MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #3

LAST TIME

- $\otimes$ OF VSPACES
  VIA QUOTIENT, UNIV. PROP.
- OPERATIONS ON LINEAR MAPS
  & TENSOR-HOM ADJUNCTION
- ALGEBRAS/$\Bbbk$ & EXAMPLES
  $\mathrm{Mat}_n(\Bbbk)$ $\quad \mathrm{End}_{\Bbbk}(V)$ $\quad T(V)$
  $\Bbbk\langle v_i \rangle_{i \in I}$ $\quad S(V)$ $\quad \Lambda(V)$

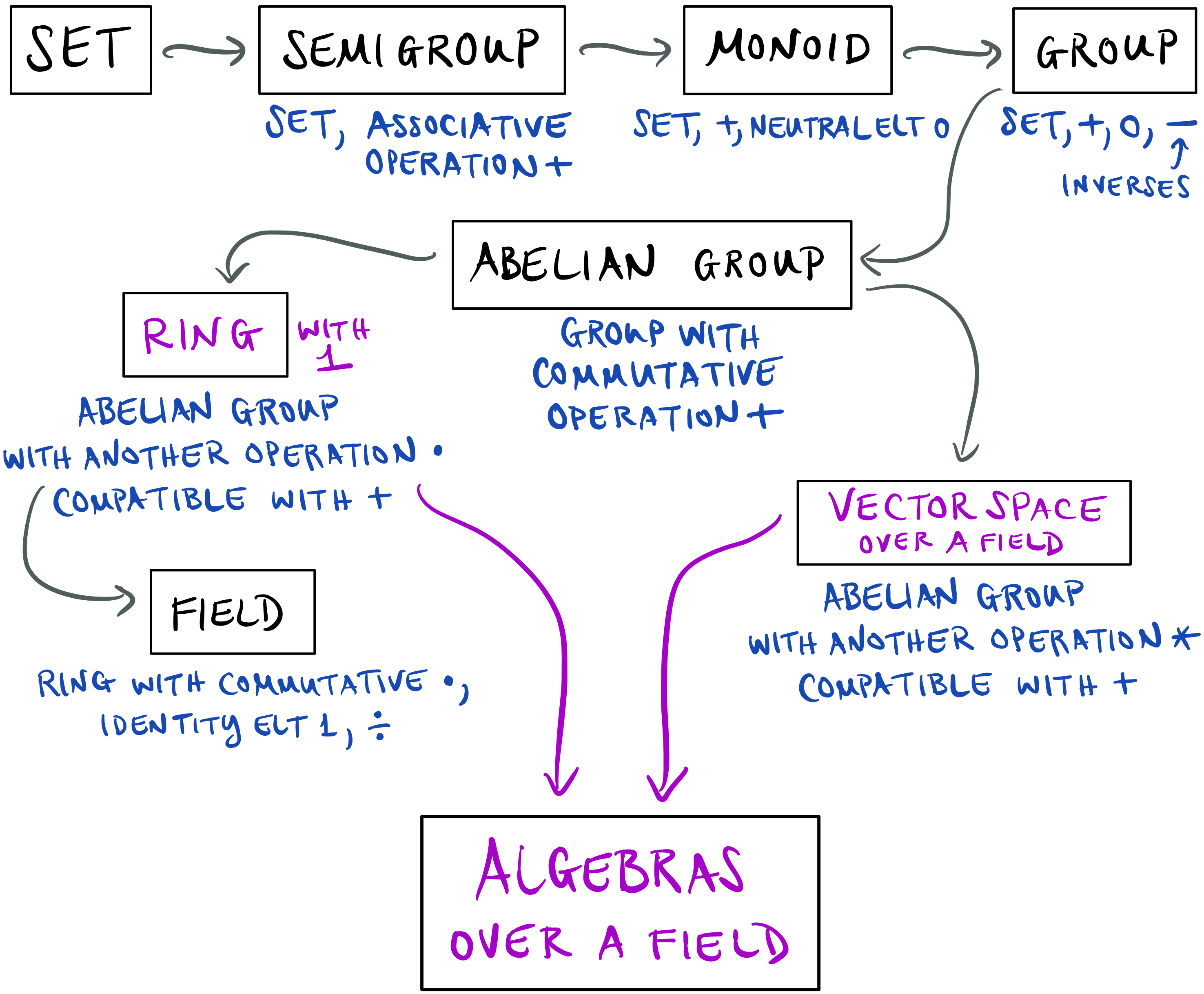
TOPICS:

I. EXAMPLES OF ALGEBRAS OVER A FIELD: $\Bbbk Q$, $\Bbbk G$ (§§1.2.5, 1.2.6)

II. REPRESENTATIONS OF ALGEBRAS & GROUPS (§§1.3.1, 1.3.4)

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS (§§1.3.2–1.3.4)

I. EXAMPLES OF ALGEBRAS OVER A FIELD: $\Bbbk Q$, $\Bbbk G$



I. EXAMPLES OF ALGEBRAS OVER A FIELD: $\Bbbk Q$, $\Bbbk G$

A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A **$\Bbbk$-ALGEBRA** IF IT COMES WITH LINEAR MAPS $m: A \otimes A \to A$ (MULTIPLICATION) & $u: \Bbbk \to A$ (UNIT)

s.t. $m(m \otimes id_A) = m(id_A \otimes m)$ (ASSOCIATIVITY) & $m(u \otimes id_A) = id_A = m(id_A \otimes u)$ (UNITALITY)

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EXAMPLE BUILT FROM A DIRECTED GRAPH

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EXAMPLE BUILT FROM A $\underbrace{\text{DIRECTED GRAPH}}_{\text{"QUIVER"}}$

$$Q = (Q_0, \; Q_1, \; s: Q_1 \longrightarrow Q_0, \; t: Q_1 \longrightarrow Q_0)$$

SET OF VERTICES

SET OF ARROWS BETWEEN VERTICES

"SOURCE" FUNCTION

"TARGET" FUNCTION

I. EXAMPLES OF ALGEBRAS OVER A FIELD: $\Bbbk Q$, $\Bbbk G$

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EXAMPLE BUILT FROM A $\underbrace{\text{DIRECTED GRAPH}}$

"QUIVER"

$$Q = (Q_0,\ Q_1,\ s: Q_1 \to Q_0,\ t: Q_1 \to Q_0)$$

SET OF VERTICES

SET OF ARROWS BETWEEN VERTICES

"SOURCE" FUNCTION

"TARGET" FUNCTION

Ex. $Q: \underset{1}{\bullet} \xrightarrow{a} \underset{2}{\bullet}$

$Q_0 = \{1, 2\}$ $\quad Q_1 = \{a\}$

$s(a) = 1$ $\quad t(a) = 2$

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EXAMPLE BUILT FROM A DIRECTED GRAPH

"QUIVER" Q

$$\parallel$$

$$(Q_0; Q_1; s, t: Q_1 \to Q_0)$$

VERTICES

ARROWS

SOURCE

TARGET

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EXAMPLE BUILT FROM A DIRECTED GRAPH

"QUIVER" Q

$$\|$$

$$(Q_0; Q_1; s, t: Q_1 \longrightarrow Q_0)$$

VERTICES

ARROWS

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TARGET

A **PATH** IN Q IS A COMPOSITION OF ARROWS IN Q (READ LEFT-TO-RIGHT)

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EXAMPLE BUILT FROM A DIRECTED GRAPH

"QUIVER" Q

$$\|$$

$$(Q_0;\ Q_1;\ s, t: Q_1 \longrightarrow Q_0)$$

VERTICES ARROWS SOURCE TARGET

A **PATH** IN Q IS A COMPOSITION OF ARROWS IN Q (READ LEFT-TO-RIGHT)

Ex. $\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$ IS THE PATH ab.

HERE, $t(a) = s(b)$

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EXAMPLE BUILT FROM A DIRECTED GRAPH

"QUIVER" Q

$\parallel$

$(Q_0; Q_1; s, t: Q_1 \to Q_0)$

VERTICES

ARROWS

SOURCE

TARGET

A **PATH** IN Q IS A COMPOSITION OF ARROWS IN Q (READ LEFT-TO-RIGHT)

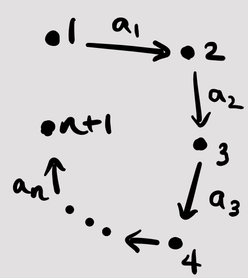
Ex. $\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$ IS THE PATH ab.

HERE, $t(a) = s(b)$

CAN ALSO FORM THE PATH

$p = a_1 a_2 \cdots a_n$ FOR $a_i \in Q_1$

WHERE $t(a_i) = s(a_{i+1})$ $\forall i = 1, \ldots, n-1$



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EXAMPLE BUILT FROM A DIRECTED GRAPH

"QUIVER" Q

$$\|$$

$$(Q_0; Q_1; s, t: Q_1 \to Q_0)$$

VERTICES

ARROWS

SOURCE

TARGET

A **PATH** IN Q IS A COMPOSITION OF ARROWS IN Q (READ LEFT-TO-RIGHT)

Ex. $\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet$ IS THE PATH ab.

