MATH 466/566
SPRING 2024

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LECTURE #4

LAST TIME

- $\mathbb{R}Q$, $\mathbb{R}G$
- REPRESENTATIONS
- (BI)MODULES

TOPICS:

I. DIRECT PRODUCT, SUM, DIRECT SUM OF ALGEBRAS & MODULES (§1.4.1)

II. TENSOR PRODUCT OF ALGEBRAS & MODULES (§§1.4.2, 1.4.4)

III. HOM AND DUAL OF ALGEBRAS & MODULES (§§1.4.3, 1.4.4)

RECALL OUR STRUCTURES OF INTEREST — $/\Bbbk$ FIELD. $\otimes := \otimes_{\Bbbk}$

A $(\Bbbk\text{-})$ ALGEBRA IS A TRIPLE

MULTIPLICATION, UNIT

$$\big(A,\ \underset{\text{MULTIPLICATION}}{m} : A \otimes A \longrightarrow A,\ \underset{\text{UNIT}}{u} : \Bbbk \longrightarrow A\big)$$

$\Bbbk$-VS

$\Bbbk$- LINEAR MAPS

SUCH THAT

$$\begin{tikzcd}
(A \otimes A) \otimes A \cong A \otimes (A \otimes A) \arrow[r, "\mathrm{id} \otimes m"] \arrow[d, "m \otimes \mathrm{id}"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

ASSOCIATIVITY

$$\begin{tikzcd}
A \cong \Bbbk \otimes A \arrow[r, "u \otimes \mathrm{id}"] \arrow[dr, "\mathrm{id}"'] \arrow[dr, phantom, bend left, "\circlearrowleft"] & A \otimes A \arrow[d, "m"] & A \otimes \Bbbk \cong A \arrow[l, "\mathrm{id} \otimes u"'] \arrow[dl, "\mathrm{id}"] \arrow[dl, phantom, bend right, "\circlearrowleft"] \\
 & A &
\end{tikzcd}$$

UNITALITY

RECALL OUR STRUCTURES OF INTEREST — / $\Bbbk$ FIELD. $\otimes := \otimes_{\Bbbk}$

A ($\Bbbk$-)ALGEBRA IS A TRIPLE

$$(\underbrace{A}_{\Bbbk\text{-VS}},\ \overbrace{m}^{\text{MULTIPLICATION}} : A \otimes A \longrightarrow A,\ \overbrace{u}^{\text{UNIT}} : \Bbbk \longrightarrow A)$$

$\Bbbk$-LINEAR MAPS

SUCH THAT

$$\begin{tikzcd}
(A \otimes A) \otimes A \cong A \otimes (A \otimes A) \arrow[r, "\mathrm{id} \otimes m"] \arrow[d, "m \otimes \mathrm{id}"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

ASSOCIATIVITY

$$\begin{tikzcd}
A \cong \Bbbk \otimes A \arrow[r, "u \otimes \mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & A \otimes A \arrow[d, "m"] & A \otimes \Bbbk \cong A \arrow[l, "\mathrm{id} \otimes u"'] \arrow[dl, "\mathrm{id}"] \\
& A &
\end{tikzcd}$$

UNITALITY

A LEFT A-MODULE IS A PAIR

$$(\underbrace{V}_{\Bbbk\text{-VS}},\ \overbrace{\triangleright}^{\text{ACTION}} : \underbrace{A}_{A\text{VS}} \otimes V \longrightarrow V)$$

LINEAR MAP

SUCH THAT

$$\begin{tikzcd}
(A \otimes A) \otimes V \cong A \otimes (A \otimes V) \arrow[r, "\mathrm{id} \otimes \triangleright"] \arrow[d, "m \otimes \mathrm{id}"'] \arrow[dr, phantom, "\circlearrowleft"] & A \otimes V \arrow[d, "\triangleright"] \\
A \otimes V \arrow[r, "\triangleright"] & V
\end{tikzcd}$$

ASSOCIATIVITY

$$\begin{tikzcd}
V \cong \Bbbk \otimes V \arrow[r, "u \otimes \mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & A \otimes V \arrow[d, "\triangleright"] \\
& V
\end{tikzcd}$$

UNITALITY

RECALL OUR STRUCTURES OF INTEREST — $/\Bbbk$ FIELD. $\otimes := \otimes_{\Bbbk}$

A ($\Bbbk$-)ALGEBRA IS A TRIPLE

$$(A,\ \underbrace{m : A\otimes A \longrightarrow A}_{\text{MULTIPLICATION}},\ \underbrace{u : \Bbbk \longrightarrow A}_{\text{UNIT}})$$

$\Bbbk$-VS $\Bbbk$-LINEAR MAPS

SUCH THAT

$$\begin{tikzcd}
(A\otimes A)\otimes A \cong A\otimes(A\otimes A) \arrow[r, "id\otimes m"] \arrow[d, "m\otimes id"'] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes A \arrow[d, "m"] \\
A\otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

ASSOCIATIVITY

$$\begin{tikzcd}
A \cong \Bbbk\otimes A \arrow[r, "u\otimes id"] \arrow[dr, "id"'] & A\otimes A \arrow[d, "m"] & A\otimes\Bbbk \cong A \arrow[l, "id\otimes u"'] \arrow[dl, "id"] \\
& A &
\end{tikzcd}$$

UNITALITY

A LEFT A-MODULE IS A PAIR

$$(V,\ \overset{\text{ACTION}}{\triangleright : A\otimes V \longrightarrow V})$$

s.t. $\triangleright(m\otimes id_V) = \triangleright(id_A \otimes \triangleright)$

$\triangleright(u\otimes id_V) = id_V$

RECALL OUR STRUCTURES OF INTEREST — $/\Bbbk$ FIELD. $\otimes := \otimes_{\Bbbk}$

A $(\Bbbk\text{-})$ **ALGEBRA** IS A TRIPLE

$$(\underset{\Bbbk\text{-VS}}{A},\ \overset{\text{MULTIPLICATION}}{m} : A \otimes A \longrightarrow A,\ \overset{\text{UNIT}}{u} : \Bbbk \longrightarrow A)$$

$\Bbbk$-VS

$\Bbbk$- LINEAR MAPS

SUCH THAT

$$\begin{tikzcd}
(A\otimes A)\otimes A \cong A\otimes(A\otimes A) \arrow[r, "id\otimes m"] \arrow[d, "m\otimes id"'] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes A \arrow[d, "m"] \\
A\otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

ASSOCIATIVITY

$$\begin{tikzcd}
A \cong \Bbbk\otimes A \arrow[r, "u\otimes id"] \arrow[dr, "id"'] & A\otimes A \arrow[d, "m"] & A\otimes\Bbbk \cong A \arrow[l, "id\otimes u"'] \arrow[dl, "id"] \\
 & A &
\end{tikzcd}$$

UNITALITY

A **LEFT A-MODULE** IS A PAIR

$$(V,\ \overset{\text{ACTION}}{\triangleright} : A\otimes V \longrightarrow V)$$

s.t. $\triangleright(m\otimes id_V) = \triangleright(id_A\otimes\triangleright)$

$\triangleright(u\otimes id_V) = id_V$

A **RIGHT A-MODULE** IS A PAIR

$$(V,\ \overset{\text{ACTION}}{\triangleleft} : V\otimes A \longrightarrow V)$$

s.t. $\triangleleft(id_V\otimes m) = \triangleleft(\triangleleft\otimes id_A)$

$\triangleleft(id_V\otimes u) = id_V$

RECALL OUR STRUCTURES OF INTEREST — $/\Bbbk$ FIELD. $\otimes := \otimes_{\Bbbk}$

A $(\Bbbk\text{-})$ ALGEBRA IS A TRIPLE

$$(\underset{\Bbbk\text{-VS}}{A},\ \overset{\text{MULTIPLICATION}}{m} : A \otimes A \longrightarrow A,\ \overset{\text{UNIT}}{u} : \Bbbk \longrightarrow A)$$

