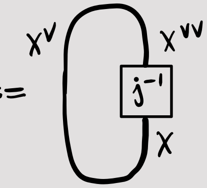
II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY
$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

PROP

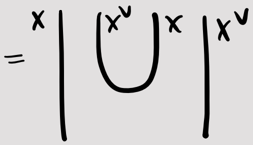
EX. $\mathcal{C}$ PIVOTAL (STRICT FOR EASE)

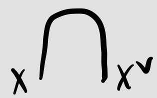
TAKE $X \in \mathcal{C}$ $\ni$.

$\dim_j X :=$ 

IS AN ISO ($\Leftrightarrow \neq 0$)

TAKE $A := X \otimes X^\vee$

WITH $m_A :=$ 

$u_A :=$ 

DEF

A IS SEPARABLE IF

$\exists\ \phi : A \longrightarrow A \otimes A$

IN A-$\mathrm{Bimod}(\mathcal{C})$

$\ni.\ m\phi = \mathrm{id}_A$

(HERE, $\triangleright_{A \otimes A} = (m_A \otimes \mathrm{id}_A)\, a^{-1}_{A,A,A}$
$\triangleleft_{A \otimes A} = (\mathrm{id}_A \otimes m_A)\, a_{A,A,A}$)

SEPARABILITY

EX.

$(\mathbb{1}, \ell_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}}) \in \mathrm{Alg}(\mathcal{C})$

IS SEPARABLE

WITH $\phi_{\mathbb{1}} := \ell_{\mathbb{1}}^{-1}$

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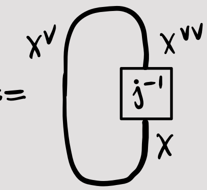
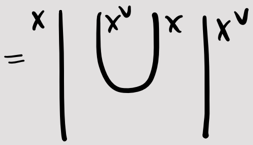
PROP

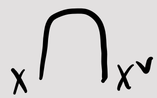
EX. $\mathcal{C}$ PIVOTAL (STRICT FOR EASE)

TAKE $X \in \mathcal{C}$ ∋.

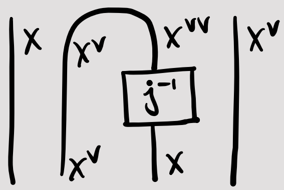
$\dim_j X :=$  IS AN ISO ($\Leftrightarrow \neq 0$)

TAKE $A := X \otimes X^\vee$

WITH $m_A :=$ 

$u_A :=$ 

THEN A IS SEPARABLE WITH

$\phi_A := (\dim_j X)^{-1}$ 

DEF

A IS SEPARABLE IF

$\exists\ \phi : A \longrightarrow A \otimes A$

IN A-$\mathrm{Bimod}(\mathcal{C})$

∋. $m\phi = id_A$

(HERE, $\triangleright_{A\otimes A} = (m_A \otimes id_A)\, a^{-1}_{A,A,A}$

$\triangleleft_{A\otimes A} = (id_A \otimes m_A)\, a_{A,A,A}$)

SEPARABILITY

EX.

$(\mathbb{1}, l_{\mathbb{1}}, id_{\mathbb{1}}) \in \mathrm{Alg}(\mathcal{C})$

IS SEPARABLE

WITH $\phi_{\mathbb{1}} := l_{\mathbb{1}}^{-1}$

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY
$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

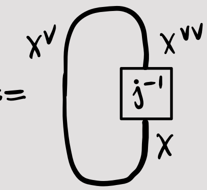
PROP SUPPOSE $\mathcal{C}$ MULTIFUSION. THEN TFAE:

- A SEPARABLE
- $A\text{-Mod}(\mathcal{C})$ IS SEMISIMPLE
- $\text{Mod-}A(\mathcal{C})$ IS SEMISIMPLE
- $A\text{-Bimod}(\mathcal{C})$ IS SEMISIMPLE

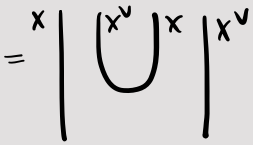
[DAVYDOV–MÜGER–NIKSHYCH–OSTRIK]

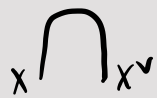
EX. $\mathcal{C}$ PIVOTAL (STRICT FOR EASE)

TAKE $X \in \mathcal{C}$ $\ni$.

$\dim_j X :=$  IS AN ISO ($\Leftrightarrow \neq 0$)

TAKE $A := X \otimes X^\vee$

WITH $m_A :=$ 

$u_A :=$ 

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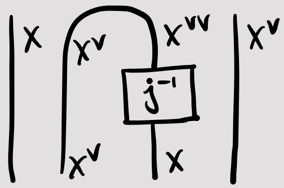
$\phi_A := (\dim_j X)^{-1}$ 

DEF

A IS SEPARABLE IF

$\exists\ \phi : A \longrightarrow A \otimes A$

IN $A\text{-Bimod}(\mathcal{C})$

$\ni$. $m\,\phi = \mathrm{id}_A$

(HERE, $\triangleright_{A \otimes A} = (m_A \otimes \mathrm{id}_A)\, a^{-1}_{A,A,A}$
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SEPARABILITY

EX.

$(\mathbb{1}, l_{\mathbb{1}}, \mathrm{id}_{\mathbb{1}}) \in \mathrm{Alg}(\mathcal{C})$

IS SEPARABLE

WITH $\phi_{\mathbb{1}} := l_{\mathbb{1}}^{-1}$

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

CONNECTEDNESS

$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$

$\Longrightarrow$

INDECOMPOSABILITY

$\ncong A_1 \boxdot A_2$ AS ALGS IN $\mathcal{C}$ FOR $A_1^{\neq 0}, A_2^{\neq 0} \in \mathrm{Alg}(\mathcal{C})$.

$\Longleftarrow$

SIMPLICITY

ONLY IDEALS ARE 0 AND A

EXACTNESS

SEMISIMPLICITY

$A\text{-}\mathrm{Mod}(\mathcal{C})$ IS A SEMISIMPLE CATEGORY

$\xLeftrightarrow[\mathcal{C}\ \text{FUSION}]{}$

SEPARABILITY

$\exists$ RIGHT INVERSE TO $m_A : A \otimes A \to A$ IN $A\text{-}\mathrm{Bimod}(\mathcal{C})$

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

CONNECTEDNESS

$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$

$\Longrightarrow$

INDECOMPOSABILITY

$\ncong A_1 \square A_2$

AS ALGS IN $\mathcal{C}$

FOR $A_1 \neq 0,\ A_2 \neq 0 \in \mathrm{Alg}(\mathcal{C})$.

$\Longleftarrow$

SIMPLICITY

ONLY IDEALS ARE 0 AND A

$\Big\Downarrow$??

EXACTNESS

SEMISIMPLICITY

A-Mod($\mathcal{C}$) IS A SEMISIMPLE CATEGORY

$\mathcal{C}$ FUSION

$\Longleftrightarrow$

SEPARABILITY

$\exists$ RIGHT INVERSE TO $m_A : A \otimes A \to A$ IN A-Bimod($\mathcal{C}$)

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

EXACTNESS

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

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DEF. A IS EXACT IF $A\text{-Mod}(\mathcal{C})$ IS AN EXACT RIGHT $\mathcal{C}$-MOD. CATEG.

THAT IS $\forall P \in \mathcal{C}$ PROJECTIVE, $\forall M \in A\text{-Mod}(\mathcal{C})$,

$$M \triangleleft_A P = M \otimes P \in A\text{-Mod}(\mathcal{C})$$

IS PROJECTIVE.

REM.

EX.

PROP.

EXACTNESS

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

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THAT IS $\forall P \in \mathcal{C}$ PROJECTIVE, $\forall M \in A$-$\mathrm{Mod}(\mathcal{C})$,

$$M \triangleleft_A P = M \otimes P \in A\text{-}\mathrm{Mod}(\mathcal{C})$$

IS PROJECTIVE.

REM.

EX. $\mathbb{1} \in \mathrm{Alg}(\mathcal{C})$ IS EXACT

PROP.

EXACTNESS

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

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$M \triangleleft_A P = M \otimes P \in A\text{-Mod}(\mathcal{C})$

IS PROJECTIVE.

REM. EXACTNESS IS MORITA INVARIANT

EX. $\mathbb{1} \in \mathrm{Alg}(\mathcal{C})$ IS EXACT

$X \otimes X^* \in \mathrm{Alg}(\mathcal{C})$ IS EXACT

PROP.

EXACTNESS

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

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$M \triangleleft_A P = M \otimes P \in A\text{-Mod}(\mathcal{C})$

IS PROJECTIVE.

REM. EXACTNESS IS MORITA INVARIANT

EX. $\mathbb{1} \in \mathrm{Alg}(\mathcal{C})$ IS EXACT

$X \otimes X^* \in \mathrm{Alg}(\mathcal{C})$ IS EXACT

PROP.

EXACTNESS $\Longleftarrow$ SEMISIMPLICITY

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

DEF. A IS EXACT IF $A\text{-Mod}(\mathcal{C})$ IS AN EXACT RIGHT $\mathcal{C}$-MOD. CATEG.

THAT IS $\forall P \in \mathcal{C}$ PROJECTIVE, $\forall M \in A\text{-Mod}(\mathcal{C})$,

$M \triangleleft_A P = M \otimes P \in A\text{-Mod}(\mathcal{C})$

IS PROJECTIVE.

REM. EXACTNESS IS MORITA INVARIANT

EX. $\mathbb{1} \in \mathrm{Alg}(\mathcal{C})$ IS EXACT

$X \otimes X^* \in \mathrm{Alg}(\mathcal{C})$ IS EXACT

PROP.

EXACTNESS $\Longleftarrow$ SEMISIMPLICITY

A SEMISIMPLE

$\Downarrow$

$A\text{-Mod}(\mathcal{C})$ SEMISIMPLE

$\Downarrow$

OBJECTS IN $A\text{-Mod}(\mathcal{C})$ ARE PROJECTIVE

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

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$A := (A,\ m_A: A\otimes A \to A,\ u_A: \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

DEF. A IS EXACT IF A-$\mathrm{Mod}(\mathcal{C})$ IS AN EXACT RIGHT $\mathcal{C}$-MOD. CATEG.

THAT IS $\forall P \in \mathcal{C}$ PROJECTIVE,
$\forall M \in A$-$\mathrm{Mod}(\mathcal{C})$,
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EX. $\mathbb{1} \in \mathrm{Alg}(\mathcal{C})$ IS EXACT

$X \otimes X^* \in \mathrm{Alg}(\mathcal{C})$ IS EXACT

PROP.
SAY $\mathcal{C}$ FINITE TENSOR.
THEN:
A IS EXACT
$\Updownarrow$
ANY RIGHT $\mathcal{C}$-MODULE FUNCTOR
$A\text{-}\mathrm{Mod}(\mathcal{C}) \longrightarrow \mathcal{M}$
IS EXACT

$\boxed{\text{EXACTNESS}} \Longleftarrow \boxed{\text{SEMISIMPLICITY}}$

A SEMISIMPLE
$\Downarrow$
A-$\mathrm{Mod}(\mathcal{C})$ SEMISIMPLE
$\Downarrow$
OBJECTS IN A-$\mathrm{Mod}(\mathcal{C})$ ARE PROJECTIVE

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

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$M \triangleleft_A P = M \otimes P \in A\text{-Mod}(\mathcal{C})$ IS PROJECTIVE.