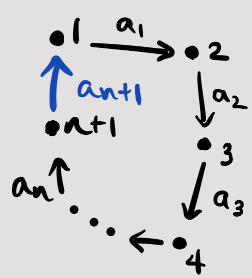
HERE, $t(a) = s(b)$

CAN ALSO FORM THE **CYCLE**

$p = a_1 a_2 \cdots a_n$ FOR $a_i \in Q_1$

WHERE $t(a_i) = s(a_{i+1})$ $\forall i = 1, \dots, n-1$

& $t(a_n) = s(a_1)$



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EXAMPLE BUILT FROM A DIRECTED GRAPH

"QUIVER" Q

$\parallel$

$(Q_0; Q_1; s, t : Q_1 \to Q_0)$

VERTICES ARROWS SOURCE TARGET

A **PATH** IN Q IS A COMPOSITION OF ARROWS IN Q (READ LEFT-TO-RIGHT)

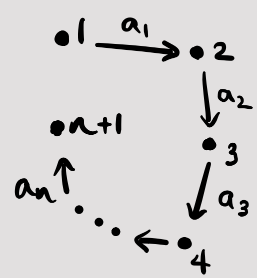
A **TRIVIAL PATH** IS A PATH OF LENGTH 0, DENOTED e_i $\qquad \bullet_i$

LENGTH n

CAN ALSO FORM THE PATH

$p = a_1 a_2 \cdots a_n$ FOR $a_i \in Q_1$

WHERE $t(a_i) = s(a_{i+1})$ $\forall i = 1, \dots, n-1$



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EXAMPLE BUILT FROM A DIRECTED GRAPH "QUIVER" Q

$$Q = (Q_0; Q_1; s, t: Q_1 \to Q_0)$$

A PATH IN Q IS A COMPOSITION OF ARROWS IN Q

THE PATH ALGEBRA $\Bbbk Q$ OF Q IS THE NOT NECESSARILY UNITAL $\Bbbk$-ALGEBRA WITH

- $\Bbbk$-VS BASIS $\equiv$ PATHS OF Q
- MULTIPLICATION $\equiv$ PATH COMPOSITION

I. EXAMPLES OF ALGEBRAS OVER A FIELD: $\Bbbk Q$, $\Bbbk G$

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EXAMPLE BUILT FROM A DIRECTED GRAPH "QUIVER" Q

$$\begin{gathered} Q \\ \| \\ (Q_0; Q_1; s, t: Q_1 \to Q_0) \end{gathered}$$

A PATH IN Q IS A COMPOSITION OF ARROWS IN Q

THE **PATH ALGEBRA $\Bbbk Q$** OF Q

IS THE $\Bbbk$-ALGEBRA WITH

- $\Bbbk$-VS BASIS $\equiv$ PATHS OF Q
- MULTIPLICATION $\equiv$ PATH COMPOSITION
- UNIT $\equiv \sum_{i \in Q_0} e_i$ WHEN $|Q_0| < \infty$

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$Q \qquad \bullet_1$

$\Bbbk Q \cong \Bbbk e_1 \cong \Bbbk$ AS ALGS

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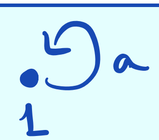
Q 

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Q $\quad \underset{1}{\bullet} \circlearrowleft a$

$\Bbbk Q \cong \Bbbk\langle a \rangle \cong \Bbbk[a]$ AS ALGS

Q $\quad 1 \bullet \xrightarrow{a} \bullet 2$

$\Bbbk Q \cong \begin{pmatrix} \Bbbk & \Bbbk \\ 0 & \Bbbk \end{pmatrix}$ AS ALGS

THE PATH ALGEBRA $\Bbbk Q$ OF Q

IS THE $\Bbbk$-ALGEBRA WITH

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$Q \qquad \bullet_1$

$\Bbbk Q \cong \Bbbk e_1 \cong \Bbbk$ AS ALGS

$Q \qquad$ $\begin{tikzcd} \underset{1}{\bullet} \arrow[loop right, "a"] \end{tikzcd}$

$\Bbbk Q \cong \Bbbk\langle a \rangle \cong \Bbbk[a]$ AS ALGS

$Q \qquad 1 \bullet \xrightarrow{a} \bullet 2$

$\Bbbk Q \cong \begin{pmatrix} \Bbbk & \Bbbk \\ 0 & \Bbbk \end{pmatrix}$ AS ALGS

THE PATH ALGEBRA $\Bbbk Q$ OF Q

PROPERTIES OFTEN GIVEN BY GRAPHICAL PROPS. OF Q

IS THE $\Bbbk$-ALGEBRA WITH $\qquad$ $\equiv$ READ $\equiv$

- $\Bbbk$-VS BASIS $\equiv$ PATHS OF Q
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$Q \qquad 1 \bullet \xrightarrow{a} \bullet 2$

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THE **PATH ALGEBRA** $\Bbbk Q$ OF Q

CAN BE DEFINED VIA UNIV. PROP.

=READ=

IS THE $\Bbbk$-ALGEBRA WITH

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EXAMPLE BUILT FROM A GROUP G

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EXAMPLE BUILT FROM A GROUP G

THE GROUP ALGEBRA $\Bbbk G$ OF $G := (G, \star, e)$

IS THE $\Bbbk$-ALGEBRA WITH

- $\Bbbk$-VS BASIS $\equiv$ ELEMENTS OF G
- MULTIPLICATION $\equiv$ GIVEN BY $\star$
- UNIT $\equiv$ e

Ex. IN $\mathbb{R}S_3$,

TAKE:

$x := 4(123)$

$y = 2(13) - 3e$

I. EXAMPLES OF ALGEBRAS OVER A FIELD: $\Bbbk Q$, $\Bbbk G$

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Ex. IN $\mathbb{R}S_3$,

TAKE:

$x := 4(123)$

$y = 2(13) - 3e$

GET:

$xy = 8(23) - 12(123)$

$yx = 8(12) - 12(123)$

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PROPERTIES OFTEN GIVEN BY PROPS. OF G

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EXAMPLE BUILT FROM A GROUP G

THE **GROUP ALGEBRA** $\Bbbk G$ OF $G := (G, \star, e)$

CAN BE DEFINED VIA UNIV. PROP.

IS THE $\Bbbk$-ALGEBRA WITH

- $\Bbbk$-VS BASIS $\equiv$ ELEMENTS OF G
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Ex. IN $\mathbb{R}S_3$,

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- UNIT $\equiv$ e

GIVEN AN ALGEBRA A, GET

$$A^\times = \left\{ a \in A \;\middle|\; \begin{array}{c} ab = ba = 1_A \\ \text{FOR SOME } b \in A \end{array} \right\}$$

GROUP OF UNITS OF A

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- UNIT $\equiv$ e

GIVEN AN ALGEBRA A, GET

$$A^\times = \left\{ a \in A \;\middle|\; \begin{array}{c} ab = ba = 1_A \\ \text{FOR SOME } b \in A \end{array} \right\}$$

GROUP OF UNITS OF A

$$\begin{array}{ccc} G & \xrightarrow[\text{GROUP MAP}]{\alpha} & (\Bbbk G)^\times \subseteq \Bbbk G \\ & & \mathrel{\rotatebox{90}{$\in$}} \\ g & \longmapsto & g \end{array}$$

I. EXAMPLES OF ALGEBRAS OVER A FIELD: $\Bbbk Q$, $\Bbbk G$

A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A $\Bbbk$-ALGEBRA IF IT COMES WITH LINEAR MAPS $m: A \otimes A \to A$ (MULTIPLICATION) & $u: \Bbbk \to A$ (UNIT)

s.t. $m(m \otimes id_A) = m(id_A \otimes m)$ (ASSOCIATIVITY) & $m(u \otimes id_A) = id_A = m(id_A \otimes u)$ (UNITALITY)

EXAMPLE BUILT FROM A GROUP G

THE GROUP ALGEBRA $\Bbbk G$ OF $G := (G, \star, e)$

CAN BE DEFINED VIA UNIV. PROP.

IS THE $\Bbbk$-ALGEBRA WITH

- $\Bbbk$-VS BASIS $\equiv$ ELEMENTS OF G
- MULTIPLICATION $\equiv$ GIVEN BY $\star$
- UNIT $\equiv$ e

GIVEN AN ALGEBRA A, GET

$$A^{\times} = \left\{ a \in A \;\middle|\; \begin{array}{c} ab = ba = 1_A \\ \text{FOR SOME } b \in A \end{array} \right\}$$

GROUP OF UNITS OF A

$$\begin{tikzcd} G \arrow[r, "\alpha"', "\text{GROUP MAP}"'] \arrow[dr, "\forall \text{ GROUP MAP } \beta"'] & (\Bbbk G)^{\times} \subseteq \Bbbk G \\ & Z^{\times} \subseteq Z \text{ ALG} \end{tikzcd}$$

I. EXAMPLES OF ALGEBRAS OVER A FIELD: $\Bbbk Q$, $\Bbbk G$

A $\Bbbk$-VSPACE $(A, +, 0, *)$ IS A **$\Bbbk$-ALGEBRA** IF IT COMES WITH LINEAR MAPS $m: A \otimes A \to A$ (MULTIPLICATION) & $u: \Bbbk \to A$ (UNIT)

s.t. $m(m \otimes id_A) = m(id_A \otimes m)$ (ASSOCIATIVITY) & $m(u \otimes id_A) = id_A = m(id_A \otimes u)$ (UNITALITY)

EXAMPLE BUILT FROM A GROUP G

THE **GROUP ALGEBRA** $\Bbbk G$ OF $G := (G, \star, e)$

CAN BE DEFINED VIA UNIV. PROP.

IS THE $\Bbbk$-ALGEBRA WITH

- $\Bbbk$-VS BASIS $\equiv$ ELEMENTS OF G
- MULTIPLICATION $\equiv$ GIVEN BY $\star$
- UNIT $\equiv$ e

GIVEN AN ALGEBRA A, GET

$$A^\times = \left\{ a \in A \;\middle|\; \begin{array}{c} ab = ba = 1_A \\ \text{FOR SOME } b \in A \end{array} \right\}$$

GROUP OF UNITS OF A

$$\begin{tikzcd}
G \arrow[r, "\alpha", "\text{GROUP MAP}"'] \arrow[d, "\forall\ \text{GROUP MAP}\ \beta"'] & (\Bbbk G)^\times \subseteq \Bbbk G \arrow[dl, dashed, "\exists!\ \text{ALGEBRA MAP}\ \gamma"] \\
Z^\times \subseteq Z \text{ ALG} &
\end{tikzcd}$$

I. EXAMPLES OF ALGEBRAS OVER A FIELD: $\Bbbk Q$, $\Bbbk G$

A $\Bbbk$-VSPACE $(A,+,0,*)$ IS A $\Bbbk$-ALGEBRA IF IT COMES WITH LINEAR MAPS $m: A\otimes A \longrightarrow A$ (MULTIPLICATION) & $u: \Bbbk \longrightarrow A$ (UNIT)

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EXAMPLE BUILT FROM A GROUP G

THE GROUP ALGEBRA $\Bbbk G$ OF $G := (G, \bigstar, e)$ CAN BE DEFINED VIA UNIV. PROP.

IS THE $\Bbbk$-ALGEBRA WITH

- $\Bbbk$-VS BASIS $\equiv$ ELEMENTS OF G
- MULTIPLICATION $\equiv$ GIVEN BY $\bigstar$
- UNIT $\equiv$ e

GIVEN AN ALGEBRA A, GET

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Z^\times \subseteq Z\ \text{ALG} &
\end{tikzcd}$$

$\cong$ AS SETS $\cong$

$$\mathrm{Hom}_{\mathrm{Group}}(G, Z^\times) \cong \mathrm{Hom}_{\mathrm{Alg}_{\Bbbk}}(\Bbbk G, Z)$$

II. REPRESENTATIONS OF ALGEBRAS & GROUPS

TAKE AN (ALGEBRAIC) STRUCTURE $\mathcal{S}$.