$\Bbbk$- LINEAR MAPS

SUCH THAT

$$m(m \otimes id_A) = m(id_A \otimes m)$$

$$m(u \otimes id_A) = id_A$$

$$m(id_A \otimes u) = id_A$$

A LEFT A-MODULE IS A PAIR

$$(V,\ \overset{\text{ACTION}}{\triangleright} : A \otimes V \longrightarrow V)$$

s.t. $\triangleright(m \otimes id_V) = \triangleright(id_A \otimes \triangleright)$

$\triangleright(u \otimes id_V) = id_V$

A RIGHT A-MODULE IS A PAIR

$$(V,\ \overset{\text{ACTION}}{\triangleleft} : V \otimes A \longrightarrow V)$$

s.t. $\triangleleft(id_V \otimes m) = \triangleleft(\triangleleft \otimes id_A)$

$\triangleleft(id_V \otimes u) = id_V$

RECALL OUR STRUCTURES OF INTEREST — $/\Bbbk$ FIELD. $\otimes := \otimes_{\Bbbk}$

A **($\Bbbk$-)ALGEBRA** IS A TRIPLE

$$\big(\underset{\Bbbk\text{-VS}}{A},\ \overset{\text{MULTIPLICATION}}{m} : A \otimes A \longrightarrow A,\ \overset{\text{UNIT}}{u} : \Bbbk \longrightarrow A\big)$$

$\Bbbk$-LINEAR MAPS

SUCH THAT

$$m(m \otimes \mathrm{id}_A) = m(\mathrm{id}_A \otimes m)$$

$$m(u \otimes \mathrm{id}_A) = \mathrm{id}_A$$

$$m(\mathrm{id}_A \otimes u) = \mathrm{id}_A$$

A **LEFT A-MODULE** IS A PAIR

$$\big(V,\ \overset{\text{ACTION}}{\triangleright} : A \otimes V \longrightarrow V\big)$$

$\ni.\ \triangleright(m \otimes \mathrm{id}_V) = \triangleright(\mathrm{id}_A \otimes \triangleright)$

$\triangleright(u \otimes \mathrm{id}_V) = \mathrm{id}_V$

A **RIGHT A-MODULE** IS A PAIR

$$\big(V,\ \overset{\text{ACTION}}{\triangleleft} : V \otimes A \longrightarrow V\big)$$

$\ni.\ \triangleleft(\mathrm{id}_V \otimes m) = \triangleleft(\triangleleft \otimes \mathrm{id}_A)$

$\triangleleft(\mathrm{id}_V \otimes u) = \mathrm{id}_V$

A **(B_1, B_2)-BIMODULE** IS A TRIPLE

$$\big(V,\ \triangleright : B_1 \otimes V \longrightarrow V,\ \triangleleft : V \otimes B_2 \longrightarrow V\big) \ \ni. \ \begin{cases} (V, \triangleright) \equiv \text{LEFT } B_1\text{-MODULE} \\ (V, \triangleleft) \equiv \text{RIGHT } B_2\text{-MODULE} \\ \triangleleft(\triangleright \otimes \mathrm{id}_{B_2}) = \triangleright(\mathrm{id}_{B_1} \otimes \triangleleft) \end{cases}$$

I. DIRECT PRODUCT, SUM, DIRECT SUM OF ALGEBRAS & MODULES

ALGEBRA A vs $m: A \otimes A \to A$ $u: \mathbb{k} \otimes A \to A$ & COMP. AXIOMS
LEFT A-MODULE V vs $\triangleright: A \otimes V \to V$ & COMP. AXIOMS
RIGHT A-MODULE V vs $\triangleleft: V \otimes A \to V$ & COMP. AXIOMS
(B_1, B_2)-BIMODULE V vs $\triangleright: B_1 \otimes V \to V$ $\triangleleft: V \otimes B_2 \to V$ & COMP. AXIOMS

I. DIRECT PRODUCT, SUM, DIRECT SUM OF ALGEBRAS & MODULES

ALGEBRA A vs $m: A \otimes A \to A$ $u: \Bbbk \otimes A \to A$ & COMP. AXIOMS
LEFT A-MODULE V vs $\triangleright: A \otimes V \to V$ & COMP. AXIOMS
RIGHT A-MODULE V vs $\triangleleft: V \otimes A \to V$ & COMP. AXIOMS
(B_1, B_2)-BIMODULE V vs $\triangleright: B_1 \otimes V \to V$ $\triangleleft: V \otimes B_2 \to V$ & COMP. AXIOMS

GIVEN LEFT A-MODULES

$$(V_1, \triangleright_1), \; (V_2, \triangleright_2), \; \dots, \; (V_r, \triangleright_r)$$

I. DIRECT PRODUCT, SUM, DIRECT SUM OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m: A \otimes A \to A$
$u: \mathbb{k} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V vs
$\triangleright: A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE
V vs
$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V vs
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

GIVEN LEFT A-MODULES

$$(V_1, \triangleright_1), \ (V_2, \triangleright_2), \ \ldots, \ (V_r, \triangleright_r)$$

THEIR **DIRECT PRODUCT** IS $V_1 \times V_2 \times \ldots \times V_r$

WITH

$$\triangleright: A \otimes (V_1 \times V_2 \times \ldots \times V_r) \longrightarrow V_1 \times V_2 \times \ldots \times V_r$$

$$a \triangleright (v_1, v_2, \ldots, v_r) := (a \triangleright_1 v_1, \ a \triangleright_2 v_2, \ \ldots, \ a \triangleright_r v_r)$$

EXERCISE 1.15
CHECK THE ASSOCIATIVITY & UNITALITY AXIOMS

I. Direct product, sum, direct sum of algebras & modules

Algebra
A vs
$m: A\otimes A \to A$
$u: \mathbb{k}\otimes A \to A$
& comp. axioms

Left A-module
V vs
$\triangleright: A\otimes V \to V$
& comp. axioms

Right A-module
V vs
$\triangleleft: V\otimes A \to V$
& comp. axioms

(B_1,B_2)-bimodule
V vs
$\triangleright: B_1\otimes V \to V$
$\triangleleft: V\otimes B_2 \to V$
& comp. axioms

Given left A-sub-modules of $(V,\tilde{\triangleright})$

$$(V_1,\triangleright_1),\ (V_2,\triangleright_2),\ \ldots,\ (V_r,\triangleright_r)$$

Their **direct product** is $V_1\times V_2\times\ldots\times V_r$

with

$$\triangleright : A\otimes(V_1\times V_2\times\ldots\times V_r) \longrightarrow V_1\times V_2\times\ldots\times V_r$$

$$a\triangleright(v_1,v_2,\ldots,v_r) := (a\triangleright_1 v_1,\ a\triangleright_2 v_2,\ \ldots,\ a\triangleright_r v_r)$$

Their **sum** is $V_1 +_V V_2 + \ldots +_V V_r$

with

$$\triangleright : A\otimes(V_1+V_2+\ldots+V_r) \longrightarrow V_1+V_2+\ldots+V_r$$

$$a\triangleright(v_1+v_2+\ldots+v_r) := (a\triangleright_1 v_1)+(a\triangleright_2 v_2)+\ldots+(a\triangleright_r v_r)$$