REM. EXACTNESS IS MORITA INVARIANT

EX. $\mathbb{1} \in \mathrm{Alg}(\mathcal{C})$ IS EXACT

$X \otimes X^* \in \mathrm{Alg}(\mathcal{C})$ IS EXACT

PROP. SAY $\mathcal{C}$ FINITE TENSOR. THEN:

A IS EXACT

$\Updownarrow$

ANY RIGHT $\mathcal{C}$-MODULE FUNCTOR $A\text{-Mod}(\mathcal{C}) \longrightarrow \mathcal{M}$ IS EXACT

EXACTNESS $\Longleftarrow$ SEMISIMPLICITY

A SEMISIMPLE

$\Downarrow$

$A\text{-Mod}(\mathcal{C})$ SEMISIMPLE

$\Downarrow$

OBJECTS IN $A\text{-Mod}(\mathcal{C})$ ARE PROJECTIVE

WOULD LIKE THIS TO BE LEFT-RIGHT SYMMETRIC. INTRINSIC CHARACTERIZATION OF EXACTNESS

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY
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SAY $\mathcal{C}$ FINITE TENSOR.
THEN:
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$A\text{-Mod}(\mathcal{C}) \longrightarrow \mathcal{M}$
IS EXACT

EXACTNESS $\Longleftarrow$ SEMISIMPLICITY

A SEMISIMPLE
$\Downarrow$
$A\text{-Mod}(\mathcal{C})$ SEMISIMPLE
$\Downarrow$
OBJECTS IN $A\text{-Mod}(\mathcal{C})$ ARE PROJECTIVE

WOULD LIKE THIS TO BE LEFT-RIGHT SYMMETRIC.
INTRINSIC CHARACTERIZATION OF EXACTNESS
... UNRESOLVED

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$

$\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1})$ TENSOR CATEGORY

$A := (A,\ m_A : A \otimes A \to A,\ u_A : \mathbb{1} \to A) \in \mathrm{Alg}(\mathcal{C})$

CONNECTEDNESS

$\dim_{\mathbb{k}} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$

$\Longrightarrow$

INDECOMPOSABILITY

$\not\cong A_1 \square A_2$

AS ALGS IN $\mathcal{C}$

FOR $A_1 \neq 0,\ A_2 \neq 0 \in \mathrm{Alg}(\mathcal{C})$.

$\Longleftarrow$

SIMPLICITY

ONLY IDEALS ARE 0 AND A

$\Downarrow$??

EXACTNESS

A-Mod($\mathcal{C}$) IS AN EXACT RIGHT $\mathcal{C}$-MODULE CATEG.

$\Longleftarrow$

SEMISIMPLICITY

A-Mod($\mathcal{C}$) IS A SEMISIMPLE CATEGORY

$\mathcal{C}$ FUSION

$\Longleftrightarrow$

SEPARABILITY

$\exists$ RIGHT INVERSE TO $m_A : A \otimes A \to A$ IN A-Bimod($\mathcal{C}$)

III. BIMODULES AND BEYOND

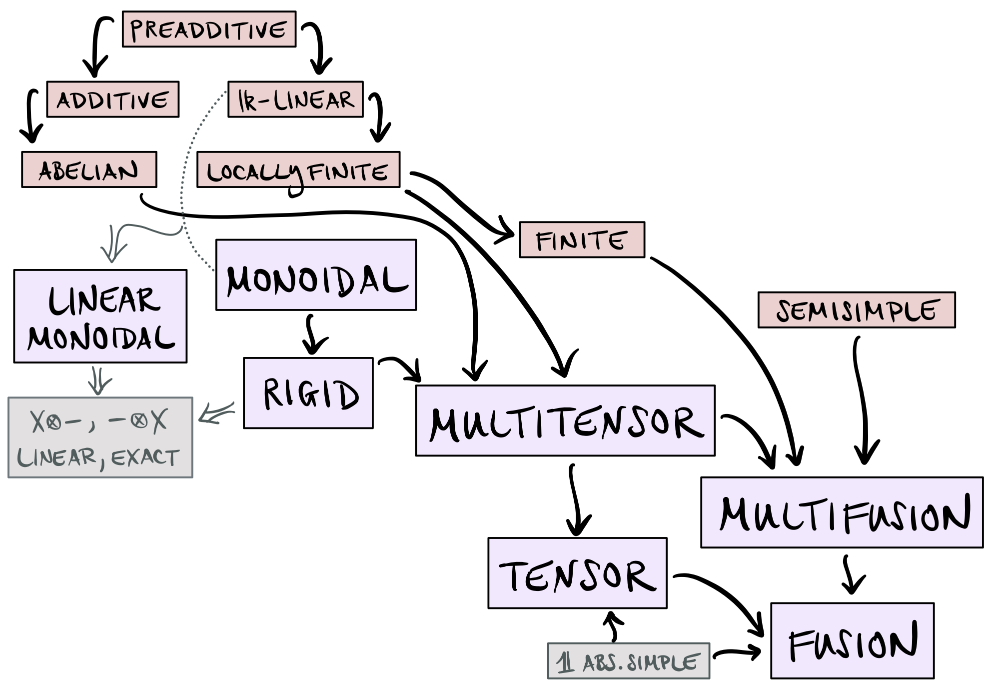
TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ TENSOR & $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

$$\boxed{\begin{array}{c}\text{LET'S STUDY WHEN} \\ \left(A\text{-}\mathrm{Bimod}(\mathcal{C}),\ \otimes_A,\ A_{reg}\right) \\ \text{IS FUSION / FINITE TENSOR.}\end{array}}$$

III. BIMODULES AND BEYOND

TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ TENSOR & $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

LET'S STUDY WHEN
$$\left(A\text{-}\mathrm{Bimod}(\mathcal{C}),\ \otimes_A,\ A_{reg}\right)$$
IS FUSION / FINITE TENSOR.



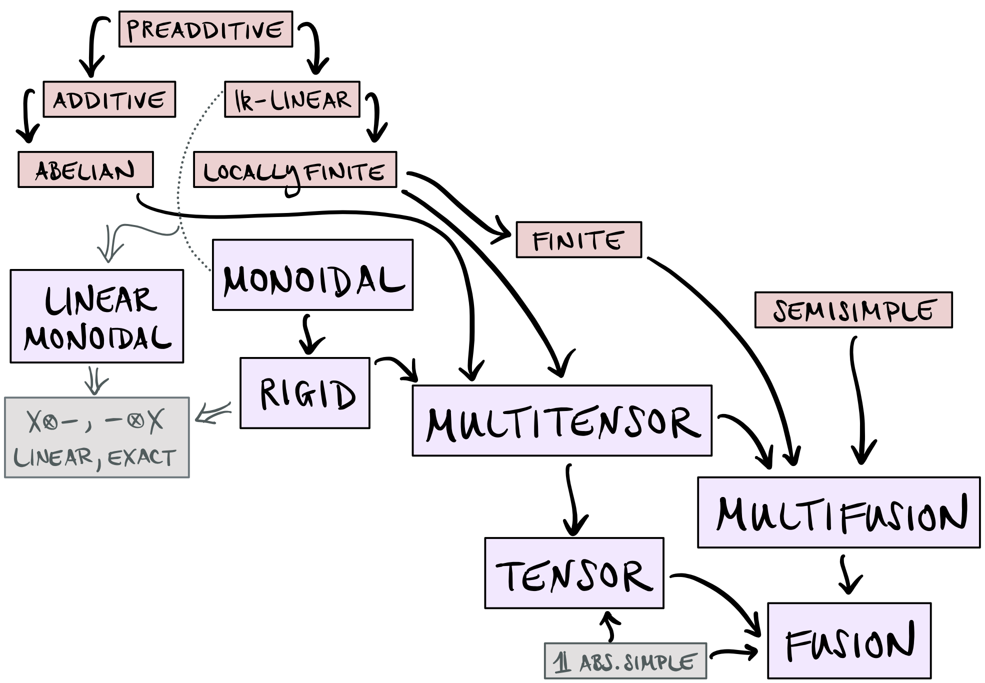
III. BIMODULES AND BEYOND

TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ TENSOR & $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

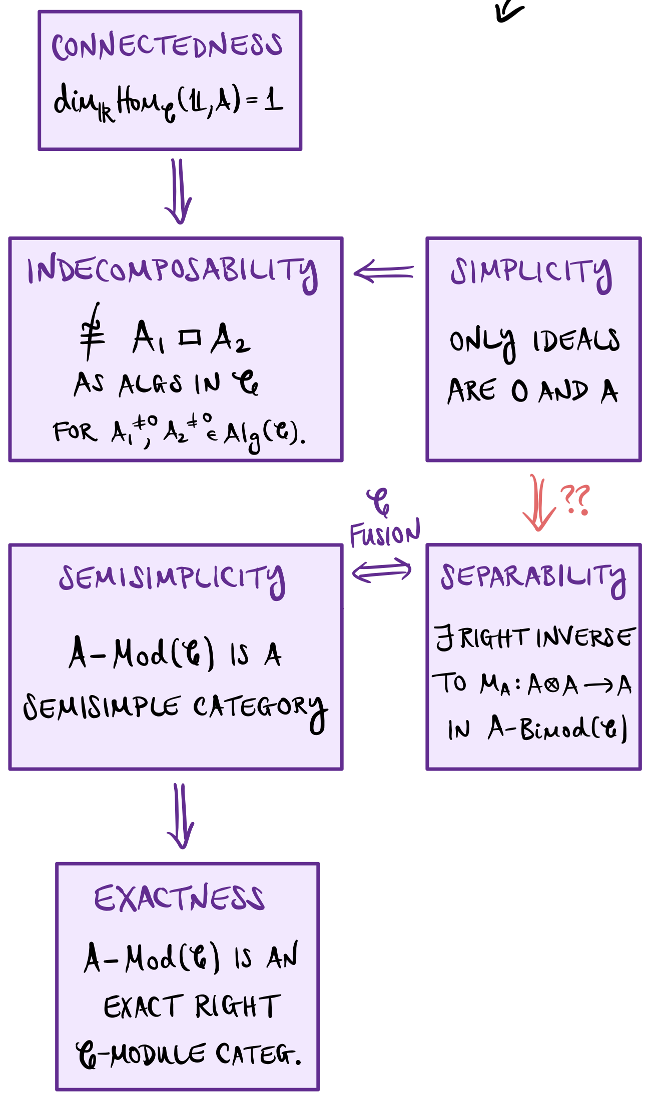
LET'S STUDY WHEN

$$\left(A\text{-}\mathrm{Bimod}(\mathcal{C}),\ \otimes_A,\ A_{\mathrm{reg}}\right)$$