E.G. GROUP, RING, ALGEBRA

A REPRESENTATION OF $\mathcal{S}$ IS ANOTHER STRUCTURE $\mathcal{U}$ $\ni$.

E.G. SET, ABELIAN GROUP, VSPACE

II. REPRESENTATIONS OF ALGEBRAS & GROUPS

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A **REPRESENTATION** OF $\mathcal{S}$ IS ANOTHER STRUCTURE $\mathcal{U}$ $\ni$.

E.G. SET, ABELIAN GROUP, VSPACE

$\underbrace{\mathrm{End}(\mathcal{U})}_{\equiv}$ HAS THE SAME STRUCTURE AS $\mathcal{S}$

[COLLECTION OF ENDOMORPHISMS OF $\mathcal{U}$]

II. REPRESENTATIONS OF ALGEBRAS & GROUPS

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A **REPRESENTATION** OF $\mathcal{S}$ IS ANOTHER STRUCTURE $\mathcal{U}$ S.T.

E.G. SET, ABELIAN GROUP, VSPACE

$\mathrm{End}(\mathcal{U})$ HAS THE SAME STRUCTURE AS $\mathcal{S}$

[COLLECTION OF ENDOMORPHISMS OF $\mathcal{U}$]

& $\exists$ STRUCTURE MAP

$$\mathcal{S} \xrightarrow{\rho} \mathrm{End}(\mathcal{U})$$

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TAKE AN (ALGEBRAIC) STRUCTURE $\mathcal{S}$.

E.G. GROUP, RING, ALGEBRA

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$\mathrm{End}(\mathcal{U})$ HAS THE SAME STRUCTURE AS $\mathcal{S}$

& $\exists$ STRUCTURE MAP

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THIS CREATES AN AVATAR OF $\mathcal{S}$ IN THE CONTEXT OF $\mathcal{U}$

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A REPRESENTATION OF $\mathcal{S}$ IS ANOTHER STRUCTURE $\mathcal{U}$ $\ni$.

E.G. SET, ABELIAN GROUP, $\boxed{\text{VSPACE}}$

$\mathrm{End}(\mathcal{U})$ HAS THE SAME STRUCTURE AS $\mathcal{S}$

$\mathrm{End}(\mathcal{U})$ $\cong$ COLLECTION OF MATRICES

& $\exists$ STRUCTURE MAP

$$\mathcal{S} \xrightarrow{\rho} \mathrm{End}(\mathcal{U})$$

COLLECTION OF MATRICES

CAN THINK ABOUT $\mathcal{S}$ VIA LINEAR ALG. (IN TERMS OF MATRICES)

THIS CREATES AN AVATAR OF $\mathcal{S}$ IN THE CONTEXT OF $\mathcal{U}$

II. REPRESENTATIONS OF ALGEBRAS & GROUPS

$S \equiv$ ALGEBRAIC STRUCTURE   E.G. GROUP, RING, ALGEBRA

A REPRESENTATION OF S IS A STRUCTURE MAP $S \xrightarrow{\rho} \mathrm{End}(U)$

FOR $U \equiv$ ANOTHER STRUCTURE   E.G. SET, ABELIAN GROUP, VSPACE
(↖ WE CHOOSE)

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(← WE CHOOSE)

TAKE A $\Bbbk$-ALGEBRA $A := (A, m, u)$.

A REPRESENTATION OF A

IS A VECTOR SPACE V EQUIPPED

WITH AN ALGEBRA MAP

$$\rho := \rho_V : A \longrightarrow \mathrm{End}_{\Bbbk}(V)$$

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$$\rho := \rho_V : A \longrightarrow \mathrm{End}_{\mathbb{k}}(V)$$

WHEN $\dim_{\mathbb{k}} V = n$

GET $\mathrm{End}_{\mathbb{k}}(V) \cong \mathrm{Mat}_n(\mathbb{k})$

CAN STUDY A VIA MATRICES

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Ex. TAKE CYCLIC GROUP

$C_2 = \langle g \mid g^2 = e \rangle$

CONSIDER THE GROUP ALG.

$\mathbb{C}C_2$

$\rho : \mathbb{C}C_2 \longrightarrow \mathrm{Mat}_2(\mathbb{C})$ GIVEN BY

$$e \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$g \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\lambda e + \mu g \longmapsto \lambda \rho(e) + \mu \rho(g) \quad \forall \lambda, \mu \in \mathbb{C}$$

"EXTENDING LINEARLY"

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ρ IS FAITHFUL

IF ρ IS INJECTIVE

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(↖ WE CHOOSE)

CAN THINK OF ρ AS S

CAPTURING $\underbrace{\text{SYMMETRIES OF } U}_{\mathrm{End}(U) \,\equiv\, \mathrm{Sym}(U)}$

$\boxed{\rho \text{ IS FAITHFUL IF } \rho \text{ IS INJECTIVE}}$

II. REPRESENTATIONS OF ALGEBRAS & GROUPS

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FOR $\mathcal{U} \equiv$ ANOTHER STRUCTURE E.G. SET, ABELIAN GROUP, VSPACE (← WE CHOOSE)

CAN THINK OF ρ AS $\mathcal{S}$ CAPTURING $\underbrace{\text{SYMMETRIES OF } \mathcal{U}}_{\mathrm{End}(\mathcal{U}) \equiv \mathrm{Sym}(\mathcal{U})}$

FAITHFULNESS ENSURES THAT $\mathcal{S}$ DOES THIS ON THE NOSE, NOT UNNECESARILY BIG

ρ IS FAITHFUL IF ρ IS INJECTIVE

II. REPRESENTATIONS OF ALGEBRAS & GROUPS

$\mathcal{S} \equiv$ ALGEBRAIC STRUCTURE E.G. GROUP, RING, ALGEBRA

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ρ IS FAITHFUL IF ρ IS INJECTIVE

Eg. 1 2 SYMMETRIES CAPTURED BY $\underbrace{C_2}_{}$

$\|$

$\{e, g\}$

DO NOTHING

SWAP

II. REPRESENTATIONS OF ALGEBRAS & GROUPS

$S \equiv$ ALGEBRAIC STRUCTURE   E.G. GROUP, RING, ALGEBRA

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FOR $U \equiv$ ANOTHER STRUCTURE   E.G. SET, ABELIAN GROUP, VSPACE

(← WE CHOOSE)

CAN THINK OF ρ AS S CAPTURING $\underbrace{\text{SYMMETRIES OF } U}_{\mathrm{End}(U) \equiv \mathrm{Sym}(U)}$

FAITHFULNESS ENSURES THAT S DOES THIS ON THE NOSE, NOT UNNECESARILY BIG

$\boxed{\rho \text{ IS FAITHFUL IF } \rho \text{ IS INJECTIVE}}$

Eg. 1 2 SYMMETRIES CAPTURED BY $C_2, C_4, C_6, \dots$

$$C_4 = \{e, g, g^2, g^3\}$$

e: DO NOTHING

g: SWAP

g^2: DO NOTHING

g^3: SWAP

II. REPRESENTATIONS OF ALGEBRAS & GROUPS

$\mathcal{S} \equiv$ ALGEBRAIC STRUCTURE E.G. GROUP, RING, ALGEBRA

A REPRESENTATION OF $\mathcal{S}$ IS A STRUCTURE MAP $\mathcal{S} \xrightarrow{\rho} \mathrm{End}(U)$

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(↑ WE CHOOSE)

CAN THINK OF ρ AS $\mathcal{S}$ CAPTURING $\underbrace{\text{SYMMETRIES OF } U}_{\mathrm{End}(U) \equiv \mathrm{Sym}(U)}$

FAITHFULNESS ENSURES THAT $\mathcal{S}$ DOES THIS ON THE NOSE, NOT UNNECESARILY BIG

ρ IS FAITHFUL IF ρ IS INJECTIVE

$\equiv$

THE ONLY ELEMENT OF $\mathcal{S}$ THAT DOES NOTHING TO U IS THE IDENTITY ELT OF $\mathcal{S}$

Eg. 1 2 SYMMETRIES CAPTURED BY THE RIGHT CHOICE C_2, $\underbrace{C_4}_{\{e,\, g,\, g^2,\, g^3\}}$, C_6, ...

$\{e, g, g^2, g^3\}$

DO NOTHING SWAP DO NOTHING SWAP

II. REPRESENTATIONS OF ALGEBRAS & GROUPS

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A REPRESENTATION OF S IS A STRUCTURE MAP $S \xrightarrow{\rho} \mathrm{End}(U)$

FOR $U \equiv$ ANOTHER STRUCTURE E.G. SET, ABELIAN GROUP, VSPACE (← WE CHOOSE)

TAKE A $\Bbbk$-ALGEBRA $A := (A, m, u)$.

A REPRESENTATION OF A

IS A VECTOR SPACE V EQUIPPED

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DEGREE/DIMENSION OF ρ

$= \dim_{\Bbbk} V$

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Ex. TAKE CYCLIC GROUP

$C_2 = \langle g \mid g^2 = e \rangle$

CONSIDER THE GROUP ALG.

$\mathbb{C}C_2$

$\rho : \mathbb{C}C_2 \longrightarrow \mathrm{Mat}_2(\mathbb{C})$ GIVEN BY

$e \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ DEGREE 2

$g \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ FAITHFUL

$\lambda e + \mu g \longmapsto \lambda \rho(e) + \mu \rho(g) \quad \forall \lambda, \mu \in \mathbb{C}$

II. REPRESENTATIONS OF ALGEBRAS & GROUPS

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Ex. TAKE CYCLIC GROUP

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CONSIDER THE GROUP ALG.