Exercise 1.15
Check the associativity & unitality axioms

I. DIRECT PRODUCT, SUM, DIRECT SUM OF ALGEBRAS & MODULES

ALGEBRA
A VS
$m: A \otimes A \to A$
$u: \mathbb{k} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V VS
$\triangleright: A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE
V VS
$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V VS
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

GIVEN LEFT A-SUBMODULES OF $(V, \tilde{\triangleright})$

$(V_1, \triangleright_1), (V_2, \triangleright_2), \ldots, (V_r, \triangleright_r)$

THEIR **DIRECT PRODUCT** IS $V_1 \times V_2 \times \ldots \times V_r$ WITH

$$\triangleright : A \otimes (V_1 \times V_2 \times \ldots \times V_r) \longrightarrow V_1 \times V_2 \times \ldots \times V_r$$

$$a \triangleright (v_1, v_2, \ldots, v_r) := (a \triangleright_1 v_1,\ a \triangleright_2 v_2, \ldots, a \triangleright_r v_r)$$

THEIR **DIRECT SUM** IS $V_1 \oplus V_2 \oplus \ldots \oplus V_r$ WITH

$$\triangleright : A \otimes (V_1 \oplus V_2 \oplus \ldots \oplus V_r) \longrightarrow V_1 \oplus V_2 \oplus \ldots \oplus V_r$$

$$a \triangleright (v_1 + v_2 + \ldots + v_r) := (a \triangleright_1 v_1) + (a \triangleright_2 v_2) + \ldots + (a \triangleright_r v_r)$$

EXERCISE 1.15
CHECK THE ASSOCIATIVITY & UNITALITY AXIOMS

I. DIRECT PRODUCT, SUM, DIRECT SUM OF ALGEBRAS & MODULES

ALGEBRA
A VS
$m: A \otimes A \to A$
$u: \mathbb{k} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V VS
$\triangleright: A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE
V VS
$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V VS
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

GIVEN LEFT A-MODULES

$$(V_1, \triangleright_1), \ (V_2, \triangleright_2), \ \dots, \ (V_r, \triangleright_r)$$

THEIR **DIRECT SUM** IS $V_1 \oplus V_2 \oplus \dots \oplus V_r$

WITH

$$\triangleright : A \otimes (V_1 \oplus V_2 \oplus \dots \oplus V_r) \longrightarrow V_1 \oplus V_2 \oplus \dots \oplus V_r$$

$$a \triangleright (v_1 + v_2 + \dots + v_r) := (a \triangleright_1 v_1) + (a \triangleright_2 v_2) + \dots + (a \triangleright_r v_r)$$

I. DIRECT PRODUCT, SUM, DIRECT SUM OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m : A \otimes A \to A$
$u : \Bbbk \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V vs
$\triangleright : A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE
V vs
$\triangleleft : V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V vs
$\triangleright : B_1 \otimes V \to V$
$\triangleleft : V \otimes B_2 \to V$
& COMP. AXIOMS

GIVEN LEFT A-MODULES

$$(V_1, \triangleright_1),\ (V_2, \triangleright_2),\ \ldots,\ (V_r, \triangleright_r)$$

THEIR **DIRECT SUM** IS $V_1 \oplus V_2 \oplus \ldots \oplus V_r$

WITH

$$\triangleright : A \otimes (V_1 \oplus V_2 \oplus \ldots \oplus V_r) \longrightarrow V_1 \oplus V_2 \oplus \ldots \oplus V_r$$

$$a \triangleright (v_1 + v_2 + \ldots + v_r) := (a \triangleright_1 v_1) + (a \triangleright_2 v_2) + \ldots + (a \triangleright_r v_r)$$

USED TO GET BUILDING BLOCKS IN MODULE THEORY

$(V \overset{\neq 0}{,} \triangleright)$ IS **DECOMPOSABLE** IF $V \cong V_1 \oplus V_2$ AS MODS

NONZERO SUBMODULES OF V

& IS **INDECOMPOSABLE** OTHERWISE.

I. Direct product, sum, direct sum of algebras & modules

Algebra
A vs
$m: A \otimes A \to A$
$u: \mathbb{k} \otimes A \to A$
& comp. axioms

Left A-module
V vs
$\triangleright: A \otimes V \to V$
& comp. axioms

Right A-module
V vs
$\triangleleft: V \otimes A \to V$
& comp. axioms

(B_1, B_2)-bimodule
V vs
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& comp. axioms

Given left A-modules

$$(V_1, \triangleright_1), \ (V_2, \triangleright_2), \ \dots, \ (V_r, \triangleright_r)$$

Their **direct sum** is $V_1 \oplus V_2 \oplus \dots \oplus V_r$

with

$$\triangleright : A \otimes (V_1 \oplus V_2 \oplus \dots \oplus V_r) \longrightarrow V_1 \oplus V_2 \oplus \dots \oplus V_r$$

$$a \triangleright (v_1 + v_2 + \dots + v_r) := (a \triangleright_1 v_1) + (a \triangleright_2 v_2) + \dots + (a \triangleright_r v_r)$$

↑ used to get **building blocks** in module theory

$(V \neq 0, \triangleright)$ is **decomposable** if $V \cong V_1 \oplus V_2$ as mods

nonzero submodules of V

& is **indecomposable** otherwise.

I. DIRECT PRODUCT, SUM, DIRECT SUM OF ALGEBRAS & MODULES

ALGEBRA

A vs

$m : A \otimes A \to A$

$u : \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs

$\triangleright : A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs

$\triangleleft : V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs

$\triangleright : B_1 \otimes V \to V$

$\triangleleft : V \otimes B_2 \to V$

& COMP. AXIOMS

GIVEN ALGEBRAS

$$(A_1, m_1, u_1),\ (A_2, m_2, u_2), \dots, (A_r, m_r, u_r)$$

I. DIRECT PRODUCT, SUM, DIRECT SUM OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m : A \otimes A \to A$
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RIGHT A-MODULE
V vs
$\triangleleft : V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V vs
$\triangleright : B_1 \otimes V \to V$
$\triangleleft : V \otimes B_2 \to V$
& COMP. AXIOMS

GIVEN ALGEBRAS

$$(A_1, m_1, u_1),\ (A_2, m_2, u_2),\ \dots,\ (A_r, m_r, u_r)$$

THEIR **DIRECT PRODUCT** IS $A_1 \times A_2 \times \dots \times A_r$

WITH

$$m((a_1, \dots, a_r) \otimes (b_1, \dots, b_r)) := (m_1(a_1 \otimes b_1), \dots, m_r(a_r \otimes b_r))$$

$$1_{A_1 \times \dots \times A_r} = (1_{A_1}, \dots, 1_{A_r})$$

I. DIRECT PRODUCT, SUM, DIRECT SUM OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m: A \otimes A \to A$
$u: \mathbb{k} \otimes A \to A$
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$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V vs
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

GIVEN ALGEBRAS

$$(A_1, m_1, u_1),\ (A_2, m_2, u_2),\ \dots,\ (A_r, m_r, u_r)$$

THEIR **DIRECT PRODUCT** IS $A_1 \times A_2 \times \dots \times A_r$

WITH

$$m((a_1, \dots, a_r) \otimes (b_1, \dots, b_r)) := (m_1(a_1 \otimes b_1), \dots, m_r(a_r \otimes b_r))$$

$$1_{A_1 \times \dots \times A_r} = (1_{A_1}, \dots, 1_{A_r})$$

$(\overset{\neq 0}{A}, m, u)$ IS **DECOMPOSABLE** IF $A \cong \overset{\neq 0}{A_1} \times \overset{\neq 0}{A_2}$ AS ALGS

& IS **INDECOMPOSABLE** OTHERWISE.