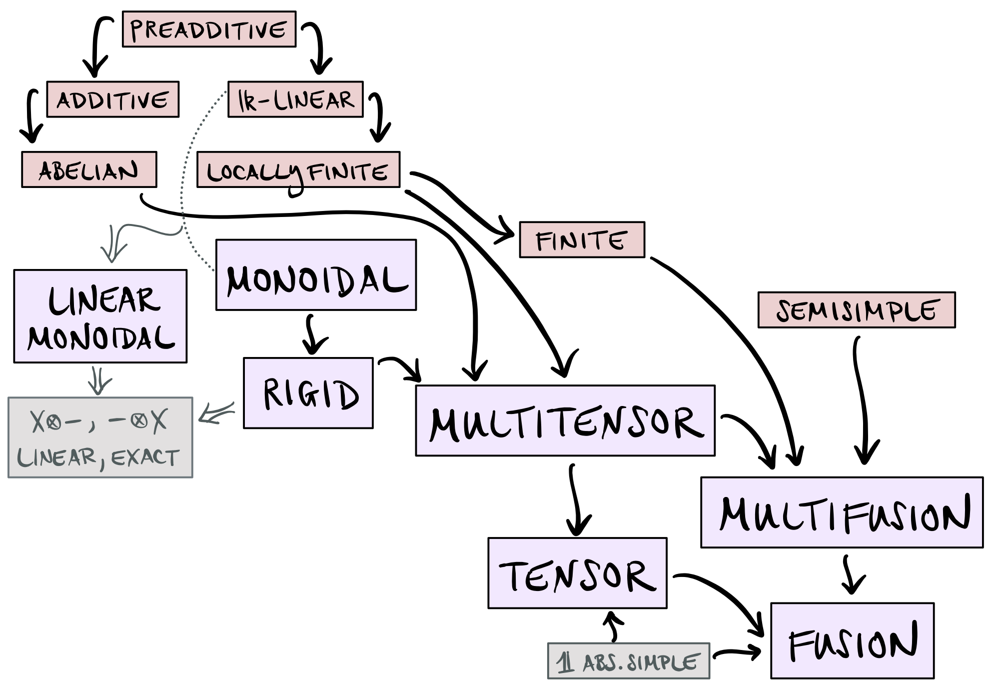
IS FUSION / FINITE TENSOR.



... WILL USE



III. BIMODULES AND BEYOND



THEOREM:

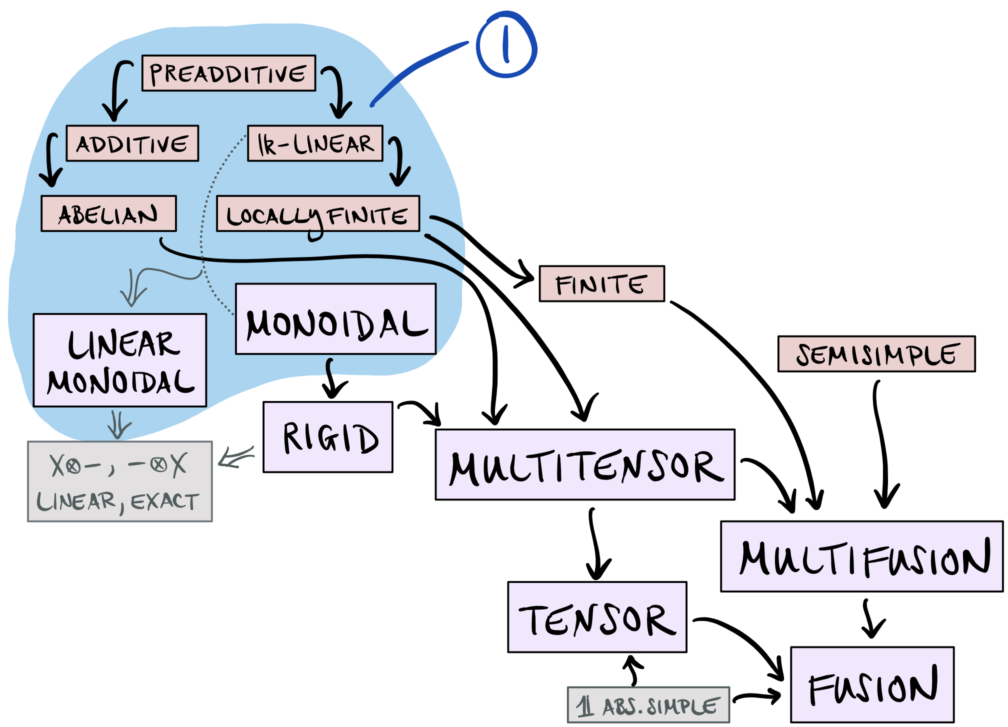
TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ TENSOR

& $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

THEN $(A\text{-Bimod}(\mathcal{C}), \otimes_A, A_{reg})$

SATISFIES:

III. BIMODULES AND BEYOND



THEOREM:

TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ TENSOR

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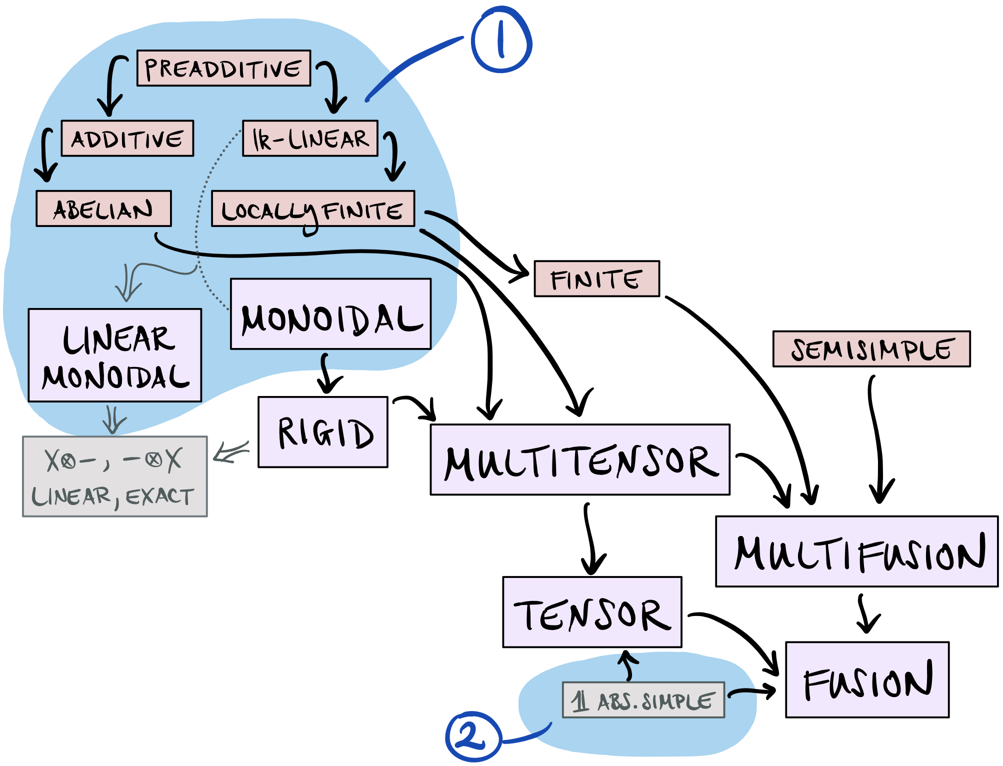
THEN $(A\text{-}\mathrm{Bimod}(\mathcal{C}), \otimes_A, A_{reg})$

SATISFIES:

1 ALWAYS ;

III. BIMODULES AND BEYOND



THEOREM:

TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ TENSOR

& $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

THEN $(A\text{-}\mathrm{Bimod}(\mathcal{C}), \otimes_A, A_{reg})$

SATISFIES:

1. ALWAYS;
2. WHEN A IS CONNECTED;

III. BIMODULES AND BEYOND

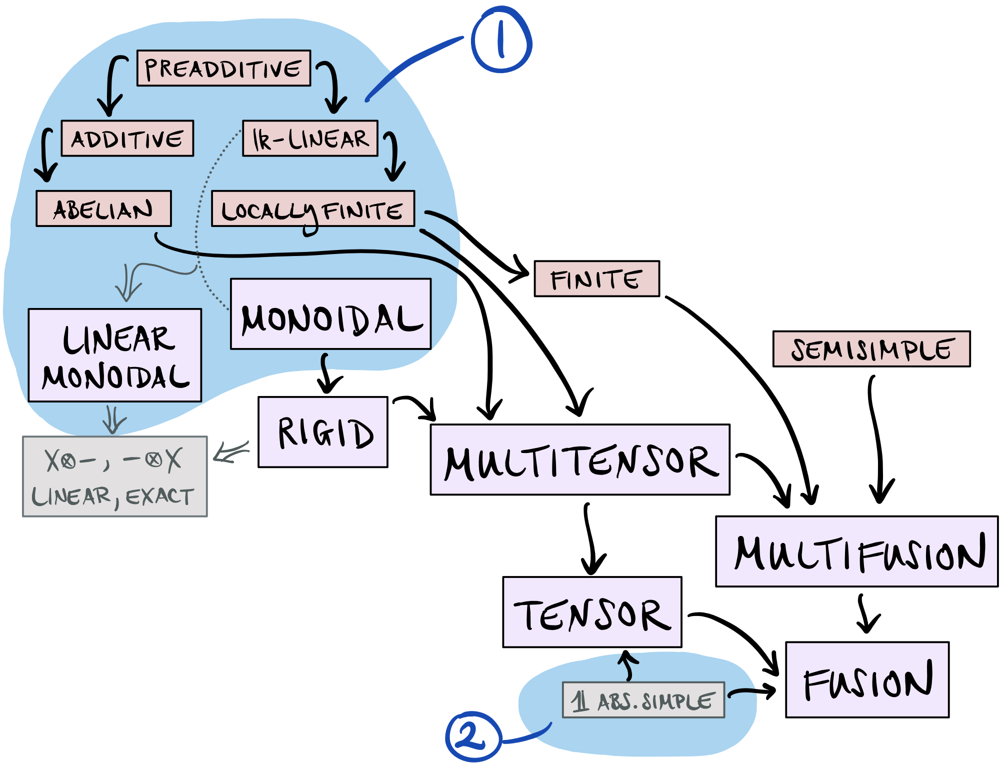
CONNECTEDNESS

$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$

$\mathrm{End}_{A\text{-Bimod}(\mathcal{C})}(A_{reg})$

IS A SUBSPACE OF

$\mathrm{End}_{A\text{-Mod}}(A_{reg}) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$



THEOREM:

TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ TENSOR & $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

THEN $(A\text{-Bimod}(\mathcal{C}), \otimes_A, A_{reg})$ SATISFIES:

1. ALWAYS ;
2. WHEN A IS CONNECTED ;

III. BIMODULES AND BEYOND

CONNECTEDNESS

$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$

$\mathrm{End}_{A\text{-Bimod}(\mathcal{C})}(A_{reg})$

IS A SUBSPACE OF

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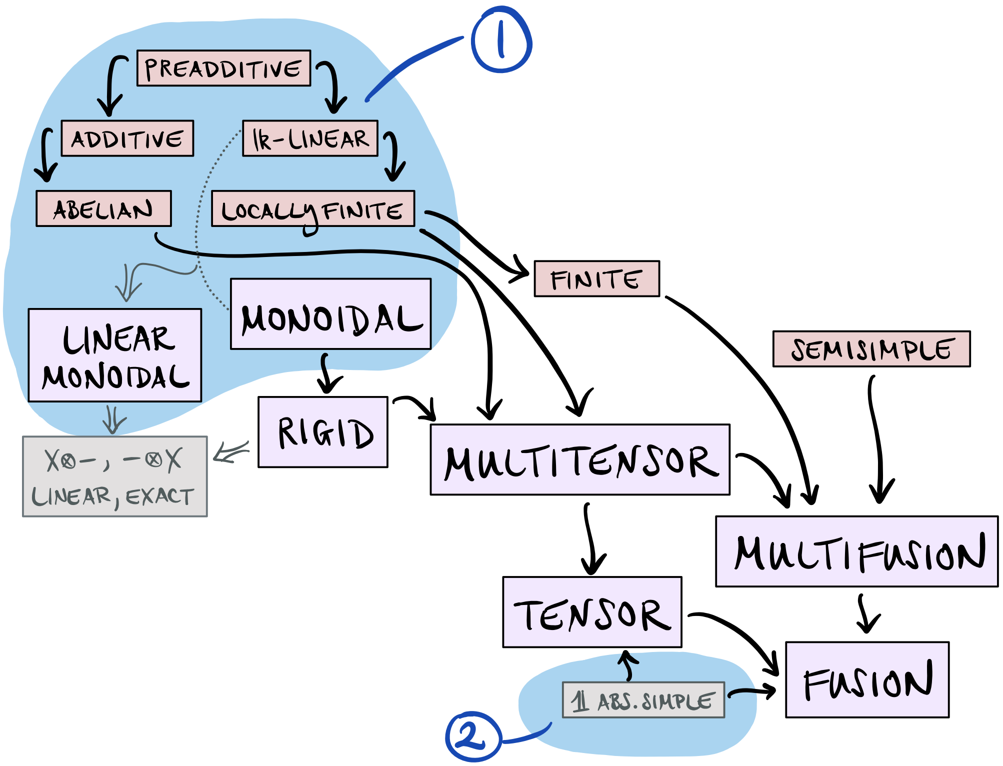
1-DIM'L



THEOREM:

TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ TENSOR

& $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

THEN $(A\text{-Bimod}(\mathcal{C}), \otimes_A, A_{reg})$

SATISFIES:

1. ALWAYS ;
2. WHEN A IS CONNECTED ;

III. BIMODULES AND BEYOND

CONNECTEDNESS

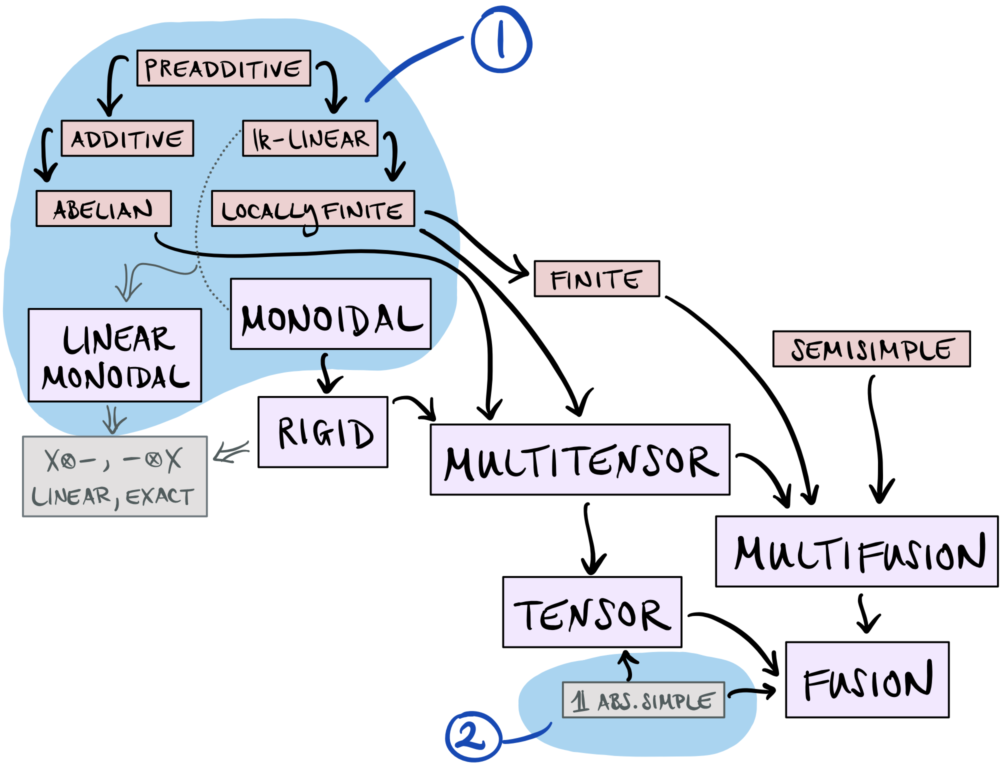
$\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$

$\mathrm{End}_{A\text{-Bimod}(\mathcal{C})}(A_{reg})$ $\leftarrow$ $\therefore$ 1-DIM'L

IS A SUBSPACE OF

$\mathrm{End}_{A\text{-Mod}}(A_{reg}) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$

1-DIM'L



THEOREM:

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& $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

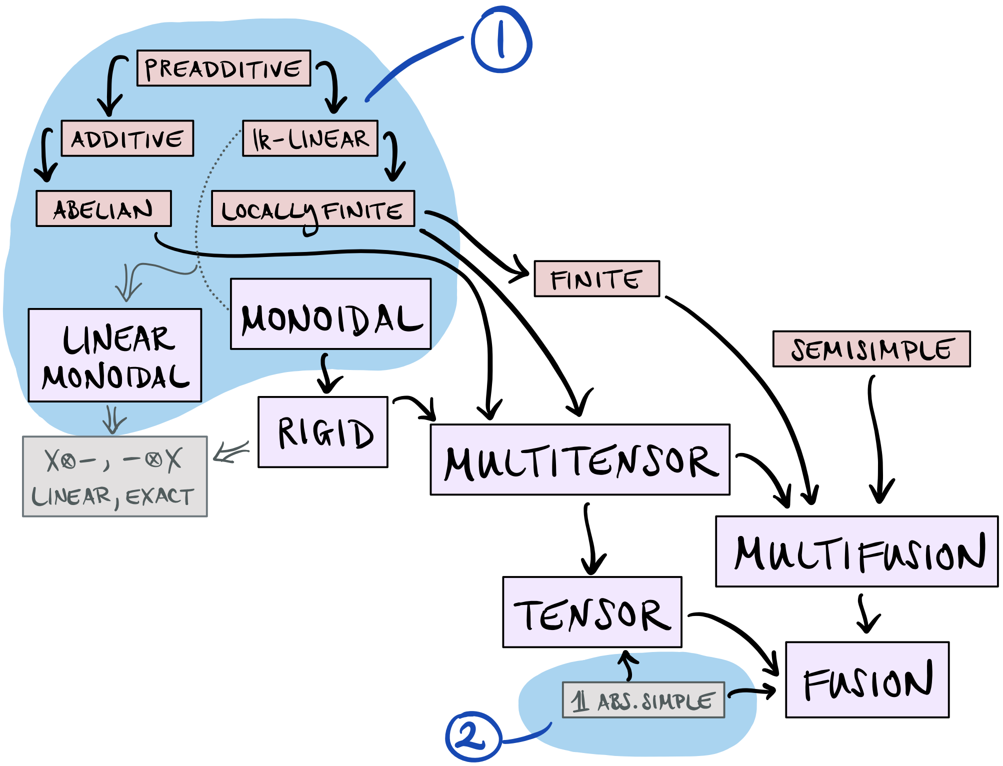
THEN $(A\text{-Bimod}(\mathcal{C}), \otimes_A, A_{reg})$

SATISFIES:

1. ALWAYS ;

2. WHEN A IS CONNECTED ;

III. BIMODULES AND BEYOND



THEOREM:

TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ TENSOR

& $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

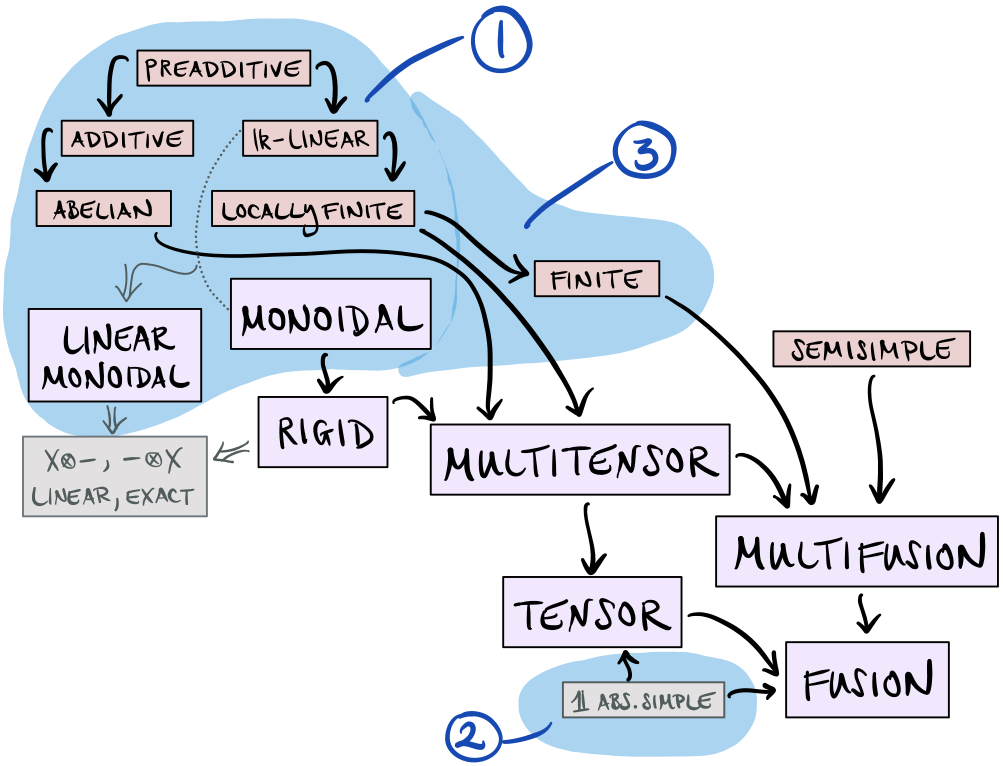
THEN $(A\text{-}\mathrm{Bimod}(\mathcal{C}), \otimes_A, A_{reg})$

SATISFIES:

1. ALWAYS;

2. WHEN A IS CONNECTED;

III. BIMODULES AND BEYOND



THEOREM:

TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ TENSOR

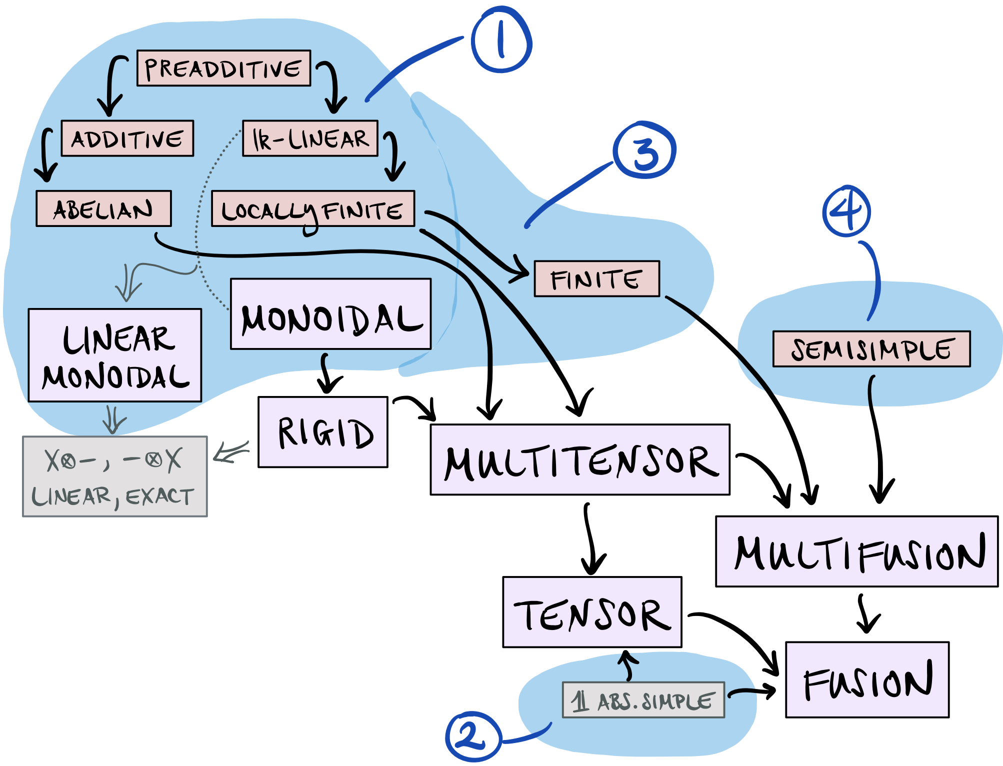
& $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

THEN $(A\text{-}\mathrm{Bimod}(\mathcal{C}), \otimes_A, A_{reg})$

SATISFIES:

1. ALWAYS ;
2. WHEN A IS CONNECTED ;
3. WHEN $\mathcal{C}$ IS FINITE ;

III. BIMODULES AND BEYOND



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TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ TENSOR

& $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

THEN $(A\text{-}\mathrm{Bimod}(\mathcal{C}), \otimes_A, A_{reg})$

SATISFIES:

① ALWAYS ;

② WHEN A IS CONNECTED ;

③ WHEN $\mathcal{C}$ IS FINITE ;

④ WHEN $\mathcal{C}$ IS SEMISIMPLE & A IS SEPARABLE ;

III. BIMODULES AND BEYOND

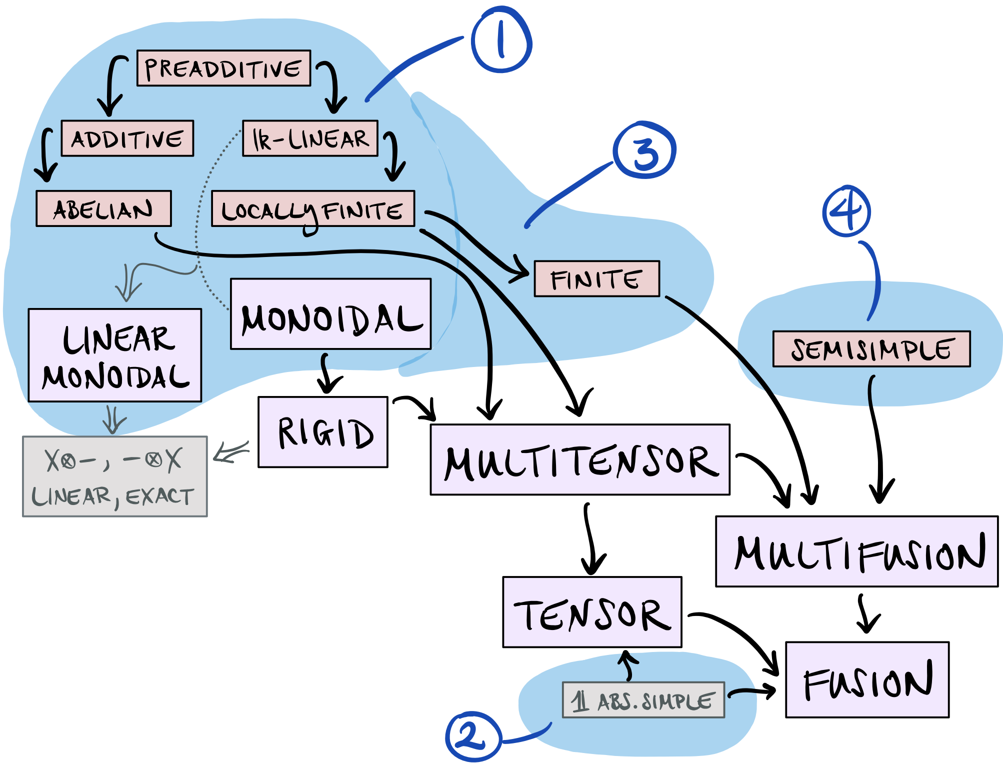
SEPARABILITY

$\exists$ RIGHT INVERSE TO $m_A : A \otimes A \to A$ IN $A\text{-Bimod}(\mathcal{C})$

PROP: SUPPOSE $\mathcal{C}$ MULTIFUSION.
THEN TFAE:

- A SEPARABLE
- $\text{Mod-}A(\mathcal{C})$ IS SEMISIMPLE
- $A\text{-Mod}(\mathcal{C})$ IS SEMISIMPLE
- $A\text{-Bimod}(\mathcal{C})$ IS SEMISIMPLE

[DAVYDOV–MÜGER–NIKSHYCH–OSTRIK]



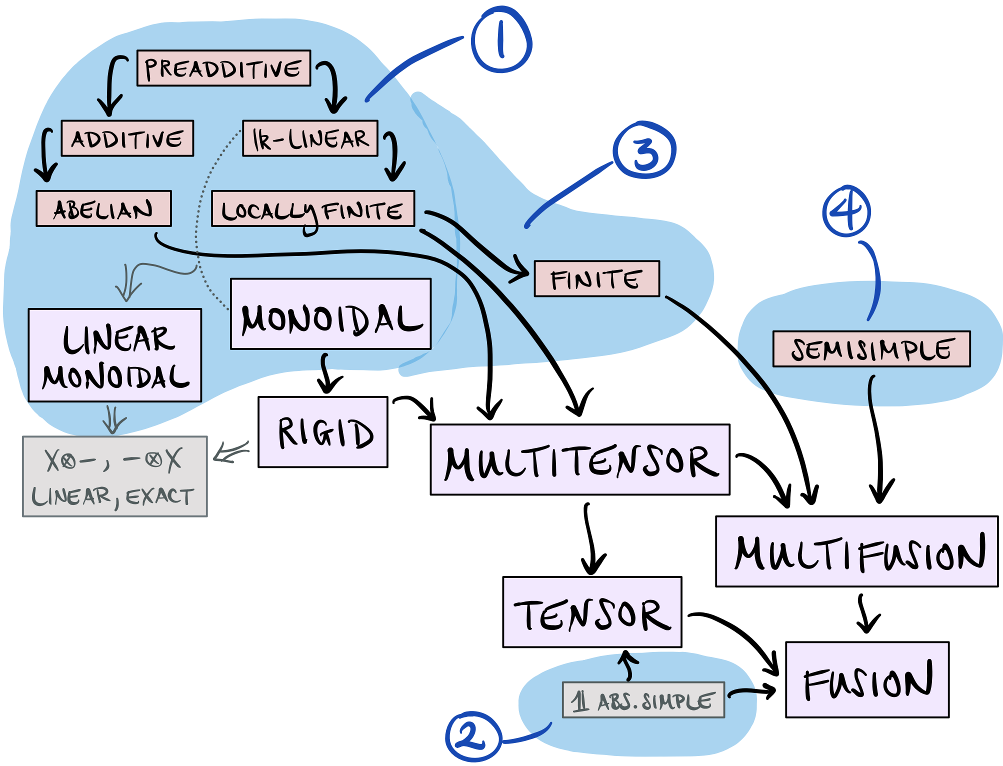
THEOREM:

TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ TENSOR & $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

THEN $(A\text{-Bimod}(\mathcal{C}), \otimes_A, A_{reg})$ SATISFIES:

1. ALWAYS ;
2. WHEN A IS CONNECTED ;
3. WHEN $\mathcal{C}$ IS FINITE ;
4. WHEN $\mathcal{C}$ IS SEMISIMPLE & A IS SEPARABLE ;

III. BIMODULES AND BEYOND



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THEN $(A\text{-}\mathrm{Bimod}(\mathcal{C}), \otimes_A, A_{reg})$ SATISFIES:

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③ WHEN $\mathcal{C}$ IS FINITE;

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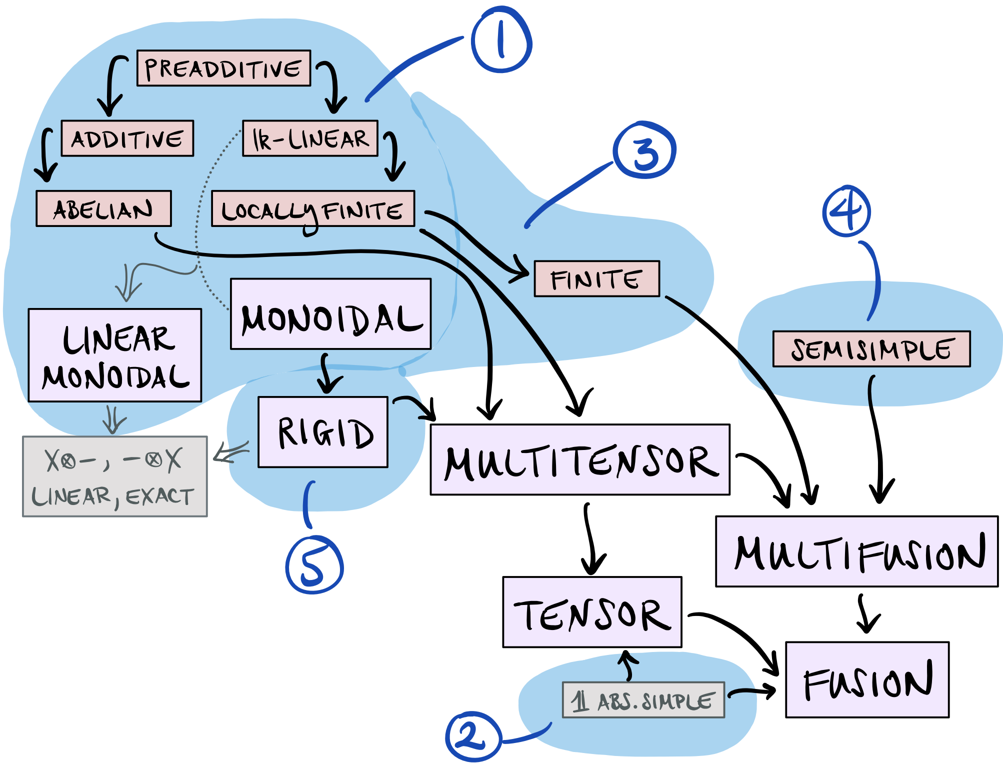
III. BIMODULES AND BEYOND