$\mathbb{C}C_4$

$\rho : \mathbb{C}C_4 \longrightarrow \mathrm{Mat}_2(\mathbb{C})$ GIVEN BY

$e, g^2 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$g, g^3 \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

& EXTEND LINEARLY

DEGREE 2

NOT FAITHFUL

II. REPRESENTATIONS OF ALGEBRAS & GROUPS

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ρ IS FAITHFUL
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Ex. TAKE $\Bbbk$-ALGEBRA (A, m, u), AND LET $A_{vs} \equiv$ UNDERLYING VS OF A.

$$\rho_{reg} : A \longrightarrow \mathrm{End}_{\Bbbk}(A_{vs})$$
$$a \longmapsto \left[A_{vs} \longrightarrow A_{vs} \right]$$

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Ex. TAKE $\mathbb{k}$-ALGEBRA (A, m, u), AND LET $A_{vs} \equiv$ UNDERLYING VS OF A.

$$\begin{aligned} \rho_{reg} : A &\longrightarrow \mathrm{End}_{\mathbb{k}}(A_{vs}) \\ a &\longmapsto \begin{bmatrix} A_{vs} \longrightarrow A_{vs} \\ b \longmapsto ab \end{bmatrix} \end{aligned}$$

$$\begin{aligned} m : A \otimes A &\longrightarrow A \\ a \otimes b &\longmapsto ab \end{aligned}$$

II. REPRESENTATIONS OF ALGEBRAS & GROUPS

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Ex. TAKE $\Bbbk$-ALGEBRA (A, m, u), AND LET $A_{vs} \equiv$ UNDERLYING VS OF A.

$$\rho_{reg} : A \longrightarrow \mathrm{End}_{\Bbbk}(A_{vs}) \equiv \textbf{REGULAR REPRESENTATION OF } A$$

$$a \longmapsto \begin{bmatrix} A_{vs} \longrightarrow A_{vs} \\ b \longmapsto ab \end{bmatrix}$$

II. REPRESENTATIONS OF ALGEBRAS & GROUPS

$S \equiv$ ALGEBRAIC STRUCTURE   E.G. GROUP, RING, ALGEBRA

A **REPRESENTATION** OF S IS A STRUCTURE MAP $S \xrightarrow{\rho} \mathrm{End}(U)$

FOR $U \equiv$ ANOTHER STRUCTURE   E.G. SET, ABELIAN GROUP, VSPACE (← WE CHOOSE)

TAKE A $\Bbbk$-ALGEBRA $A := (A, m, u)$.

A **REPRESENTATION** OF A

IS A VECTOR SPACE V EQUIPPED

WITH AN ALGEBRA MAP

$$\rho := \rho_V : A \longrightarrow \mathrm{End}_{\Bbbk}(V)$$

DEGREE/DIMENSION OF ρ
$= \dim_{\Bbbk} V$

ρ IS **FAITHFUL**
IF ρ IS INJECTIVE

Ex. TAKE $\Bbbk$-ALGEBRA (A, m, u), AND LET $A_{vs} \equiv$ UNDERLYING VS OF A.

$$\rho_{reg} : A \longrightarrow \mathrm{End}_{\Bbbk}(A_{vs}) \qquad \equiv \textbf{REGULAR REPRESENTATION OF } A$$

$$a \longmapsto \begin{bmatrix} A_{vs} \longrightarrow A_{vs} \\ b \longmapsto ab \end{bmatrix}$$

- $\deg(\rho_{reg}) = \dim_{\Bbbk} A_{vs}$
- FAITHFUL

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MORPHISM

SUBSTRUCTURE

QUOTIENT STRUCTURE

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MORPHISM

You do!

PICK ONE & GUESS THE DEFINITION

HINT: A REP'N IS A VSPACE WITH EXTRA STUFF

SUBSTRUCTURE

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A SUBREP'N OF (V, ρ_V)

IS A SUBSPACE W OF V s.t.

$$\cdots\ W \hookrightarrow V \xrightarrow{\rho(a)} V\ \cdots$$

$\forall a \in A$

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MORPHISM

A SUBREP'N OF (V, ρ_V)

IS A SUBSPACE W OF V $\ni$.

$$\mathrm{IMAGE}\left(W \hookrightarrow V \xrightarrow{\rho(a)} V \right) \subseteq W \quad \forall a \in A$$

QUOTIENT STRUCTURE

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MORPHISM

A SUBREP'N OF (V, ρ_V)

IS A SUBSPACE W OF V ∋.

$$\mathrm{IMAGE}\left(W \hookrightarrow V \xrightarrow{\rho(a)} V\right) \subseteq W \quad \forall a \in A$$

A QUOTIENT REP'N OF (V, ρ_V)

IS A QUOTIENT SPACE V/W OF V ∋.

$$\cdots V \xrightarrow{\rho(a)} V \twoheadrightarrow V/W \cdots \quad \forall a \in A$$

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A **QUOTIENT REP'N** OF (V, ρ_V)

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$$W \subseteq \mathrm{KER}\left(V \xrightarrow{\rho(a)} V \twoheadrightarrow V/W\right) \quad \forall a \in A$$

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$$\rho := \rho_V : A \longrightarrow \mathrm{End}_{\Bbbk}(V)$$

A **REP'N MORPHISM**

$(V, \rho_V) \longrightarrow (V', \rho_{V'})$

IS A LINEAR MAP $\phi : V \longrightarrow V'$

∋.

$$\begin{tikzcd} V \arrow[r, "\rho(a)"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[d, "\phi"] \\ V' \arrow[r, "\rho'(a)"'] & V' \end{tikzcd} \quad \forall a \in A$$

A **SUBREP'N** OF (V, ρ_V)

IS A SUBSPACE W OF V ∋.

$$\mathrm{IMAGE}\left(W \hookrightarrow V \xrightarrow{\rho(a)} V \right) \subseteq W \quad \forall a \in A$$

A **QUOTIENT REP'N** OF (V, ρ_V)

IS A QUOTIENT SPACE V/W OF V ∋.

$$W \subseteq \ker\left(V \xrightarrow{\rho(a)} V \twoheadrightarrow V/W\right) \quad \forall a \in A$$

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$$\rho := \rho_V : A \longrightarrow \mathrm{End}_{\Bbbk}(V)$$

AN **EQUIVALENCE OF REP'NS**

$$(V, \rho_V) \cong (V', \rho_{V'})$$

IS AN INVERTIBLE LINEAR MAP $\phi : V \longrightarrow V'$

$\ni.$

$$\begin{tikzcd} V \arrow[r, "\rho(a)"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[d, "\phi"] \\ V' \arrow[r, "\rho'(a)"'] & V' \end{tikzcd} \quad \forall a \in A$$

↑ WHEN TWO REP'NS ARE CONSIDERED THE SAME

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Ex. $A = \mathbb{C}C_2$

$C_2 = \langle g \mid g^2 = e \rangle$

$V = \mathbb{C}^2$

$\rho : \mathbb{C}C_2 \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$

$e \longmapsto \left[\left\{ \begin{smallmatrix} x \\ y \end{smallmatrix} \right. \mapsto \left\{ \begin{smallmatrix} x \\ y \end{smallmatrix} \right. \right]$

$g \longmapsto \left[\left\{ \begin{smallmatrix} x \\ y \end{smallmatrix} \right. \mapsto \left\{ \begin{smallmatrix} y \\ x \end{smallmatrix} \right. \right]$

& EXTENDING LINEARLY

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& EXTENDING LINEARLY

$\rho' : \mathbb{C}C_2 \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$

$e \longmapsto \left[\left\{\begin{smallmatrix}x\\y\end{smallmatrix}\right. \mapsto \left\{\begin{smallmatrix}x\\y\end{smallmatrix}\right.\right]$

$g \longmapsto \left[\left\{\begin{smallmatrix}x\\y\end{smallmatrix}\right. \mapsto \left\{\begin{smallmatrix}-x\\y\end{smallmatrix}\right.\right]$

& EXTENDING LINEARLY

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$$\begin{tikzcd} V \arrow[r, "a"] \arrow[d, "\phi"'] \arrow[dr, phantom, "2"] & V \arrow[d, "\phi"] \\ V' \arrow[r, "\rho'(a)"'] & V' \end{tikzcd}$$

Ex. $(\mathbb{C}^2, \rho) \cong (\mathbb{C}^2, \rho')$

VIA $\phi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$

$$\left\{ \begin{matrix} x \\ y \end{matrix} \right. \longmapsto \left\{ \begin{matrix} x+y \\ y-x \end{matrix} \right.$$

$\rho : \mathbb{C}C_2 \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$

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$x \mapsto y$

$a = e$ ✓

$a = g$: $x+y$, $y-x$ ✓

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$a = e$ ✓

$a = g$: $y \mapsto x$, $y - x$, $x + y$ ✓

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USED CHANGE OF BASIS

$\rho : \mathbb{C}C_2 \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$

$e \longmapsto \left[\left\{\begin{matrix} x \\ y \end{matrix}\right. \mapsto \left\{\begin{matrix} x \\ y \end{matrix}\right.\right]$

$g \longmapsto \left[\left\{\begin{matrix} x \\ y \end{matrix}\right. \mapsto \left\{\begin{matrix} y \\ x \end{matrix}\right.\right]$

& EXTENDING LINEARLY

$\rho' : \mathbb{C}C_2 \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}^2)$

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A REPRESENTATION OF A IS A VECTOR SPACE V EQUIPPED WITH AN ALGEBRA MAP

$$\rho := \rho_V : A \longrightarrow \mathrm{End}_{\Bbbk}(V)$$

Ex. TAKE CYCLIC GROUP $C_2 = \langle g \mid g^2 = e \rangle$

CONSIDER THE GROUP ALG. $\mathbb{C}C_2$

$\rho : \mathbb{C}C_2 \longrightarrow \mathrm{Mat}_2(\mathbb{C})$ GIVEN BY

$$e \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$g \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

DEGREE 2
FAITHFUL

$$\lambda e + \mu g \longmapsto \lambda \rho(e) + \mu \rho(g) \quad \forall \lambda, \mu \in \mathbb{C}$$

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$$\rho := \rho_V : A \longrightarrow \mathrm{End}_{\Bbbk}(V)$$

TAKE A GROUP G

A REPRESENTATION OF G

IS A VECTOR SPACE V EQUIPPED

WITH A GROUP MAP

$$\rho := \rho_V : G \longrightarrow GL(V) \equiv \mathrm{Aut}_{\Bbbk}(V)$$

EX. TAKE CYCLIC GROUP

$$C_2 = \langle g \mid g^2 = e \rangle$$

CONSIDER THE GROUP ALG.

$$\mathbb{C}C_2$$

$\rho : \mathbb{C}C_2 \longrightarrow \mathrm{Mat}_2(\mathbb{C}^2)$ GIVEN BY

$$e \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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- $\deg(\rho_V) := \dim_{\mathbb{R}} V$
- ρ FAITHFUL IF INJECTIVE
- MORPHISMS, SUBREPS & QUOTIENT REPS DEFINED LIKEWISE