↑

BUILDING BLOCKS FOR ALGEBRAS

I. DIRECT PRODUCT, SUM, DIRECT SUM OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m : A \otimes A \to A$
$u : \mathbb{k} \otimes A \to A$
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(B_1, B_2)-BIMODULE
V vs
$\triangleright : B_1 \otimes V \to V$
$\triangleleft : V \otimes B_2 \to V$
& comp. axioms

GIVEN ALGEBRAS

$$(A_1, m_1, u_1),\ (A_2, m_2, u_2),\ \ldots,\ (A_r, m_r, u_r)$$

THEIR **DIRECT PRODUCT** IS $A_1 \times A_2 \times \cdots \times A_r$

WITH

$$m((a_1, \ldots, a_r) \otimes (b_1, \ldots, b_r)) := (m_1(a_1 \otimes b_1), \ldots, m_r(a_r \otimes b_r))$$

$$1_{A_1 \times \cdots \times A_r} = (1_{A_1}, \ldots, 1_{A_r})$$

$(A^{\neq 0}, m, u)$ IS **DECOMPOSABLE** IF $A \cong A_1^{\neq 0} \times A_2^{\neq 0}$ AS ALGS

& IS **INDECOMPOSABLE** OTHERWISE.

EXER. 1.15 SHOW $(A^{\neq 0}, m, u)$ IS INDECOMPOSABLE

$\Leftrightarrow$ THE ONLY ELTS $\underbrace{e \in Z(A) \text{ s.t. } e^2 = e}_{\text{“CENTRAL IDEMPOTENTS”}}$ ARE 0_A & 1_A

I. Direct product, sum, direct sum of algebras & modules

Algebra
A vs
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$u: \Bbbk \otimes A \to A$
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$\triangleleft: V \otimes A \to V$
& comp. axioms

(B_1,B_2)-bimodule
V vs
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& comp. axioms

Given algebras

$$(A_1, m_1, u_1),\ (A_2, m_2, u_2),\ \dots,\ (A_r, m_r, u_r)$$

Their **direct product** is $A_1 \times A_2 \times \dots \times A_r$

with

$$m((a_1,\dots,a_r)\otimes(b_1,\dots,b_r)) := (m_1(a_1 \otimes b_1), \dots, m_r(a_r \otimes b_r))$$

$$1_{A_1 \times \dots \times A_r} = (1_{A_1}, \dots, 1_{A_r})$$

$(A^{\neq 0}, m, u)$ is **decomposable** if $A \cong A_1^{\neq 0} \times A_2^{\neq 0}$ as algs

& is **indecomposable** otherwise.

Exer. 1.15 Show $(A^{\neq 0}, m, u)$ is indecomposable

$\Leftrightarrow$ the only elts $e \in Z(A)$ s.t. $e^2 = e$ are 0_A & 1_A

Ex. $\Bbbk Q$ indecomp $\Leftrightarrow$ Q connected

I. DIRECT PRODUCT, SUM, DIRECT SUM OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m: A \otimes A \to A$
$u: \mathbb{k} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V vs
$\triangleright: A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE
V vs
$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V vs
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

GIVEN ALGEBRAS

$$(A_1, m_1, u_1),\ (A_2, m_2, u_2),\ \dots,\ (A_r, m_r, u_r)$$

THEIR **DIRECT PRODUCT** IS $A_1 \times A_2 \times \dots \times A_r$

WITH

$$m((a_1, \dots, a_r) \otimes (b_1, \dots, b_r)) := (m_1(a_1 \otimes b_1), \dots, m_r(a_r \otimes b_r))$$

$$1_{A_1 \times \dots \times A_r} = (1_{A_1}, \dots, 1_{A_r})$$

(A_i, m_i, u_i) IS A NONUNITAL SUBALGEBRA OF $A_1 \times \dots \times A_i \times \dots \times A_r$ $\forall i = 1, \dots, r$

I. DIRECT PRODUCT, SUM, DIRECT SUM OF ALGEBRAS & MODULES

ALGEBRA

A VS

$m: A \otimes A \to A$

$u: \Bbbk \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V VS

$\triangleright: A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V VS

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V VS

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

GIVEN ALGEBRAS

$$(A_1, m_1, u_1),\ (A_2, m_2, u_2), \ldots, (A_r, m_r, u_r)$$

WITH A_i SUBSPACE OF A $\forall i$

THEIR **DIRECT PRODUCT** IS $A_1 \times A_2 \times \cdots \times A_r$

WITH

$$m((a_1, \ldots, a_r) \otimes (b_1, \ldots, b_r)) := (m_1(a_1 \otimes b_1), \ldots, m_r(a_r \otimes b_r))$$

$$1_{A_1 \times \cdots \times A_r} = (1_{A_1}, \ldots, 1_{A_r})$$

(A_i, m_i, u_i) IS A NONUNITAL SUBALGEBRA OF $A_1 \times \cdots \times A_i \times \cdots \times A_r$ $\forall i = 1, \ldots, r$

I. DIRECT PRODUCT, SUM, DIRECT SUM OF ALGEBRAS & MODULES

ALGEBRA

A vs

$m : A \otimes A \to A$

$u : \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs

$\triangleright : A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs

$\triangleleft : V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs

$\triangleright : B_1 \otimes V \to V$

$\triangleleft : V \otimes B_2 \to V$

& COMP. AXIOMS

GIVEN ALGEBRAS

$$(A_1, m_1, u_1), (A_2, m_2, u_2), \ldots, (A_r, m_r, u_r)$$

WITH A_i SUBSPACE OF A $\forall i$

THEIR **DIRECT PRODUCT** IS $A_1 \times A_2 \times \cdots \times A_r$

WITH

$$m((a_1, \ldots, a_r) \otimes (b_1, \ldots, b_r)) := (m_1(a_1 \otimes b_1), \ldots, m_r(a_r \otimes b_r))$$

$$1_{A_1 \times \cdots \times A_r} = (1_{A_1}, \ldots, 1_{A_r})$$

(A_i, m_i, u_i) IS A NONUNITAL SUBALGEBRA OF $A_1 \times \cdots \times A_i \times \cdots \times A_r$ $\forall i = 1, \ldots, r$

THEIR **DIRECT (SUM)** OF UNDERLYING VSPACES IS AN ALGEBRA IF $(m_A)|_{A_i \otimes A_j} = 0$ $i \neq j$

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

ALGEBRA

A vs

$m: A \otimes A \to A$

$u: \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_A V$

$\triangleright: A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1} V_{B_2}$

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

CONSIDER THE NOTATION

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{k}}$

ALGEBRA A vs $m : A \otimes A \to A$ $u : \mathbb{k} \otimes A \to A$ & COMP. AXIOMS
LEFT A-MODULE V vs $\quad {}_A V$ $\triangleright : A \otimes V \to V$ & COMP. AXIOMS
RIGHT A-MODULE V vs $\quad V_A$ $\triangleleft : V \otimes A \to V$ & COMP. AXIOMS
(B_1, B_2)-BIMODULE V vs $\quad {}_{B_1} V_{B_2}$ $\triangleright : B_1 \otimes V \to V$ $\triangleleft : V \otimes B_2 \to V$ & COMP. AXIOMS

$$\boxed{{}_{B_1}V \otimes W_{B_2} \quad \equiv \quad (B_1, B_2)\text{-BIMODULE}}$$