THEOREM:

TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ TENSOR & $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

THEN $(A\text{-Bimod}(\mathcal{C}), \otimes_A, A_{reg})$ SATISFIES:

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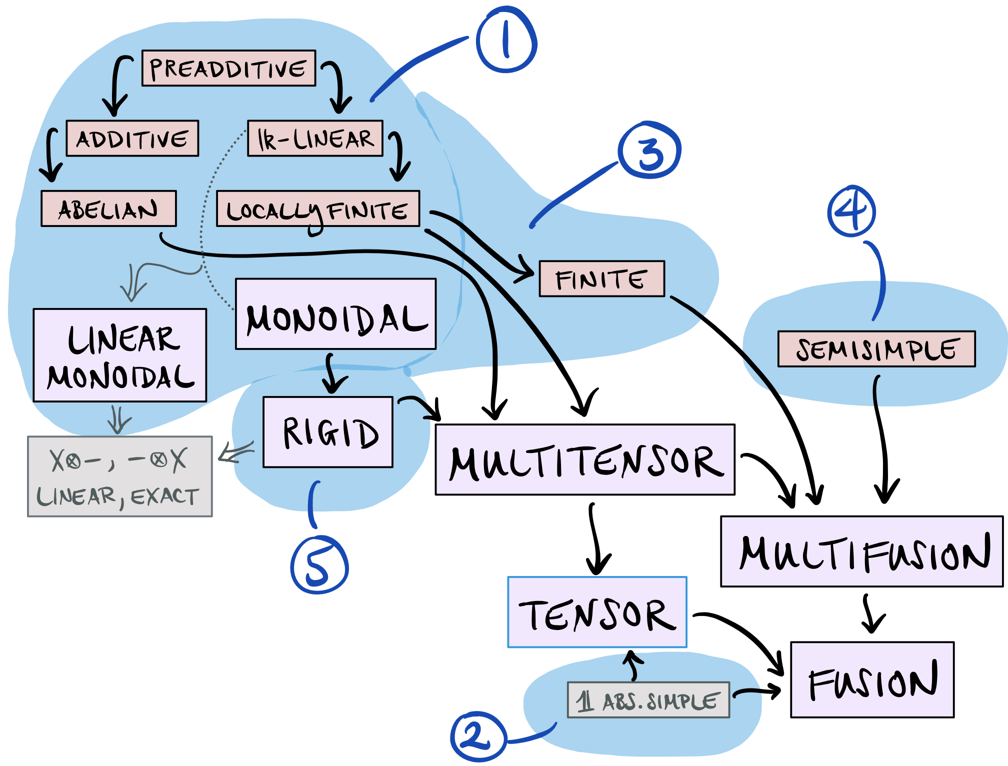
2. WHEN A IS CONNECTED ;
3. WHEN $\mathcal{C}$ IS FINITE ;
4. WHEN $\mathcal{C}$ IS SEMISIMPLE & A IS SEPARABLE ;
5. WHEN $\mathcal{C}$ IS FINITE & A IS EXACT.

III. BIMODULES AND BEYOND

EXACTNESS

A-Mod($\mathcal{C}$) IS AN EXACT RIGHT $\mathcal{C}$-MODULE CATEG.

PROP.
SAY $\mathcal{C}$ FINITE TENSOR.
THEN:
A IS EXACT
$\Updownarrow$
ANY RIGHT $\mathcal{C}$-MODULE FUNCTOR A-Mod($\mathcal{C}$) $\longrightarrow \mathcal{M}$ IS EXACT



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TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ TENSOR & $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

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ANY RIGHT $\mathcal{C}$-MODULE FUNCTOR A-Mod($\mathcal{C}$) $\longrightarrow \mathcal{M}$ IS EXACT

A-Bimod($\mathcal{C}$)

$\otimes \,$ SI   GEN'D EW THM

$\mathrm{Rex}_{\mathrm{Mod}\text{-}\mathcal{C}}(\text{A-Mod}(\mathcal{C}), \text{A-Mod}(\mathcal{C}))$

$\otimes \,$ SI

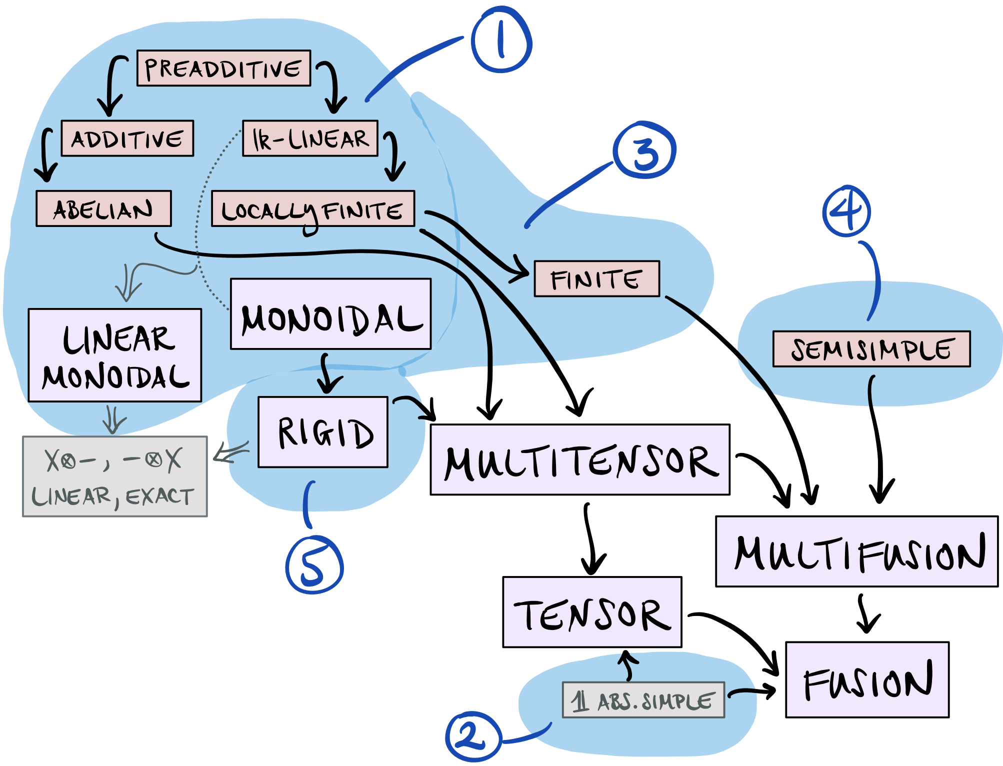
$\mathrm{Ex}_{\mathrm{Mod}\text{-}\mathcal{C}}(\text{A-Mod}(\mathcal{C}), \text{A-Mod}(\mathcal{C}))$

THEOREM:

TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ TENSOR & $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

THEN $(\text{A-Bimod}(\mathcal{C}), \otimes_A, A_{reg})$ SATISFIES:

1. ALWAYS;
2. WHEN A IS CONNECTED;
3. WHEN $\mathcal{C}$ IS FINITE;
4. WHEN $\mathcal{C}$ IS SEMISIMPLE & A IS SEPARABLE;
5. WHEN $\mathcal{C}$ IS FINITE & A IS EXACT.



III. BIMODULES AND BEYOND

EXACTNESS

$A\text{-Mod}(\mathcal{C})$ IS AN EXACT RIGHT $\mathcal{C}$-MODULE CATEG.

PROP. SAY $\mathcal{C}$ FINITE TENSOR.
THEN:
A IS EXACT
$\Updownarrow$
ANY RIGHT $\mathcal{C}$-MODULE FUNCTOR $A\text{-Mod}(\mathcal{C}) \longrightarrow \mathcal{M}$ IS EXACT

$A\text{-Bimod}(\mathcal{C})$

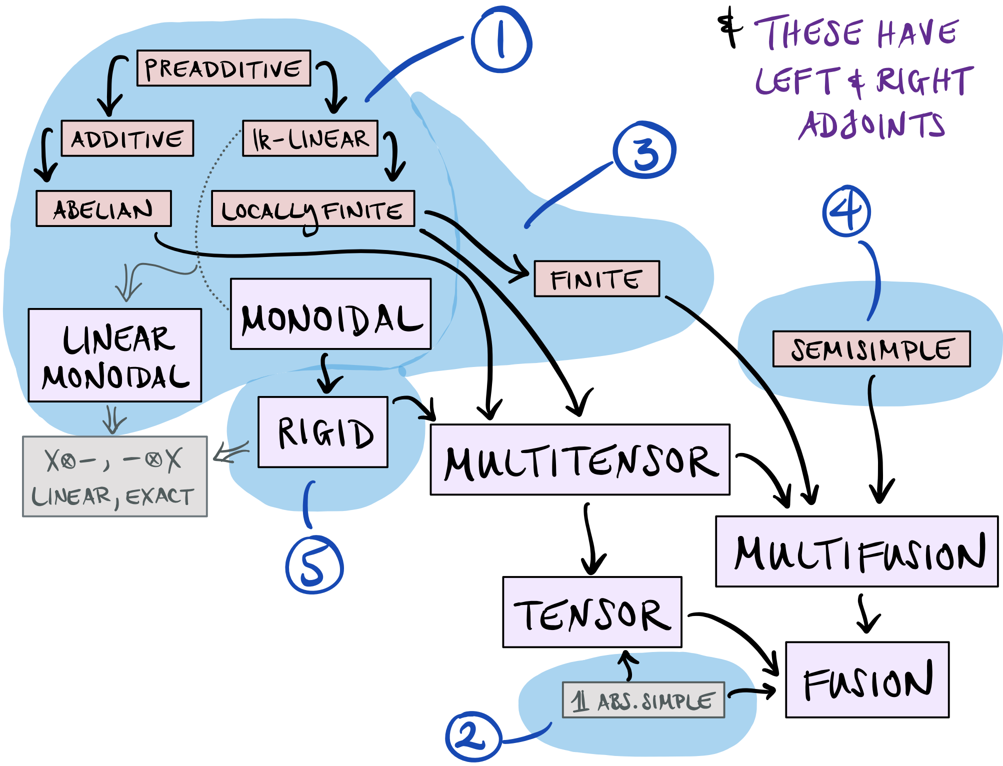
$\otimes \cong$ GEN'D EW THM

$\text{Rex}_{\text{Mod-}\mathcal{C}}(A\text{-Mod}(\mathcal{C}), A\text{-Mod}(\mathcal{C}))$

$\otimes \cong$

$\text{Ex}_{\text{Mod-}\mathcal{C}}(A\text{-Mod}(\mathcal{C}), A\text{-Mod}(\mathcal{C}))$

& THESE HAVE LEFT & RIGHT ADJOINTS



THEOREM:

TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ TENSOR
& $(A, m, u) \in \text{Alg}(\mathcal{C})$.