TAKE A GROUP G

A REPRESENTATION OF G

IS A VECTOR SPACE V EQUIPPED

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DEGREE 2

FAITHFUL

$$\lambda e + \mu g \longmapsto \lambda \rho(e) + \mu \rho(g) \quad \forall \lambda, \mu \in \mathbb{C}$$

II. REPRESENTATIONS OF ALGEBRAS & GROUPS

$S \equiv$ ALGEBRAIC STRUCTURE   E.G. GROUP, RING, ALGEBRA

A REPRESENTATION OF S IS A STRUCTURE MAP $S \xrightarrow{\rho} \mathrm{End}(U)$

FOR $U \equiv$ ANOTHER STRUCTURE   E.G. SET, ABELIAN GROUP, VSPACE (← WE CHOOSE)

- $\deg(\rho_V) := \dim_{\mathbb{R}} V$
- ρ FAITHFUL IF INJECTIVE
- MORPHISMS, SUBREPS & QUOTIENT REPS DEFINED LIKEWISE

TAKE A GROUP G

A REPRESENTATION OF G IS A VECTOR SPACE V EQUIPPED WITH A GROUP MAP

$$\rho := \rho_V : G \longrightarrow GL(V) \equiv \mathrm{Aut}_{\mathbb{R}}(V)$$

Ex. TAKE CYCLIC GROUP $C_2 = \langle g \mid g^2 = e \rangle$

$\rho : C_2 \longrightarrow GL_2(\mathbb{C})$ GIVEN BY

$$e \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$g \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

DEGREE 2

FAITHFUL

II. REPRESENTATIONS OF ALGEBRAS & GROUPS

$S \equiv$ ALGEBRAIC STRUCTURE E.G. GROUP, RING, ALGEBRA

A REPRESENTATION OF S IS A STRUCTURE MAP $S \xrightarrow{\rho} \mathrm{End}(U)$

FOR $U \equiv$ ANOTHER STRUCTURE E.G. SET, ABELIAN GROUP, VSPACE
(← WE CHOOSE)

- $\deg(\rho_V) := \dim_{\mathbb{R}} V$
- ρ FAITHFUL IF INJECTIVE

THERE'S A DIFFERENCE BETWEEN FAITHFULNESS FOR GROUPS AND FOR GROUP ALGS.

≡ SEE §1.3.4 ≡

TAKE A GROUP G

A REPRESENTATION OF G IS A VECTOR SPACE V EQUIPPED WITH A GROUP MAP

$$\rho := \rho_V : G \longrightarrow GL(V) \equiv \mathrm{Aut}_{\mathbb{R}}(V)$$

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$$e \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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DEGREE 2

FAITHFUL

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

$S \equiv$ ALGEBRAIC STRUCTURE   E.G. GROUP, RING, ALGEBRA

$U \equiv$ ANOTHER STRUCTURE   E.G. SET, ABELIAN GROUP, VSPACE

WANT S TO CAPTURE SYMMETRIES OF U

$$\boxed{\begin{array}{c}\text{A REPRESENTATION OF } S \\ \text{IS } U \text{ EQUIPPED WITH} \\ \text{A STRUCTURE MAP} \\ S \xrightarrow{\rho} \mathrm{End}(U)\end{array}}$$

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

$S \equiv$ ALGEBRAIC STRUCTURE   E.G. GROUP, RING, ALGEBRA

$U \equiv$ ANOTHER STRUCTURE   E.G. SET, ABELIAN GROUP, VSPACE

WANT S TO CAPTURE SYMMETRIES OF U

A REPRESENTATION OF S IS U EQUIPPED WITH A STRUCTURE MAP $S \xrightarrow{\rho} \mathrm{End}(U)$	AN S-MODULE IS U EQUIPPED WITH AN "ACTION" MAP $S \times U \xrightarrow{\triangleright} U$ COMPATIBLE WITH THE STRUCTURE OF S

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

$S \equiv$ ALGEBRAIC STRUCTURE   E.G. GROUP, RING, ALGEBRA

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WANT S TO CAPTURE SYMMETRIES OF U

REPACKAGING

A REPRESENTATION OF S IS U EQUIPPED WITH A STRUCTURE MAP $S \xrightarrow{\rho} \mathrm{End}(U)$	AN S-MODULE IS U EQUIPPED WITH AN "ACTION" MAP $S \times U \xrightarrow{\triangleright} U$ COMPATIBLE WITH THE STRUCTURE OF S

$$[\rho : S \longrightarrow \mathrm{End}(U)] \longmapsto$$

$$\longleftarrow\!\!| \; [\triangleright : S \times U \longrightarrow U]$$

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

$S \equiv$ ALGEBRAIC STRUCTURE E.G. GROUP, RING, ALGEBRA

$U \equiv$ ANOTHER STRUCTURE E.G. SET, ABELIAN GROUP, VSPACE

WANT S TO CAPTURE SYMMETRIES OF U

REPACKAGING

A REPRESENTATION OF S IS U EQUIPPED WITH A STRUCTURE MAP $S \xrightarrow{\rho} \mathrm{End}(U)$	AN S-MODULE IS U EQUIPPED WITH AN "ACTION" MAP $S \times U \xrightarrow{\triangleright} U$ COMPATIBLE WITH THE STRUCTURE OF S

$$[\rho : S \longrightarrow \mathrm{End}(U)] \longmapsto s \triangleright u := \rho(s)(u) \quad \forall s \in S,\ u \in U$$

$$\rho(s)(u) := s \triangleright u \quad \forall s \in S,\ u \in U \longleftarrow [\triangleright : S \times U \longrightarrow U]$$

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

$\mathcal{S} \equiv$ ALGEBRAIC STRUCTURE E.G. GROUP, RING, ALGEBRA

$\mathcal{U} \equiv$ ANOTHER STRUCTURE E.G. SET, ABELIAN GROUP, VSPACE

WANT $\mathcal{S}$ TO CAPTURE SYMMETRIES OF $\mathcal{U}$

GET BIJECTION (SEE EXERCISE 1.12)

A REPRESENTATION OF $\mathcal{S}$ IS $\mathcal{U}$ EQUIPPED WITH A STRUCTURE MAP $\mathcal{S} \xrightarrow{\rho} \operatorname{End}(\mathcal{U})$	AN $\mathcal{S}$-MODULE IS $\mathcal{U}$ EQUIPPED WITH AN "ACTION" MAP $\mathcal{S} \times \mathcal{U} \xrightarrow{\triangleright} \mathcal{U}$ COMPATIBLE WITH THE STRUCTURE OF $\mathcal{S}$

$$\left[\rho : \mathcal{S} \longrightarrow \operatorname{End}(\mathcal{U})\right] \longmapsto \quad s \triangleright u := \rho(s)(u) \quad \forall s \in \mathcal{S},\, u \in \mathcal{U}$$

$$\rho(s)(u) := s \triangleright u \quad \forall s \in \mathcal{S},\, u \in \mathcal{U} \quad \longleftarrow \left[\triangleright : \mathcal{S} \times \mathcal{U} \longrightarrow \mathcal{U}\right]$$

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

$S \equiv$ ALGEBRAIC STRUCTURE   E.G. GROUP, RING, ALGEBRA

$U \equiv$ ANOTHER STRUCTURE   E.G. SET, ABELIAN GROUP, VSPACE

WANT S TO CAPTURE SYMMETRIES OF U

GET BIJECTION (SEE EXERCISE 1.12)

A REPRESENTATION OF S IS U EQUIPPED WITH A STRUCTURE MAP $S \xrightarrow{\rho} \mathrm{End}(U)$	AN S-MODULE IS U EQUIPPED WITH AN "ACTION" MAP $S \times U \xrightarrow{\triangleright} U$ COMPATIBLE WITH THE STRUCTURE OF S

NICE FOR STUDYING SYMMETRIES VIA

PROPERTIES OF S (E.G. FAITHFULNESS)

PROPERTIES OF U (E.G. DEGREE)

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

$\mathcal{S} \equiv$ ALGEBRA $A = (A,\ m: A \otimes A \to A,\ u: \mathbb{k} \to A)$

$\mathcal{U} \equiv$ VECTOR SPACE V

A REPRESENTATION OF $\mathcal{S}$ IS $\mathcal{U}$ EQUIPPED WITH A STRUCTURE MAP

$$\mathcal{S} \xrightarrow{\rho} \mathrm{End}(\mathcal{U})$$

AN $\mathcal{S}$-MODULE IS $\mathcal{U}$ EQUIPPED WITH AN "ACTION" MAP $\mathcal{S} \times \mathcal{U} \xrightarrow{\triangleright} \mathcal{U}$ COMPATIBLE WITH THE STRUCTURE OF $\mathcal{S}$

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

$\mathcal{S} \equiv$ ALGEBRA $A = (A,\ m: A \otimes A \to A,\ u: \Bbbk \to A)$

$\mathcal{U} \equiv$ VECTOR SPACE V

A REPRESENTATION OF A
IS V EQUIPPED WITH
AN ALGEBRA MAP
$\rho := \rho_V : A \longrightarrow \mathrm{End}_{\Bbbk}(V)$

A REPRESENTATION OF $\mathcal{S}$
IS $\mathcal{U}$ EQUIPPED WITH
A STRUCTURE MAP
$\mathcal{S} \xrightarrow{\rho} \mathrm{End}(\mathcal{U})$

AN $\mathcal{S}$-MODULE
IS $\mathcal{U}$ EQUIPPED WITH
AN "ACTION" MAP $\mathcal{S} \times \mathcal{U} \xrightarrow{\triangleright} \mathcal{U}$
COMPATIBLE WITH THE STRUCTURE OF $\mathcal{S}$

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

$\mathcal{S} \equiv$ ALGEBRA $A = (A,\ m : A \otimes A \to A,\ u : \Bbbk \to A)$

$\mathcal{U} \equiv$ VECTOR SPACE V

A **REPRESENTATION** OF A

IS V EQUIPPED WITH

AN ALGEBRA MAP

$\rho := \rho_V : A \longrightarrow \mathrm{End}_{\Bbbk}(V)$

A **LEFT A-MODULE**

IS V EQUIPPED WITH

A LINEAR MAP $\triangleright : A \otimes V \longrightarrow V$

s.t.