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{R}}$

ALGEBRA
A vs
$m: A \otimes A \to A$
$u: \mathbb{k} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V vs $\quad {}_A V$
$\triangleright: A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE
V vs $\quad V_A$
$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V vs $\quad {}_{B_1}V_{B_2}$
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

$$\boxed{{}_{B_1}V \otimes W_{B_2} \equiv (B_1, B_2)\text{-BIMODULE}}$$

$=$ $V \otimes W$ AS A VECTOR SPACE, WITH

$$\begin{tikzcd}
\triangleright_{V\otimes W} : B_1 \otimes (V \otimes W) \arrow[rr] \arrow[dr, "\cong"'] & \arrow[d, phantom, "\underline{\text{DEF}}"] & V \otimes W \\
& (B_1 \otimes V) \otimes W \arrow[ur, "\triangleright_V \otimes \mathrm{id}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\triangleleft_{V\otimes W} : (V \otimes W) \otimes B_2 \arrow[rr] \arrow[dr, "\cong"'] & \arrow[d, phantom, "\underline{\text{DEF}}"] & V \otimes W \\
& V \otimes (W \otimes B_2) \arrow[ur, "\mathrm{id} \otimes \triangleleft_W"'] &
\end{tikzcd}$$

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{R}}$

ALGEBRA
A vs
$m : A \otimes A \to A$
$u : \mathbb{R} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE ${}_A V$
V vs
$\triangleright : A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE V_A
V vs
$\triangleleft : V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE ${}_{B_1}V_{B_2}$
V vs
$\triangleright : B_1 \otimes V \to V$
$\triangleleft : V \otimes B_2 \to V$
& COMP. AXIOMS

$$\boxed{{}_{B_1}V \otimes W_{B_2} \equiv (B_1, B_2)\text{-BIMODULE}}$$

$=$ $V \otimes W$ AS A VECTOR SPACE, WITH

$$\begin{tikzcd}
\triangleright_{V\otimes W} : B_1 \otimes (V \otimes W) \arrow[rr] \arrow[dr, "\cong"'] & \arrow[d, phantom, "\underline{\mathrm{DEF}}"] & V \otimes W \\
& (B_1 \otimes V) \otimes W \arrow[ur, "\triangleright_V \otimes \mathrm{id}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
\triangleleft_{V\otimes W} : (V \otimes W) \otimes B_2 \arrow[rr] \arrow[dr, "\cong"'] & \arrow[d, phantom, "\underline{\mathrm{DEF}}"] & V \otimes W \\
& V \otimes (W \otimes B_2) \arrow[ur, "\mathrm{id} \otimes \triangleleft_W"'] &
\end{tikzcd}$$

EXER. 1.16 CHECK THAT $(V \otimes W, \triangleright, \triangleleft)$ ABOVE IS INDEED A (B_1, B_2)-BIMODULE

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{R}}$

ALGEBRA
A VS
$m: A \otimes A \to A$
$u: \mathbb{R} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE $\quad {}_AV$
V VS
$\triangleright: A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE $\quad V_A$
V VS
$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE $\quad {}_{B_1}V_{B_2}$
V VS
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

$$\boxed{V \otimes_A W := V_A \otimes {}_AW \equiv \text{VECTOR SPACE}}$$

DEFINED BY UNIVERSAL PROPERTY

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{R}}$

ALGEBRA
A VS
$m: A \otimes A \to A$
$u: \mathbb{R} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V VS
$\triangleright: A \otimes V \to V$
& COMP. AXIOMS
${}_A V$

RIGHT A-MODULE
V VS
$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS
V_A

(B_1, B_2)-BIMODULE
V VS
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS
${}_{B_1} V_{B_2}$

$$\boxed{V \otimes_A W := V_A \otimes {}_A W \equiv \text{VECTOR SPACE}}$$

DEFINED BY UNIVERSAL PROPERTY

RECALL: $V \otimes W$ DEFINED BY

$$V \times W \xrightarrow{\alpha \text{ BILINEAR}} V \otimes W$$

THE (UNIQUE) VSPACE $V \otimes W$ ATTACHED TO $V \times W$ VIA BILINEAR α

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{R}}$

ALGEBRA
A vs
$m: A \otimes A \to A$
$u: \mathbb{R} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V vs $\quad {}_A V$
$\triangleright: A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE
V vs $\quad V_A$
$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V vs $\quad {}_{B_1}V_{B_2}$
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

$$\boxed{V \otimes_A W := V_A \otimes {}_A W \equiv \text{VECTOR SPACE}}$$

DEFINED BY UNIVERSAL PROPERTY

RECALL: $V \otimes W$ DEFINED BY

$$\begin{tikzcd}
V \times W \arrow[r, "\alpha \text{ BILINEAR}"] \arrow[dr, "\forall \text{ BILINEAR MAP } \beta"'] & V \otimes W \\
& Z \text{ vs}
\end{tikzcd}$$

THE (UNIQUE) VSPACE $V \otimes W$ ATTACHED TO $V \times W$ VIA BILINEAR α

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{R}}$

ALGEBRA

A vs

$m : A \otimes A \to A$

$u : \mathbb{R} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE ${}_AV$

V vs

$\triangleright : A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE V_A

V vs

$\triangleleft : V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE ${}_{B_1}V_{B_2}$

V vs

$\triangleright : B_1 \otimes V \to V$

$\triangleleft : V \otimes B_2 \to V$

& COMP. AXIOMS

$$\boxed{V \otimes_A W := V_A \otimes {}_AW \equiv \text{VECTOR SPACE}}$$

DEFINED BY UNIVERSAL PROPERTY

RECALL: $V \otimes W$ DEFINED BY

$$\begin{tikzcd} V \times W \arrow[rr, "\alpha\ \text{BILINEAR}"] \arrow[dr, "\forall\ \text{BILINEAR MAP}\ \beta"'] & & V \otimes W \arrow[dl, dashed, "\exists!\ \text{LINEAR MAP}\ \gamma"] \\ & Z\ \text{vs} & \end{tikzcd}$$

THE (UNIQUE) VSPACE $V \otimes W$ ATTACHED TO $V \times W$ VIA BILINEAR α

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{R}}$

ALGEBRA

A vs

$m : A \otimes A \to A$

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& COMP. AXIOMS

LEFT A-MODULE $\quad {}_AV$

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& COMP. AXIOMS

RIGHT A-MODULE $\quad V_A$

V vs

$\triangleleft : V \otimes A \to V$

& COMP. AXIOMS

(B_1,B_2)-BIMODULE $\quad {}_{B_1}V_{B_2}$

V vs

$\triangleright : B_1 \otimes V \to V$

$\triangleleft : V \otimes B_2 \to V$

& COMP. AXIOMS

$$\boxed{V \otimes_A W := V_A \otimes {}_AW \equiv \text{VECTOR SPACE}}$$

DEFINED BY UNIVERSAL PROPERTY

RECALL: $V \otimes W$ DEFINED BY

THE (UNIQUE) VSPACE

ATTACHED TO $V \times W$ VIA BILINEAR α

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha\ \text{BILINEAR}"] \arrow[dr, "\forall\ \text{BILINEAR MAP}\ \beta"'] \arrow[rr, phantom, shift right=6ex, "?"] & & V \otimes W \arrow[dl, dashed, "\exists!\ \text{LINEAR MAP}\ \gamma"] \\
& Z\ \text{vs} &
\end{tikzcd}$$

$$\mathrm{Bilin}(V \times W, Z) \cong \mathrm{Hom}_{\mathbb{R}}(V \otimes W, Z)$$