THEN $(A\text{-Bimod}(\mathcal{C}), \otimes_A, A_{reg})$ SATISFIES:

1. ALWAYS ;
2. WHEN A IS CONNECTED ;
3. WHEN $\mathcal{C}$ IS FINITE ;
4. WHEN $\mathcal{C}$ IS SEMISIMPLE & A IS SEPARABLE ;
5. WHEN $\mathcal{C}$ IS FINITE & A IS EXACT.

III. BIMODULES AND BEYOND

EXACTNESS

A-$\mathrm{Mod}(\mathcal{C})$ IS AN EXACT RIGHT $\mathcal{C}$-MODULE CATEG.

GET LEFT & RIGHT DUALS

$$A\text{-}\mathrm{Bimod}(\mathcal{C})$$

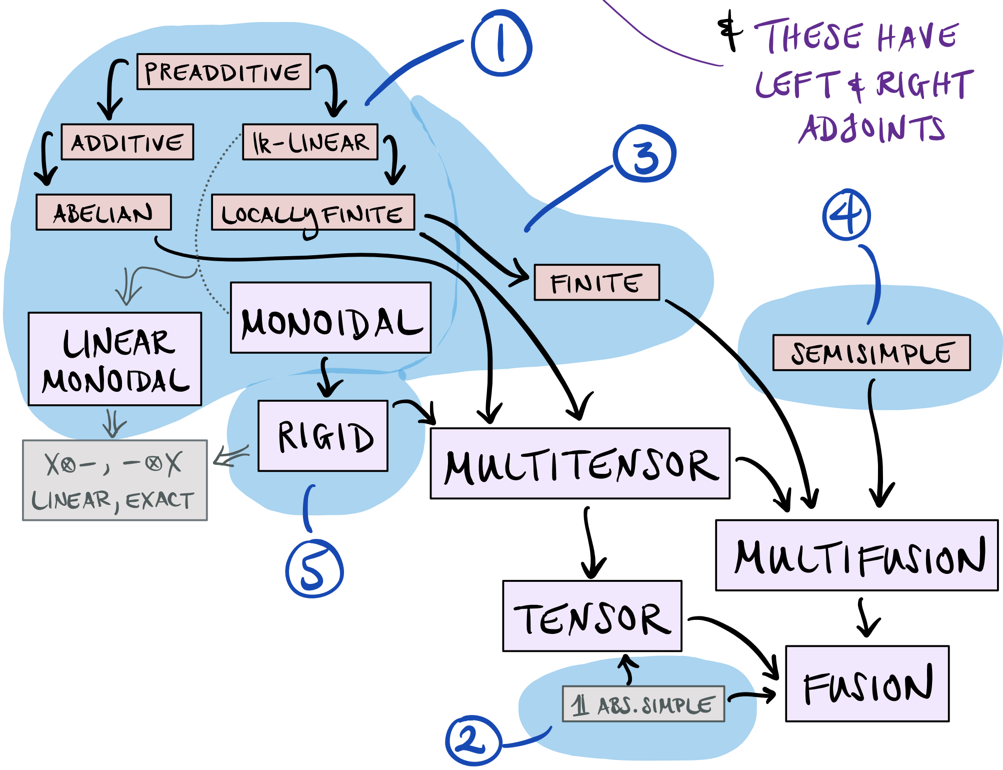
$\otimes \simeq$ GEN'D EW THM

$$\mathrm{Rex}_{\mathrm{Mod}\text{-}\mathcal{C}}(A\text{-}\mathrm{Mod}(\mathcal{C}), A\text{-}\mathrm{Mod}(\mathcal{C}))$$

$\otimes \simeq$

$$\mathrm{Ex}_{\mathrm{Mod}\text{-}\mathcal{C}}(A\text{-}\mathrm{Mod}(\mathcal{C}), A\text{-}\mathrm{Mod}(\mathcal{C}))$$

& THESE HAVE LEFT & RIGHT ADJOINTS



THEOREM:

TAKE $(\mathcal{C}, \otimes, \mathbb{1})$ TENSOR & $(A, m, u) \in \mathrm{Alg}(\mathcal{C})$.

THEN $(A\text{-}\mathrm{Bimod}(\mathcal{C}), \otimes_A, A_{reg})$ SATISFIES:

1. ALWAYS ;
2. WHEN A IS CONNECTED ;
3. WHEN $\mathcal{C}$ IS FINITE ;
4. WHEN $\mathcal{C}$ IS SEMISIMPLE & A IS SEPARABLE ;
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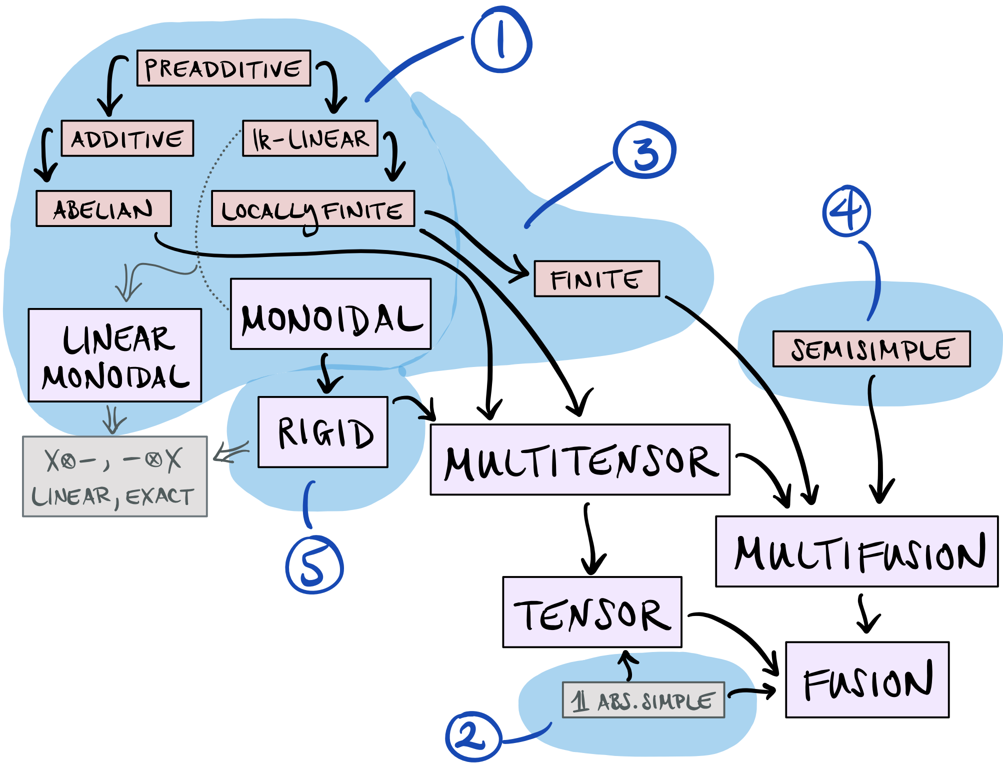
III. BIMODULES AND BEYOND



THEOREM:

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THEN $(A\text{-Bimod}(\mathcal{C}), \otimes_A, A_{reg})$ SATISFIES:

1. ALWAYS;
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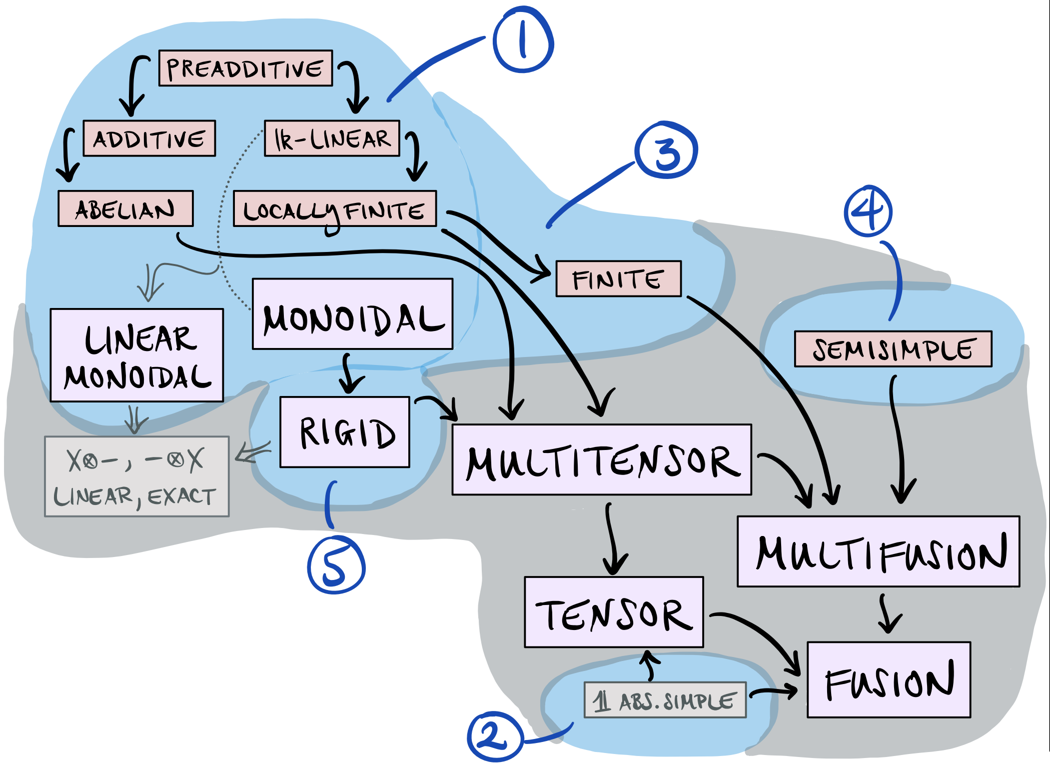
3. WHEN $\mathcal{C}$ IS FINITE;
4. WHEN $\mathcal{C}$ IS SEMISIMPLE & A IS SEPARABLE;
5. WHEN $\mathcal{C}$ IS FINITE & A IS EXACT.

III. BIMODULES AND BEYOND

NOW WE GET THE REST...



THEOREM:

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3. WHEN $\mathcal{C}$ IS FINITE ;
4. WHEN $\mathcal{C}$ IS SEMISIMPLE & A IS SEPARABLE ;
5. WHEN $\mathcal{C}$ IS FINITE & A IS EXACT.