$$\begin{tikzcd}
A \otimes (A \otimes V) \arrow[r, "\mathrm{id} \otimes \triangleright"] \arrow[d, "\cong"'] & A \otimes V \arrow[dd, "\triangleright"] \\
(A \otimes A) \otimes V \arrow[d, "m \otimes \mathrm{id}"'] \arrow[r, phantom, "\text{comp. w/ } m_A"] & \\
A \otimes V \arrow[r, "\triangleright"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes V \arrow[r, "u \otimes \mathrm{id}"] \arrow[d, "\cong"'] & A \otimes V \arrow[dd, "\triangleright"] \\
V \arrow[dr, "\mathrm{id}"'] \arrow[r, phantom, "\text{comp. w/ } u_A"] & \\
& V
\end{tikzcd}$$

A **REPRESENTATION** OF $\mathcal{S}$

IS $\mathcal{U}$ EQUIPPED WITH

A STRUCTURE MAP

$\mathcal{S} \xrightarrow{\rho} \mathrm{End}(\mathcal{U})$

AN $\mathcal{S}$-**MODULE**

IS $\mathcal{U}$ EQUIPPED WITH

AN "ACTION" MAP $\mathcal{S} \times \mathcal{U} \xrightarrow{\triangleright} \mathcal{U}$

COMPATIBLE WITH THE STRUCTURE OF $\mathcal{S}$

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

$\mathcal{S} \equiv$ ALGEBRA $A = (A,\ m: A\otimes A \to A,\ u: \Bbbk \to A)$

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A REPRESENTATION OF A
IS V EQUIPPED WITH
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$\rho := \rho_V : A \longrightarrow \mathrm{End}_{\Bbbk}(V)$

A LEFT A-MODULE
IS V EQUIPPED WITH
A LINEAR MAP $\triangleright : A\otimes V \longrightarrow V$

$\ni.$

$$\begin{tikzcd} A\otimes(A\otimes V) \arrow[r,"\mathrm{id}\otimes\triangleright"] \arrow[d,"\cong"'] & A\otimes V \arrow[dd,"\triangleright"] \\ (A\otimes A)\otimes V \arrow[d,"m\otimes\mathrm{id}"'] \arrow[r, phantom, "\circlearrowleft\ \text{comp. w/ } m_A"] & \\ A\otimes V \arrow[r,"\triangleright"] & V \end{tikzcd}
\qquad
\begin{tikzcd} \Bbbk\otimes V \arrow[r,"u\otimes\mathrm{id}"] \arrow[d,"\cong"'] & A\otimes V \arrow[d,"\triangleright"] \\ V \arrow[r,"\mathrm{id}"'] \arrow[ur, phantom, "\circlearrowleft\ \text{comp. w/ } u_A"] & V \end{tikzcd}$$

RIGHT A-MODULE $(V,\ \triangleleft : V\otimes A \to V)$ DEFINED LIKEWISE

A REPRESENTATION OF $\mathcal{S}$
IS $\mathcal{U}$ EQUIPPED WITH
A STRUCTURE MAP
$\mathcal{S} \xrightarrow{\rho} \mathrm{End}(\mathcal{U})$

AN $\mathcal{S}$-MODULE
IS $\mathcal{U}$ EQUIPPED WITH
AN "ACTION" MAP $\mathcal{S} \times \mathcal{U} \xrightarrow{\triangleright} \mathcal{U}$
COMPATIBLE WITH THE STRUCTURE OF $\mathcal{S}$

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

$\mathcal{S} \equiv$ ALGEBRA $A = (A,\ m : A \otimes A \longrightarrow A,\ u : \Bbbk \longrightarrow A)$

$\mathcal{U} \equiv$ VECTOR SPACE V

A REPRESENTATION OF A

IS V EQUIPPED WITH

AN ALGEBRA MAP

$\rho := \rho_V : A \longrightarrow \mathrm{End}_{\Bbbk}(V)$

A LEFT A-MODULE

IS V EQUIPPED WITH

A LINEAR MAP $\triangleright : A \otimes V \longrightarrow V$

s.t.

$$\begin{tikzcd}
A \otimes (A \otimes V) \arrow[r, "id \otimes \triangleright"] \arrow[d, "\cong"'] & A \otimes V \arrow[dd, "\triangleright"] \\
(A \otimes A) \otimes V \arrow[d, "m \otimes id"'] \arrow[r, phantom, "= \text{ comp. w/ } m_A"] & \\
A \otimes V \arrow[r, "\triangleright"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes V \arrow[r, "u \otimes id"] \arrow[d, "\cong"'] & A \otimes V \arrow[d, "\triangleright"] \\
V \arrow[r, "id"'] \arrow[ur, phantom, "= \text{ comp. w/ } u_A"] & V
\end{tikzcd}$$

Ex. TAKE CYCLIC GROUP

$C_2 = \langle g \mid g^2 = e \rangle$

CONSIDER THE GROUP ALG.

$\mathbb{C}C_2$

$\rho : \mathbb{C}C_2 \longrightarrow \mathrm{Mat}_2(\Bbbk)$ GIVEN BY

$e \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$g \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\lambda e + \mu g \longmapsto \lambda \rho(e) + \mu \rho(g) \quad \forall \lambda, \mu \in \mathbb{C}$

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

$\mathcal{S} \equiv$ ALGEBRA $A = (A,\ m: A \otimes A \longrightarrow A,\ u: \Bbbk \longrightarrow A)$

$\mathcal{U} \equiv$ VECTOR SPACE V

A REPRESENTATION OF A IS V EQUIPPED WITH AN ALGEBRA MAP

$$\rho := \rho_V : A \longrightarrow \mathrm{End}_{\Bbbk}(V)$$

A LEFT A-MODULE IS V EQUIPPED WITH A LINEAR MAP $\triangleright : A \otimes V \longrightarrow V$

$\ni.$

$$\begin{tikzcd}
A \otimes (A \otimes V) \arrow[r, "\mathrm{id} \otimes \triangleright"] \arrow[d, "\cong"'] & A \otimes V \arrow[dd, "\triangleright"] \\
(A \otimes A) \otimes V \arrow[d, "m \otimes \mathrm{id}"'] & \\
A \otimes V \arrow[r, "\triangleright"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes V \arrow[r, "u \otimes \mathrm{id}"] \arrow[d, "\cong"'] & A \otimes V \arrow[d, "\triangleright"] \\
V \arrow[r, "\mathrm{id}"'] & V
\end{tikzcd}$$

Ex. TAKE CYCLIC GROUP

$$C_2 = \langle g \mid g^2 = e \rangle$$

CONSIDER THE GROUP ALG.

$$\mathbb{C}C_2$$

$\triangleright : \mathbb{C}C_2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ GIVEN BY

$$(e, \{\begin{smallmatrix} x \\ y \end{smallmatrix}) \longmapsto \{\begin{smallmatrix} x \\ y \end{smallmatrix}$$

$$(g, \{\begin{smallmatrix} x \\ y \end{smallmatrix}) \longmapsto \{\begin{smallmatrix} y \\ x \end{smallmatrix}$$

$$(\lambda e + \mu g, \{\begin{smallmatrix} x \\ y \end{smallmatrix})) \longmapsto \lambda(e \triangleright \{\begin{smallmatrix} x \\ y \end{smallmatrix}) + \mu(g \triangleright \{\begin{smallmatrix} x \\ y \end{smallmatrix}) \quad \forall \lambda, \mu \in \mathbb{C}$$

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

ALGEBRA $A=(A,m,u)$ VECTOR SPACE V

- $\dim(V,\triangleright) \equiv \dim_{\mathbb{R}} V$

  INSTEAD OF "DEGREE"

A LEFT A-MODULE

IS V EQUIPPED WITH

A LINEAR MAP $\triangleright : A\otimes V \longrightarrow V$

s.t.

$$
\begin{tikzcd}
A\otimes(A\otimes V) \arrow[r, "id\otimes\triangleright"] \arrow[d, "\cong"'] & A\otimes V \arrow[dd, "\triangleright"] \\
(A\otimes A)\otimes V \arrow[d, "m\otimes id"'] & \\
A\otimes V \arrow[r, "\triangleright"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk\otimes V \arrow[r, "u\otimes id"] \arrow[d, "\cong"'] & A\otimes V \arrow[d, "\triangleright"] \\
V \arrow[r, "id"'] & V
\end{tikzcd}
$$

Ex. TAKE CYCLIC GROUP

$C_2 = \langle g \mid g^2 = e\rangle$

CONSIDER THE GROUP ALG.

$\mathbb{C}C_2$

$\triangleright : \mathbb{C}C_2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ GIVEN BY

$\left(e, \left\{\begin{matrix}x\\y\end{matrix}\right.\right) \longmapsto \left\{\begin{matrix}x\\y\end{matrix}\right.$ DEGREE 2

$\left(g, \left\{\begin{matrix}x\\y\end{matrix}\right.\right) \longmapsto \left\{\begin{matrix}y\\x\end{matrix}\right.$

$\left(\lambda e+\mu g, \left\{\begin{matrix}x\\y\end{matrix}\right.\right) \mapsto \lambda\left(e\triangleright\left\{\begin{matrix}x\\y\end{matrix}\right.\right)+\mu\left(g\triangleright\left\{\begin{matrix}x\\y\end{matrix}\right.\right) \quad \forall \lambda,\mu\in\mathbb{C}$

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

ALGEBRA $A=(A,m,u)$ VECTOR SPACE V

A **LEFT A-MODULE** IS V EQUIPPED WITH A LINEAR MAP $\rhd : A\otimes V \longrightarrow V$ s.t.

$$
\begin{tikzcd}
A\otimes(A\otimes V) \arrow[r, "\mathrm{id}\otimes\rhd"] \arrow[d, "\cong"] & A\otimes V \arrow[dd, "\rhd"] \\
(A\otimes A)\otimes V \arrow[d, "m\otimes \mathrm{id}"'] \arrow[r, phantom, "\circlearrowright"] & \\
A\otimes V \arrow[r, "\rhd"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk\otimes V \arrow[r, "u\otimes \mathrm{id}"] \arrow[d, "\cong"] & A\otimes V \arrow[d, "\rhd"] \\
V \arrow[r, "\mathrm{id}"] \arrow[ur, phantom, "\circlearrowright"] & V
\end{tikzcd}
$$

- **$\dim(V,\rhd)$** $\equiv \dim_{\Bbbk} V$

- $(V,\rhd)$ IS **FAITHFUL** IF $\nexists\, I \neq 0$ IDEAL OF A s.t.

$$\left(V,\ \begin{matrix} A/I \otimes V \xrightarrow{\bar{\rhd}} V \\ (a+I)\otimes v \longmapsto (a\rhd v)+I \end{matrix}\right)$$