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{k}}$

ALGEBRA

A vs

$m: A \otimes A \to A$

$u: \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_AV$

$\triangleright: A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1}V_{B_2}$

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

$$\boxed{V \otimes_A W := V_A \otimes {}_AW \equiv \text{VECTOR SPACE}}$$

DEFINED BY UNIVERSAL PROPERTY

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha \text{ bilinear}"] \arrow[dr, "\forall \text{ bilinear map } \beta"'] & \arrow[d, phantom, "\circlearrowleft"] & V \otimes W \arrow[dl, dashed, "\exists! \text{ linear map } \gamma"] \\
& Z_{vs} &
\end{tikzcd}$$

$$\mathrm{Bilin}(V \times W, Z) \cong \mathrm{Hom}_{\mathbb{k}}(V \otimes W, Z)$$

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{k}}$

ALGEBRA
A VS
$m : A \otimes A \to A$
$u : \mathbb{k} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V VS $\quad {}_A V$
$\triangleright : A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE
V VS $\quad V_A$
$\triangleleft : V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V VS $\quad {}_{B_1} V_{B_2}$
$\triangleright : B_1 \otimes V \to V$
$\triangleleft : V \otimes B_2 \to V$
& COMP. AXIOMS

$$\boxed{V \otimes_A W := V_A \otimes {}_A W \equiv \text{VECTOR SPACE}}$$

DEFINED BY UNIVERSAL PROPERTY

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha \text{ BILINEAR}"] \arrow[dr, "\forall \text{ BILINEAR MAP } \beta"'] & \circlearrowleft & V \otimes W \arrow[dl, dashed, "\exists! \text{ LINEAR MAP } \gamma"] \\
& Z_{\text{VS}} &
\end{tikzcd}$$

$\mathrm{Bilin}(V \times W, Z)$
$\cong \mathrm{Hom}_{\mathbb{k}}(V \otimes W, Z)$

$$V \times W \xrightarrow{\alpha \text{ A-BALANCED}} V \otimes_A W$$

THE (UNIQUE) VSPACE

ATTACHED TO $V \times W$
VIA A-BALANCED α

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{R}}$

ALGEBRA

A vs

$m : A \otimes A \to A$

$u : \mathbb{R} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_AV$

$\triangleright : A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft : V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1}V_{B_2}$

$\triangleright : B_1 \otimes V \to V$

$\triangleleft : V \otimes B_2 \to V$

& COMP. AXIOMS

$$\boxed{V \otimes_A W := V_A \otimes {}_AW \equiv \text{VECTOR SPACE}}$$

DEFINED BY UNIVERSAL PROPERTY

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha\ \text{BILINEAR}"] \arrow[dr, "\forall\ \text{BILINEAR MAP } \beta"'] & \arrow[d, phantom, "\circlearrowright"] & V \otimes W \arrow[dl, dashed, "\exists!\ \text{LINEAR MAP } \gamma"] \\
& Z_{\text{vs}} &
\end{tikzcd}$$

$$\mathrm{Bilin}(V \times W, Z) \cong \mathrm{Hom}_{\mathbb{R}}(V \otimes W, Z)$$

$$V \times W \xrightarrow{\alpha\ \text{A-BALANCED}} V \otimes_A W$$

THE (UNIQUE) VSPACE ATTACHED TO $V \times W$ VIA A-BALANCED α

$\Updownarrow$

α BILINEAR

&

$$\alpha(v \triangleleft a, w) = \alpha(v, a \triangleright w)$$

$\forall a \in A,\ v \in V,\ w \in W$

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\Bbbk}$

ALGEBRA

A vs

$m: A \otimes A \to A$

$u: \Bbbk \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_AV$

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& COMP. AXIOMS

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V vs $\quad V_A$

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1}V_{B_2}$

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

$$\boxed{V \otimes_A W := V_A \otimes {}_AW \equiv \text{VECTOR SPACE}}$$

DEFINED BY UNIVERSAL PROPERTY

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha \text{ BILINEAR}"] \arrow[dr, "\forall \text{ BILINEAR MAP } \beta"'] & \arrow[d, phantom, "\circlearrowright"] & V \otimes W \arrow[dl, dashed, "\exists! \text{ LINEAR MAP } \gamma"] \\
 & Z_{vs} &
\end{tikzcd}$$

$$\mathrm{Bilin}(V \times W, Z) \cong \mathrm{Hom}_{\Bbbk}(V \otimes W, Z)$$

$$\begin{tikzcd}
V \times W \arrow[r, "\alpha \text{ A-BALANCED}"] \arrow[dr, "\forall \text{ A-BALANCED MAP } \beta"'] & V \otimes_A W \\
 & Z_{vs}
\end{tikzcd}$$

THE (UNIQUE) VSPACE

ATTACHED TO $V \times W$ VIA A-BALANCED α

$\equiv$

α BILINEAR

&

$\alpha(v \triangleleft a, w) = \alpha(v, a \triangleright w)$

$\forall a \in A,\ v \in V,\ w \in W$

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\Bbbk}$

ALGEBRA

A vs

$m: A \otimes A \to A$

$u: \Bbbk \otimes A \to A$

& comp. axioms

LEFT A-MODULE

V vs $\quad {}_A V$

$\triangleright: A \otimes V \to V$

& comp. axioms

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft: V \otimes A \to V$

& comp. axioms

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1}V_{B_2}$

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& comp. axioms

$$\boxed{V \otimes_A W := V_A \otimes {}_A W \equiv \text{VECTOR SPACE}}$$

DEFINED BY UNIVERSAL PROPERTY

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha \text{ bilinear}"] \arrow[dr, "\forall \text{ bilinear map } \beta"'] & \arrow[d, phantom, "\circlearrowright"] & V \otimes W \arrow[dl, dashed, "\exists! \text{ linear map } \gamma"] \\
& Z_{vs} &
\end{tikzcd}$$

$$\mathrm{Bilin}(V \times W, Z) \cong \mathrm{Hom}_{\Bbbk}(V \otimes W, Z)$$

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha \text{ A-balanced}"] \arrow[dr, "\forall \text{ A-balanced map } \beta"'] & & V \otimes_A W \arrow[dl, dashed, "\exists! \text{ linear map } \gamma"] \\
& Z_{vs} &
\end{tikzcd}$$

THE (UNIQUE) VSPACE

ATTACHED TO $V \times W$ VIA A-BALANCED α

$\equiv$

α BILINEAR

&

$\alpha(v \triangleleft a, w) = \alpha(v, a \triangleright w)$

$\forall a \in A,\ v \in V,\ w \in W$

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{R}}$

ALGEBRA

A vs

$m: A \otimes A \to A$

$u: \mathbb{R} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE $\quad {}_A V$

V vs

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& COMP. AXIOMS

RIGHT A-MODULE $\quad V_A$

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$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE $\quad {}_{B_1}V_{B_2}$

V vs

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

$$\boxed{V \otimes_A W := V_A \otimes {}_A W \equiv \text{VECTOR SPACE}}$$

DEFINED BY UNIVERSAL PROPERTY

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha\ \text{BILINEAR}"] \arrow[dr, "\forall\ \text{BILINEAR MAP}\ \beta"'] & \arrow[d, phantom, "\circlearrowright"] & V \otimes W \arrow[dl, dashed, "\exists!\ \text{LINEAR MAP}\ \gamma"] \\
& Z_{vs} &
\end{tikzcd}$$

$$\mathrm{Bilin}(V \times W, Z) \cong \mathrm{Hom}_{\mathbb{R}}(V \otimes W, Z)$$