IV. CATEGORICAL MORITA EQUIVALENCE

FOR $\Bbbk$-ALGEBRAS

$A \underset{\text{MORITA}}{\sim} B$

"EQUIVALENCE" OF ALGEBRAS

$\overset{\text{DEF}}{\equiv}$

$A\text{-Mod}_{\Bbbk} \simeq B\text{-Mod}_{\Bbbk}$

MORITA'S THM $\Updownarrow$

$\exists$ BIMODULES$/\Bbbk$: ${}_AP_B$, ${}_BQ_A$

s.t. $\begin{cases} P \otimes_B Q \cong A_{reg} \text{ IN } A\text{-Bimod}_{\Bbbk} \\ Q \otimes_A P \cong B_{reg} \text{ IN } B\text{-Bimod}_{\Bbbk} \end{cases}$

$\Updownarrow$

$\exists$ FIN. GEN. PROJ. $M \in A\text{-Mod}_{\Bbbk}$

WITH $\text{Hom}_{A\text{-Mod}_{\Bbbk}}(M, -)$ FAITHFUL

s.t. $B^{op} \cong \text{End}_{A\text{-Mod}}(M)$

AS $\Bbbk$-ALGEBRAS

WANT

TENSOR CATEGS.

$\mathcal{C} \underset{\text{MORITA}}{\sim} \mathcal{D}$

$\overset{\text{DEF}}{\equiv}$

$\mathcal{C}\text{-Mod} \simeq \mathcal{D}\text{-Mod}$

IV. CATEGORICAL MORITA EQUIVALENCE

FOR $\Bbbk$-ALGEBRAS

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MORITA'S THM $\Updownarrow$

$\exists$ BIMODULES$/\Bbbk$: ${}_A P_B$, ${}_B Q_A$

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$\Updownarrow$

$\exists$ FIN. GEN. PROJ. $M \in A\text{-Mod}_{\Bbbk}$ WITH $\mathrm{Hom}_{A\text{-Mod}_{\Bbbk}}(M,-)$ FAITHFUL

s.t. $B^{op} \cong \mathrm{End}_{A\text{-Mod}}(M)$ AS $\Bbbk$-ALGEBRAS

WANT

TENSOR CATEGS.

$\mathcal{C} \underset{\text{MORITA}}{\sim} \mathcal{D}$

$\overset{\text{DEF}}{\Longleftrightarrow}$

$\mathcal{C}\text{-Mod} \simeq \mathcal{D}\text{-Mod}$

... CAN MIMIC VIA BIMODULE CATEGORIES

$\exists$ ${}_{\mathcal{C}}\mathcal{P}_{\mathcal{D}}$, ${}_{\mathcal{D}}\mathcal{Q}_{\mathcal{C}}$

+ CONDITIONS

... SHOOT FOR TENSOR EQUIVALENCE

$\mathcal{D}^{\otimes op} \simeq \mathrm{Rex}_{\mathcal{C}\text{-Mod}}(\mathcal{M},\mathcal{M})$

IV. CATEGORICAL MORITA EQUIVALENCE

FOR $\Bbbk$-ALGEBRAS

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$\mathcal{D}^{\otimes op} \simeq \mathrm{Rex}_{\mathcal{C}\text{-Mod}}(\mathcal{M}, \mathcal{M})$

VIA ALGEBRAS IN $\otimes$ CATS.

CAN USE OSTRIK'S THEOREM (OR A GENERALIZATION)

ON MODULE CATEGORIES OVER FUSION CATEGS.

(OVER FINITE TENSOR CATEGS.)

⋮

$\mathcal{M} \simeq \text{Mod-}A(\mathcal{C})$

FOR SOME $A \in \mathrm{Alg}(\mathcal{C})$

IV. CATEGORICAL MORITA EQUIVALENCE

FOR $\Bbbk$-ALGEBRAS

$A \underset{\text{MORITA}}{\sim} B$
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$\Updownarrow$

$\exists$ FIN. GEN. PROJ. $M \in A\text{-Mod}_{\Bbbk}$
WITH $\text{Hom}_{A\text{-Mod}_{\Bbbk}}(M,-)$ FAITHFUL
$\ni$. $B^{op} \cong \text{End}_{A\text{-Mod}}(M)$
AS $\Bbbk$-ALGEBRAS

WANT

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$\mathcal{C} \underset{\text{MORITA}}{\sim} \mathcal{D}$
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... CAN MIMIC VIA BIMODULE CATEGORIES
$\exists$ ${}_{\mathcal{C}}\mathcal{P}_{\mathcal{D}}$, ${}_{\mathcal{D}}\mathcal{Q}_{\mathcal{C}}$
+ CONDITIONS

... SHOOT FOR TENSOR EQUIVALENCE
$\mathcal{D}^{\otimes op} \simeq \text{Rex}_{\mathcal{C}\text{-Mod}}(\mathcal{M},\mathcal{M}) \simeq \big(A\text{-Bimod}(\mathcal{C})\big)^{\otimes op}$

VIA ALGEBRAS IN $\otimes$ CATS.

CAN USE
OSTRIK'S THEOREM
(OR A GENERALIZATION)

ON MODULE CATEGORIES OVER FUSION CATEGS.
(OVER FINITE TENSOR CATEGS.)

$\vdots$

$\mathcal{M} \simeq \text{Mod-}A(\mathcal{C})$
FOR SOME $A \in \text{Alg}(\mathcal{C})$

[GENERALIZED EILENBERG–WATTS THEOREM]

IV. CATEGORICAL MORITA EQUIVALENCE

TENSOR CATEGS. $\mathcal{C}, \mathcal{D}$ ARE CATEGORICALLY MORITA EQUIVALENT IF

$$\mathcal{D}^{\otimes op} \simeq \mathrm{Rex}_{\mathcal{C}\text{-Mod}}(\mathcal{M}, \mathcal{M})$$

AS TENSOR CATEGS, FOR SOME EXACT $\mathcal{M} \in \mathcal{C}\text{-Mod}$.

WANT

TENSOR CATEGS.

$$\mathcal{C} \underset{\text{MORITA}}{\sim} \mathcal{D}$$

$\Updownarrow$ DEF

$$\mathcal{C}\text{-Mod} \simeq \mathcal{D}\text{-Mod}$$

... CAN MIMIC VIA BIMODULE CATEGORIES $\exists\ {}_{\mathcal{C}}\mathcal{P}_{\mathcal{D}},\ {}_{\mathcal{D}}\mathcal{Q}_{\mathcal{C}}$ + CONDITIONS

... SHOOT FOR TENSOR EQUIVALENCE

$$\mathcal{D}^{\otimes op} \simeq \mathrm{Rex}_{\mathcal{C}\text{-Mod}}(\mathcal{M}, \mathcal{M}) \simeq (A\text{-Bimod}(\mathcal{C}))^{\otimes op}$$

VIA ALGEBRAS IN $\otimes$ CATS.

CAN USE OSTRIK'S THEOREM (OR A GENERALIZATION) ON MODULE CATEGORIES OVER FUSION CATEGS. (OVER FINITE TENSOR CATEGS.)

$\vdots$

$$\mathcal{M} \simeq \text{Mod-}A(\mathcal{C})$$

FOR SOME $A \in \mathrm{Alg}(\mathcal{C})$

[GENERALIZED EILENBERG–WATTS THEOREM]

IV. CATEGORICAL MORITA EQUIVALENCE

TENSOR CATEGS. $\mathcal{C}$, $\mathcal{D}$ ARE

CATEGORICALLY MORITA EQUIVALENT

IF

$$\mathcal{D}^{\otimes op} \simeq \mathrm{Rex}_{\mathcal{C}\text{-Mod}}(\mathcal{M}, \mathcal{M})$$

AS TENSOR CATEGS,

FOR SOME EXACT $\mathcal{M} \in \mathcal{C}\text{-Mod}$.

$\therefore$ IN THE SETTING OF OSTRIK'S THEOREM (OR A GENERALIZATION)

$$\mathcal{M} \simeq \mathrm{Mod}\text{-}A(\mathcal{C})$$

FOR SOME $A \in \mathrm{Alg}(\mathcal{C})$

$$\simeq (A\text{-}\mathrm{Bimod}(\mathcal{C}))^{\otimes op}$$

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FOR SOME $A \in \mathrm{Alg}(\mathcal{C})$

$$\simeq \left(A\text{-Bimod}(\mathcal{C})\right)^{\otimes op}$$

KNOW WHEN FUSION / FINITE TENSOR

IV. CATEGORICAL MORITA EQUIVALENCE

TENSOR CATEGS. $\mathcal{C}, \mathcal{D}$ ARE **CATEGORICALLY MORITA EQUIVALENT** IF

$$\mathcal{D}^{\otimes op} \simeq \mathrm{Rex}_{\mathcal{C}\text{-Mod}}(\mathcal{M}, \mathcal{M})$$

AS TENSOR CATEGS, FOR SOME EXACT $\mathcal{M} \in \mathcal{C}$-Mod.

IN THE SETTING OF OSTRIK'S THEOREM (OR A GENERALIZATION)

$$\mathcal{M} \simeq \text{Mod-}A(\mathcal{C})$$

FOR SOME $A \in \mathrm{Alg}(\mathcal{C})$

$$\simeq (A\text{-Bimod}(\mathcal{C}))^{\otimes op}$$

KNOW WHEN FUSION / FINITE TENSOR

EXAMPLE G FINITE GROUP

$\mathcal{C} = \mathrm{Vec}_G$ & $\mathcal{D} = G\text{-Mod}$ ARE CATEGORICALLY MORITA EQUIVALENT BECAUSE

$$G\text{-Mod}^{\otimes op} \underset{\text{TENS}}{\simeq} \mathrm{Rex}_{\mathrm{Vec}_G\text{-Mod}}(\mathrm{Vec}, \mathrm{Vec}).$$

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AS TENSOR CATEGS, FOR SOME EXACT $\mathcal{M} \in \mathcal{C}$-Mod.

IN THE SETTING OF OSTRIK'S THEOREM (OR A GENERALIZATION)

$$\mathcal{M} \simeq \text{Mod-}A(\mathcal{C}) \quad \text{FOR SOME } A \in \mathrm{Alg}(\mathcal{C})$$

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= KEY REFERENCE =

§7.12 OF BOOK "TENSOR CATEGORIES" BY ETINGOF-GELAKI-NIKSHYCH-OSTRIK

IV. CATEGORICAL MORITA EQUIVALENCE

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VIA BIMODULE CATEGORIES ${}_{\mathcal{C}}\mathcal{P}_{\mathcal{D}}$, ${}_{\mathcal{D}}\mathcal{Q}_{\mathcal{C}}$ + CONDITIONS

SEE E-N-O'S PAPER "FUSION CATS. AND HOMOTOPY THY"

MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #22

END OF VOLUME I

TOPICS:

I. NOTIONS OF SAMENESS (RECAP)

II. PROPERTIES OF ALGEBRAS IN $\mathcal{C}$ (§4.9)

III. BIMODULES AND BEYOND (§4.10.1)

IV. CATEGORICAL MORITA EQUIVALENCE (§4.10.2)

Lecture #22 keywords: categorically Morita equivalent, category of bimodules, connected algebra, exact algebra, indecomposable algebra, semisimple algebra, separable algebra, simple algebra