IS A LEFT MODULE OVER A/I

Ex. TAKE CYCLIC GROUP $C_2=\langle g \mid g^2=e\rangle$

CONSIDER THE GROUP ALG. $\mathbb{C}C_2$

$\rhd : \mathbb{C}C_2\times\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ GIVEN BY

$\left(e, \left\{\begin{matrix}x\\y\end{matrix}\right.\right) \longmapsto \left\{\begin{matrix}x\\y\end{matrix}\right.$ DEGREE 2

$\left(g, \left\{\begin{matrix}x\\y\end{matrix}\right.\right) \longmapsto \left\{\begin{matrix}y\\x\end{matrix}\right.$ FAITHFUL

$\left(\lambda e+\mu g, \left\{\begin{matrix}x\\y\end{matrix}\right.\right) \mapsto \lambda\left(e\rhd\left\{\begin{matrix}x\\y\end{matrix}\right.\right)+\mu\left(g\rhd\left\{\begin{matrix}x\\y\end{matrix}\right.\right)$ $\forall \lambda,\mu\in\mathbb{C}$

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

ALGEBRA $A = (A, m, u)$ VECTOR SPACE V

A **LEFT A-MODULE** IS V EQUIPPED WITH A LINEAR MAP $\triangleright : A \otimes V \longrightarrow V$ ∋

$$\begin{tikzcd} A\otimes(A\otimes V) \arrow[r, "id\otimes\triangleright"] & A\otimes V \arrow[dd, "\triangleright"] \\ (A\otimes A)\otimes V \arrow[u, "\cong"] \arrow[d, "m\otimes id"'] \arrow[r, phantom, "2"] & \\ A\otimes V \arrow[r, "\triangleright"] & V \end{tikzcd} \qquad \begin{tikzcd} \Bbbk\otimes V \arrow[r, "u\otimes id"] & A\otimes V \arrow[dd, "\triangleright"] \\ V \arrow[u, "\cong"] \arrow[ddr, phantom, "2"] \arrow[dr, "id"'] & \\ & V \end{tikzcd}$$

- $\dim(V, \triangleright) \equiv \dim_{\Bbbk} V$

- $(V, \triangleright)$ IS **FAITHFUL** IF $\nexists\, I \neq 0$ IDEAL OF A ∋

$$\left(V,\ \begin{matrix} A/I \otimes V \xrightarrow{\overline{\triangleright}} V \\ (a+I)\otimes v \longmapsto (a\triangleright v)+I \end{matrix} \right)$$

IS A LEFT MODULE OVER A/I

Ex. TAKE $\Bbbk$-ALGEBRA (A, m, u), AND LET $A_{vs} \equiv$ UNDERLYING VS OF A.

$$\triangleright_{reg} : \begin{matrix} A\otimes A_{vs} \longrightarrow A_{vs} \\ (a, b) \longmapsto ab \end{matrix}$$

$\equiv$ **REGULAR LEFT A-MODULE**

- $\dim(\triangleright_{reg}) = \dim_{\Bbbk} A_{vs}$
- FAITHFUL

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

ALGEBRA $A = (A, m, u)$ VECTOR SPACE V

MORPHISMS

A **LEFT A-MODULE** IS V EQUIPPED WITH A LINEAR MAP $\triangleright : A \otimes V \longrightarrow V$

s.t.

$$
\begin{tikzcd}
A \otimes (A \otimes V) \arrow[r, "id \otimes \triangleright"] \arrow[d, "\cong"'] & A \otimes V \arrow[dd, "\triangleright"] \\
(A \otimes A) \otimes V \arrow[d, "m \otimes id"'] & \\
A \otimes V \arrow[r, "\triangleright"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes V \arrow[r, "u \otimes id"] \arrow[d, "\cong"'] & A \otimes V \arrow[d, "\triangleright"] \\
V \arrow[r, "id"'] & V
\end{tikzcd}
$$

SUBSTRUCTURES

QUOTIENT STRUCTURES

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

ALGEBRA $A = (A, m, u)$ VECTOR SPACE V

MORPHISMS

A LEFT A-MODULE MAP

$$(V, \triangleright) \longrightarrow (V', \triangleright')$$

IS A LINEAR MAP $\phi : V \to V'$ s.t.

$$\begin{tikzcd}
A \otimes V \arrow[r, "\triangleright"] \arrow[d, "\mathrm{id} \otimes \phi"'] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[d, "\phi"] \\
A \otimes V' \arrow[r, "\triangleright'"] & V'
\end{tikzcd}$$

A LEFT A-MODULE

IS V EQUIPPED WITH

A LINEAR MAP $\triangleright : A \otimes V \longrightarrow V$

s.t.

$$\begin{tikzcd}
A \otimes (A \otimes V) \arrow[r, "\mathrm{id} \otimes \triangleright"] \arrow[d, phantom, "\cong" rotate=90] & A \otimes V \arrow[dd, "\triangleright"] \\
(A \otimes A) \otimes V \arrow[d, "m \otimes \mathrm{id}"'] \arrow[r, phantom, "\circlearrowleft"] & \\
A \otimes V \arrow[r, "\triangleright"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes V \arrow[r, "u \otimes \mathrm{id}"] \arrow[d, phantom, "\cong" rotate=90] & A \otimes V \arrow[dd, "\triangleright"] \\
V \arrow[dr, "\mathrm{id}"'] \arrow[r, phantom, "\circlearrowleft"] & \\
& V
\end{tikzcd}$$

SUBSTRUCTURES

QUOTIENT STRUCTURES

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

ALGEBRA $A = (A, m, u)$ VECTOR SPACE V

A LEFT A-MODULE

IS V EQUIPPED WITH

A LINEAR MAP $\triangleright : A \otimes V \longrightarrow V$

s.t.

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A \otimes (A \otimes V) \arrow[r, "\mathrm{id} \otimes \triangleright"] \arrow[d, phantom, "\cong" rotate=90] & A \otimes V \arrow[dd, "\triangleright"] \\
(A \otimes A) \otimes V \arrow[d, "m \otimes \mathrm{id}"'] \arrow[r, phantom, "\circlearrowleft"] & \\
A \otimes V \arrow[r, "\triangleright"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes V \arrow[r, "u \otimes \mathrm{id}"] \arrow[d, phantom, "\cong" rotate=90] & A \otimes V \arrow[dd, "\triangleright"] \\
V \arrow[drr, "\mathrm{id}"'] \arrow[r, phantom, "\circlearrowleft"] & \\
& V
\end{tikzcd}$$

MORPHISMS

A LEFT A-MODULE MAP

$$(V, \triangleright) \longrightarrow (V', \triangleright')$$

IS A LINEAR MAP $\phi : V \to V'$ s.t.

$$\phi \circ \triangleright = \triangleright' \circ (\mathrm{id}_A \otimes \phi)$$

$(V, \triangleright) \cong (V', \triangleright')$ ISOMORPHISM

WHEN ϕ IS INVERTIBLE

SUBSTRUCTURES

QUOTIENT STRUCTURES

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

ALGEBRA $A = (A, m, u)$ VECTOR SPACE V

MORPHISMS

A LEFT A-MODULE MAP

$$(V, \triangleright) \longrightarrow (V', \triangleright')$$

IS A LINEAR MAP $\phi : V \to V'$ $\ni$.

$$\phi \circ \triangleright = \triangleright' \circ (\mathrm{id}_A \otimes \phi)$$

$(V, \triangleright) \cong (V', \triangleright')$ ISOMORPHISM WHEN ϕ IS INVERTIBLE

A LEFT A-MODULE IS V EQUIPPED WITH A LINEAR MAP $\triangleright : A \otimes V \longrightarrow V$

$\ni$.

$$\begin{tikzcd}
A \otimes (A \otimes V) \arrow[r, "\mathrm{id} \otimes \triangleright"] \arrow[d, "\cong"'] & A \otimes V \arrow[dd, "\triangleright"] \\
(A \otimes A) \otimes V \arrow[d, "m \otimes \mathrm{id}"'] \arrow[r, phantom, "\circlearrowleft"] & \\
A \otimes V \arrow[r, "\triangleright"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes V \arrow[r, "u \otimes \mathrm{id}"] \arrow[d, "\cong"'] & A \otimes V \arrow[d, "\triangleright"] \\
V \arrow[r, "\mathrm{id}"'] \arrow[ur, phantom, "\circlearrowleft"] & V
\end{tikzcd}$$

PLEASE READ ABOUT

SUBSTRUCTURES $\longleftarrow$ SUBMODULES

&

QUOTIENT STRUCTURES $\longleftarrow$ QUOTIENT MODULES

IN §1.3.2

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

ALGEBRA $A = (A, m, u)$ VECTOR SPACE V

MORPHISMS

A LEFT A-MODULE MAP

$$(V, \triangleright) \longrightarrow (V', \triangleright')$$

IS A LINEAR MAP $\phi : V \to V'$ $\ni$.

$$\begin{tikzcd}
A \otimes V \arrow[r, "\triangleright"] \arrow[d, "\mathrm{id} \otimes \phi"'] \arrow[dr, phantom, "\circlearrowleft"] & V \arrow[d, "\phi"] \\
A \otimes V' \arrow[r, "\triangleright'"] & V'
\end{tikzcd}$$

A LEFT A-MODULE

IS V EQUIPPED WITH

A LINEAR MAP $\triangleright : A \otimes V \longrightarrow V$

$\ni$.

$$\begin{tikzcd}
A \otimes (A \otimes V) \arrow[r, "\mathrm{id} \otimes \triangleright"] \arrow[d, phantom, "\cong" rotate=90] & A \otimes V \arrow[dd, "\triangleright"] \\
(A \otimes A) \otimes V \arrow[d, "m \otimes \mathrm{id}"'] \arrow[r, phantom, "\circlearrowleft"] & \\
A \otimes V \arrow[r, "\triangleright"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes V \arrow[r, "u \otimes \mathrm{id}"] \arrow[d, phantom, "\cong" rotate=90] & A \otimes V \arrow[d, "\triangleright"] \\
V \arrow[r, "\mathrm{id}"'] \arrow[ur, phantom, "\circlearrowleft"] & V
\end{tikzcd}$$

THINK ABOUT ANALOGOUS NOTIONS FOR RIGHT A-MODULES

PLEASE READ ABOUT

SUBSTRUCTURES $\longleftarrow$ SUBMODULES

&

QUOTIENT STRUCTURES $\longleftarrow$ QUOTIENT MODULES

IN §1.3.2

II. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

ALGEBRA $A = (A, m, u)$ VECTOR SPACE V

MORPHISMS

A LEFT A-MODULE MAP

$$(V, \triangleright) \longrightarrow (V', \triangleright')$$

IS A LINEAR MAP $\phi : V \to V'$ $\ni$.