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha\ \text{A-BALANCED}"] \arrow[dr, "\forall\ \text{A-BALANCED MAP}\ \beta"'] & \arrow[d, phantom, "\circlearrowright"] & V \otimes_A W \arrow[dl, dashed, "\exists!\ \text{LINEAR MAP}\ \gamma"] \\
& Z_{vs} &
\end{tikzcd}$$

A-BALANCED α

$|||$

α BILINEAR

&

$\alpha(v \triangleleft a, w) = \alpha(v, a \triangleright w)$

$\forall a \in A,\ v \in V,\ w \in W$

$$\text{A-Balan}(V \times W, Z) \cong \mathrm{Hom}_{\mathbb{R}}(V \otimes_A W, Z)$$

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{R}}$

ALGEBRA

A vs

$m : A \otimes A \to A$

$u : \mathbb{R} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_A V$

$\triangleright : A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft : V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1} V_{B_2}$

$\triangleright : B_1 \otimes V \to V$

$\triangleleft : V \otimes B_2 \to V$

& COMP. AXIOMS

$$\boxed{V \otimes_A W := V_A \otimes {}_A W \equiv \text{VECTOR SPACE}}$$

↑ IS REALIZED AS A QUOTIENT SPACE

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha\ \text{BILINEAR}"] \arrow[dr, "\forall\ \text{BILINEAR MAP } \beta"'] \arrow[r, phantom, shift right=6ex, "\circlearrowleft"] & & V \otimes W \arrow[dl, dashed, "\exists!\ \text{LINEAR MAP } \gamma"] \\
& Z_{\text{vs}} &
\end{tikzcd}$$

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha\ \text{A-BALANCED}"] \arrow[dr, "\forall\ \text{A-BALANCED MAP } \beta"'] \arrow[r, phantom, shift right=6ex, "\circlearrowleft"] & & V \otimes_A W \arrow[dl, dashed, "\exists!\ \text{LINEAR MAP } \gamma"] \\
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$$V\otimes W := \frac{\mathrm{SPAN}_{\mathbb{R}}\langle (v,w)\rangle_{v\in V,\, w\in W}}{\left(\begin{array}{c} (v+v',w)-(v,w)-(v',w) \\ (v,w+w')-(v,w)-(v,w') \\ (\lambda v,w)-(v,\lambda w) \end{array} \right)} \qquad \begin{array}{r} v,v'\in V \\ w,w'\in W \\ \lambda\in\mathbb{R} \end{array}$$

$$\begin{tikzcd}
V\times W \arrow[rr, "\alpha \text{ A-BALANCED}"] \arrow[dr, "\forall \text{ A-BALANCED MAP } \beta"'] & \arrow[d, phantom, "\circlearrowright"] & V\otimes_A W \arrow[dl, dashed, "\exists! \text{ LINEAR MAP } \gamma"] \\
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\end{tikzcd}$$

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$\triangleright : B_1 \otimes V \to V$

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& COMP. AXIOMS

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$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha\ \text{A-BALANCED}"] \arrow[dr, "\forall\ \text{A-BALANCED MAP } \beta"'] & \arrow[d, phantom, "\circlearrowright"] & V \otimes_A W \arrow[dl, dashed, "\exists!\ \text{LINEAR MAP } \gamma"] \\
& Z_{\text{vs}} &
\end{tikzcd}$$

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& COMP. AXIOMS

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V vs

$\triangleright : A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE $\quad V_A$

V vs

$\triangleleft : V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE $\quad {}_{B_1} V_{B_2}$

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$\triangleright : B_1 \otimes V \to V$

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SIMPLE TENSORS.....

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha\ \text{A-BALANCED}"] \arrow[dr, "\forall\ \text{A-BALANCED MAP}\ \beta"'] & \arrow[d, phantom, "\circlearrowleft"] & V \otimes_A W \arrow[dl, dashed, "\exists!\ \text{LINEAR MAP}\ \gamma"] \\
 & Z_{vs} &
\end{tikzcd}$$

$(v, w) \longmapsto v \otimes_A w$

α BILINEAR &

$\alpha(v \triangleleft a, w) = \alpha(v, a \triangleright w)$

$$V \otimes_A W := \frac{\mathrm{SPAN}_{\mathbb{R}} \langle (v, w) \rangle_{v \in V,\, w \in W}}{\left(\begin{array}{l} (v+v', w) - (v, w) - (v', w) \\ (v, w+w') - (v, w) - (v, w') \\ (\lambda v, w) - (v, \lambda w) \\ (v \triangleleft a, w) - (v, a \triangleright w) \end{array} \right)} \qquad \begin{array}{r} v, v' \in V \\ w, w' \in W \\ \lambda \in \mathbb{R} \\ a \in A \end{array}$$

SATISFY:

$(v \triangleleft a) \otimes_A w = v \otimes_A (a \triangleright v)$

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{R}}$

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A vs
$m : A \otimes A \to A$
$u : \mathbb{R} \otimes A \to A$
& COMP. AXIOMS

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V vs $\quad {}_A V$
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(B_1, B_2)-BIMODULE
V vs $\quad {}_{B_1} V_{B_2}$
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$\triangleleft : V \otimes B_2 \to V$
& COMP. AXIOMS

$$V \otimes_A W := V_A \otimes {}_A W \equiv \text{VECTOR SPACE}$$

EXERCISE 1.17

CAN DEFINE MODULES
VIA ABELIAN GROUPS $M, N, \ldots$
INSTEAD VECTOR SPACES $V, W, \ldots$

GET
ABELIAN GROUP
$M \otimes_A N$

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha", "\text{A-BALANCED}"'] \arrow[dr, "\beta"'] \arrow[rr, phantom, shift right=8, "\circlearrowright"] & & V \otimes_A W \arrow[dl, dashed, "\gamma"] \\
& Z \text{ vs} &
\end{tikzcd}$$

$\forall$ A-BALANCED MAP β $\qquad$ $\exists!$ LINEAR MAP γ

α BILINEAR &
$\alpha(v \triangleleft a, w) = \alpha(v, a \triangleright w)$

$$V \otimes_A W := \frac{\operatorname{Span}_{\mathbb{R}} \langle (v,w) \rangle_{v \in V, w \in W}}{\left(\begin{array}{ll} v, v' \in V & (v+v', w) - (v,w) - (v',w) \\ w, w' \in W & (v, w+w') - (v,w) - (v,w') \\ \lambda \in \mathbb{R} & (\lambda v, w) - (v, \lambda w) \\ a \in A & (v \triangleleft a, w) - (v, a \triangleright w) \end{array} \right)}$$

$$(v \triangleleft a) \otimes_A w = v \otimes_A (a \triangleright v)$$

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

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ALGEBRA

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& COMP. AXIOMS

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V VS $\quad {}_A V$

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& COMP. AXIOMS

RIGHT A-MODULE

V VS $\quad V_A$

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V VS $\quad {}_{B_1} V_{B_2}$

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

$$V \otimes_A W := V_A \otimes {}_A W \equiv \text{VECTOR SPACE}$$

EXERCISE 1.17

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \; ??$$

GET ABELIAN GROUP $M \otimes_A N$

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha"', "\text{A-BALANCED}"] \arrow[dr, "\beta"'] & \arrow[d, phantom, "\circlearrowright"] & V \otimes_A W \arrow[dl, dashed, "\gamma"] \\
& Z_{\text{VS}} &
\end{tikzcd}$$