$$\begin{tikzcd}
A \otimes V \arrow[r, "\triangleright"] \arrow[d, "\mathrm{id} \otimes \phi"'] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "\phi"] \\
A \otimes V' \arrow[r, "\triangleright'"] & V'
\end{tikzcd}$$

A LEFT A-MODULE

IS V EQUIPPED WITH

A LINEAR MAP $\triangleright : A \otimes V \longrightarrow V$

$\ni$.

$$\begin{tikzcd}
A \otimes (A \otimes V) \arrow[r, "\mathrm{id} \otimes \triangleright"] \arrow[d, "\cong" sloped] & A \otimes V \arrow[dd, "\triangleright"] \\
(A \otimes A) \otimes V \arrow[d, "m \otimes \mathrm{id}"'] \arrow[r, phantom, "\circlearrowright"] & \\
A \otimes V \arrow[r, "\triangleright"] & V
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes V \arrow[r, "u \otimes \mathrm{id}"] \arrow[d, "\cong" sloped] & A \otimes V \arrow[dd, "\triangleright"] \\
V \arrow[dr, "\mathrm{id}"'] \arrow[r, phantom, "\circlearrowright"] & \\
& V
\end{tikzcd}$$

PLEASE READ ABOUT

SUBSTRUCTURES $\longleftarrow$ SUBMODULES

&

QUOTIENT STRUCTURES $\longleftarrow$ QUOTIENT MODULES

IN §1.3.2

THINK ABOUT ANALOGOUS NOTIONS FOR RIGHT A-MODULES

& MODULES OVER GROUPS G IN §1.3.4

III. MODULES AND $\underbrace{\text{BIMODULES}}$ OVER ALGEBRAS & GROUPS

PUTTING LEFT & RIGHT MODULES TOGETHER —

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

PUTTING LEFT & RIGHT MODULES TOGETHER —

TAKE ALGEBRAS (B_1, m_1, u_1) & (B_2, m_2, u_2)

A (B_1, B_2)-BIMODULE

IS A VECTOR SPACE V

EQUIPPED WITH LINEAR MAPS

$\triangleright : B_1 \otimes V \to V$ AND $\triangleleft : V \otimes B_2 \to V$

.∋. $(V, \triangleright) \equiv$ LEFT B_1-MODULE

$(V, \triangleleft) \equiv$ RIGHT B_2-MODULE

&

$$\begin{tikzcd}
(B_1 \otimes V) \otimes B_2 \arrow[r, "\cong"] \arrow[d, "\triangleright \otimes id"'] & B_1 \otimes (V \otimes B_2) \arrow[r, "id \otimes \triangleleft"] \arrow[d, phantom, "\circlearrowright"] & B_1 \otimes V \arrow[d, "\triangleright"] \\
V \otimes B_2 \arrow[rr, "\triangleleft"] & & V
\end{tikzcd}$$

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

PUTTING LEFT & RIGHT MODULES TOGETHER —

TAKE ALGEBRAS (B_1, m_1, u_1) & (B_2, m_2, u_2)

A (B_1, B_2)-BIMODULE

IS A VECTOR SPACE* V

EQUIPPED WITH LINEAR MAPS**

$\triangleright : B_1 \otimes V \to V$ AND $\triangleleft : V \otimes B_2 \to V$

s.t. $(V, \triangleright) \equiv$ LEFT B_1-MODULE

$(V, \triangleleft) \equiv$ RIGHT B_2-MODULE

&

$$\begin{tikzcd}
 & B_1 \otimes (V \otimes B_2) \arrow[r, "\mathrm{id} \otimes \triangleleft"] & B_1 \otimes V \arrow[dd, "\triangleright"] \\
(B_1 \otimes V) \otimes B_2 \arrow[ur, "\cong"] \arrow[d, "\triangleright \otimes \mathrm{id}"'] & & \\
V \otimes B_2 \arrow[rr, "\triangleleft"] & & V
\end{tikzcd}$$

* JUST OUR CHOICE HERE. COULD BE A { SET, ABELIAN GROUP

$\triangleright, \triangleleft$

** { FUNCTION, GROUP MAP

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

PUTTING LEFT & RIGHT MODULES TOGETHER —

TAKE ALGEBRAS (B_1, m_1, u_1) & (B_2, m_2, u_2)

A (B_1, B_2)-BIMODULE

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s.t. $(V, \triangleright) \equiv$ LEFT B_1-MODULE

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& $\triangleleft \circ (\triangleright \otimes id_{B_2}) = \triangleright \circ (id_{B_1} \otimes \triangleleft)$

THIS IS CALLED AN A-BIMODULE

WHEN $B_1 = B_2 =: A$ (ALGEBRA).

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

PUTTING LEFT & RIGHT MODULES TOGETHER —

TAKE ALGEBRAS (B_1, m_1, u_1) & (B_2, m_2, u_2)

A (B_1, B_2)-BIMODULE IS A VECTOR SPACE V EQUIPPED WITH LINEAR MAPS

$\triangleright : B_1 \otimes V \to V$ AND $\triangleleft : V \otimes B_2 \to V$

s.t. $(V, \triangleright) \equiv$ LEFT B_1-MODULE

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THIS IS CALLED AN A-BIMODULE WHEN $B_1 = B_2 =: A$ (ALGEBRA).

- A MAP OF (B_1, B_2)-BIMODULES $(V, \triangleright, \triangleleft) \longrightarrow (V', \triangleright', \triangleleft')$ IS A LINEAR MAP $\phi : V \to V'$ THAT IS A MAP OF $\begin{cases} \text{LEFT } B_1\text{-MODULES \&} \\ \text{RIGHT } B_2\text{-MODULES} \end{cases}$

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

PUTTING LEFT & RIGHT MODULES TOGETHER —

TAKE ALGEBRAS (B_1, m_1, u_1) & (B_2, m_2, u_2)

A (B_1, B_2)-BIMODULE

IS A VECTOR SPACE V

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  - LEFT B_1-MODULES &
  - RIGHT B_2-MODULES

CAN DEFINE

AN ISOMORPHISM OF BIMODULES

SUBBIMODULES

QUOTIENT BIMODULES

IN A SIMILAR MANNER

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

PUTTING LEFT & RIGHT MODULES TOGETHER —

TAKE ALGEBRAS (B_1, m_1, u_1) & (B_2, m_2, u_2)

A (B_1, B_2)-BIMODULE

IS A VECTOR SPACE V

EQUIPPED WITH LINEAR MAPS

$\triangleright : B_1 \otimes V \to V$ AND $\triangleleft : V \otimes B_2 \to V$

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THIS IS CALLED AN A-BIMODULE

WHEN $B_1 = B_2 =: A$ (ALGEBRA).

- A MAP OF (B_1, B_2)-BIMODULES

  $(V, \triangleright, \triangleleft) \longrightarrow (V', \triangleright', \triangleleft')$

  IS A LINEAR MAP $\phi : V \to V'$

  THAT IS A MAP OF $\begin{cases} \text{LEFT } B_1\text{-MODULES \&} \\ \text{RIGHT } B_2\text{-MODULES} \end{cases}$

CAN DEFINE

REGULAR A-BIMODULE

DIMENSION

$= \dim_{\mathbb{R}} V$

IN A SIMILAR MANNER

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS

PUTTING LEFT & RIGHT MODULES TOGETHER —

TAKE ALGEBRAS (B_1, m_1, u_1) & (B_2, m_2, u_2)

A (B_1, B_2)-BIMODULE

IS A VECTOR SPACE V

EQUIPPED WITH LINEAR MAPS

$\triangleright : B_1 \otimes V \to V$ AND $\triangleleft : V \otimes B_2 \to V$

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& $\triangleleft \circ (\triangleright \otimes id_{B_2}) = \triangleright \circ (id_{B_1} \otimes \triangleleft)$

THIS IS CALLED AN A-BIMODULE

WHEN $B_1 = B_2 =: A$ (ALGEBRA).

BUT FAITHFULNESS

IS A

ONE-SIDED NOTION

CAN DEFINE

REGULAR A-BIMODULE

DIMENSION

$= \dim_{\mathbb{k}} V$

IN A SIMILAR MANNER

MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #3

TOPICS:

I. EXAMPLES OF ALGEBRAS OVER A FIELD: $\Bbbk Q$, $\Bbbk G$ (§§1.2.5, 1.2.6)

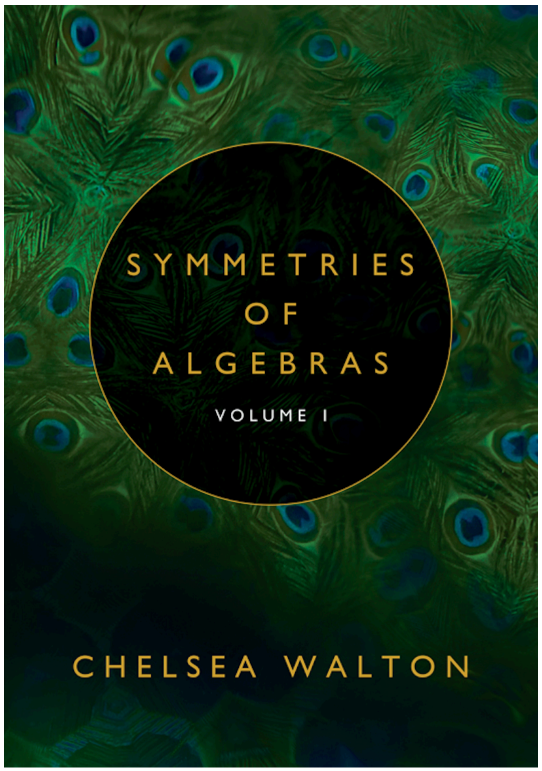
II. REPRESENTATIONS OF ALGEBRAS & GROUPS (§§1.3.1, 1.3.4)

III. MODULES AND BIMODULES OVER ALGEBRAS & GROUPS (§§1.3.2–1.3.4)

NEXT TIME: OPERATIONS ON ALGEBRAS & MODULES

Enjoy this lecture?
You'll enjoy the textbook!

C. Walton's "Symmetries of Algebras, Volume 1" (2024)



Lecture #3 keywords: bimodule over an algebra, faithfulness, group algebra, module over an algebra, path algebra, quiver, representation of an algebra