$\forall$ A-BALANCED MAP β

$\exists!$ LINEAR MAP γ

α BILINEAR &

$$\alpha(v \triangleleft a, w) = \alpha(v, a \triangleright w)$$

$$V \otimes_A W := \frac{\mathrm{SPAN}_{\mathbb{R}} \langle (v,w) \rangle_{v \in V, w \in W}}{\begin{array}{r|c} v, v' \in V & (v+v', w) - (v,w) - (v',w) \\ w, w' \in W & (v, w+w') - (v,w) - (v,w') \\ \lambda \in \mathbb{R} & (\lambda v, w) - (v, \lambda w) \\ a \in A & (v \triangleleft a, w) - (v, a \triangleright w) \end{array}}$$

$$(v \triangleleft a) \otimes_A w = v \otimes_A (a \triangleright v)$$

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{R}}$

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$$\boxed{V \otimes_A W := V_A \otimes {}_A W \equiv \text{VECTOR SPACE}}$$

EXERCISE 1.17

$$\mathbb{Q} \otimes_{\underbrace{\mathbb{Z}}_{\text{ABELIAN GROUP}}} \mathbb{Q} \cong \, ??$$

$$\frac{a}{b} \otimes_{\mathbb{Z}} \frac{c}{d}$$

$a, b, c, d \in \mathbb{Z}$

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha", "\text{A-BALANCED}"'] \arrow[dr, "\beta"'] & & V \otimes_A W \arrow[dl, dashed, "\gamma"] \\
& Z \text{ vs} \arrow[u, phantom, "\circlearrowright"] &
\end{tikzcd}$$

$\forall$ A-BALANCED MAP β

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α BILINEAR &

$$\alpha(v \triangleleft a, w) = \alpha(v, a \triangleright w)$$

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$v, v' \in V$, $w, w' \in W$, $\lambda \in \mathbb{R}$, $a \in A$

$$(v \triangleleft a) \otimes_A w = v \otimes_A (a \triangleright v)$$

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EXERCISE 1.17

ABELIAN GROUP

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \,??$$

$$\frac{a}{b} \otimes_{\mathbb{Z}} \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{d} \otimes_{\mathbb{Z}} \frac{c}{d}$$

$a, b, c, d \in \mathbb{Z}$

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha", "\text{A-BALANCED}"'] \arrow[dr, "\beta"'] & & V \otimes_A W \arrow[dl, dashed, "\gamma"] \\
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$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong ??$$

$$\frac{a}{b} \otimes_{\mathbb{Z}} \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{d} \otimes_{\mathbb{Z}} \frac{c}{d} \underset{\uparrow \,\mathbb{Z}\text{-BALAN}}{=} \frac{a}{b} \cdot \frac{1}{d} \otimes_{\mathbb{Z}} c$$

$a, b, c, d \in \mathbb{Z}$

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha", "\text{A-BALANCED}"'{above, yshift=8pt}] \arrow[dr, "\beta"'] & & V \otimes_A W \arrow[dl, dashed, "\gamma"] \\
& Z \text{ vs} &
\end{tikzcd}$$

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$$\frac{a}{b} \otimes_{\mathbb{Z}} \frac{c}{d} = \frac{a}{b}\cdot\frac{d}{d} \otimes_{\mathbb{Z}} \frac{c}{d} \underset{\mathbb{Z}\text{-BALAN}}{=} \frac{a}{b}\cdot\frac{1}{d} \otimes_{\mathbb{Z}} c \underset{\mathbb{Z}\text{-BALAN}}{=} \frac{ac}{bd} \otimes_{\mathbb{Z}} \frac{1}{1}$$

$a, b, c, d \in \mathbb{Z}$

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha", "\text{A-BALANCED}"'] \arrow[dr, "\beta"'] & \arrow[d, phantom, "\circlearrowright"] & V \otimes_A W \arrow[dl, dashed, "\gamma"] \\
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$v, v' \in V$
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& COMP. AXIOMS

$$V \otimes_A W := V_A \otimes {}_A W \equiv \text{VECTOR SPACE}$$

EXERCISE 1.17

USE UNIVERSAL PROP. FOR PROOF

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \text{ AS ABELIAN GROUPS}$$

$$\frac{a}{b} \otimes_{\mathbb{Z}} \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{d} \otimes_{\mathbb{Z}} \frac{c}{d} \underset{\mathbb{Z}\text{-BALAN}}{\uparrow=} \frac{a}{b} \cdot \frac{1}{d} \otimes_{\mathbb{Z}} c \underset{\mathbb{Z}\text{-BALAN}}{\uparrow=} \frac{ac}{bd} \otimes_{\mathbb{Z}} \frac{1}{1}$$

JUST A COMPUTATION

$a, b, c, d \in \mathbb{Z}$

$$\begin{tikzcd} V \times W \arrow[rr, "\alpha", "\text{A-BALANCED}"'] \arrow[dr, "\beta"'] & & V \otimes_A W \arrow[dl, dashed, "\gamma"] \\ & Z \text{ vs} & \end{tikzcd}$$

$\forall$ A-BALANCED MAP β $\qquad$ $\circlearrowright$ $\qquad$ $\exists!$ LINEAR MAP γ

α BILINEAR &

$$\alpha(v \triangleleft a, w) = \alpha(v, a \triangleright w)$$

$$V \otimes_A W := \frac{\text{SPAN}_{\mathbb{R}} \langle (v, w) \rangle_{v \in V, w \in W}}{\left(\begin{array}{l} (v+v', w) - (v, w) - (v', w) \\ (v, w+w') - (v, w) - (v, w') \\ (\lambda v, w) - (v, \lambda w) \\ (v \triangleleft a, w) - (v, a \triangleright w) \end{array} \right)}$$

$v, v' \in V$, $w, w' \in W$, $\lambda \in \mathbb{R}$, $a \in A$

$$(v \triangleleft a) \otimes_A w = v \otimes_A (a \triangleright v)$$

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{R}}$

ALGEBRA

A vs

$m: A \otimes A \to A$

$u: \mathbb{R} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_AV$

$\triangleright: A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)–BIMODULE

V vs $\quad {}_{B_1}V_{B_2}$

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

$$\boxed{V \otimes_A W := V_A \otimes {}_AW \equiv \text{VECTOR SPACE}}$$

EXERCISE 1.17

USE UNIVERSAL PROP. FOR PROOF

SHOW: $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ AS ABELIAN GROUPS

$$\frac{a}{b} \otimes_{\mathbb{Z}} \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{d} \otimes_{\mathbb{Z}} \frac{c}{d} \underset{\mathbb{Z}\text{-BALAN}}{\overset{\uparrow}{=}} \frac{a}{b} \cdot \frac{1}{d} \otimes_{\mathbb{Z}} c \underset{\mathbb{Z}\text{-BALAN}}{\overset{\uparrow}{=}} \frac{ac}{bd} \otimes_{\mathbb{Z}} \frac{1}{1}$$

$a, b, c, d \in \mathbb{Z}$

$$\begin{tikzcd}
V \times W \arrow[r, "\alpha", "\text{A-BALANCED}"'] \arrow[dr, "\beta"'] & V \otimes_A W \arrow[d, dashed, "\gamma"] \\
& Z \text{ vs}
\end{tikzcd}$$

$\forall$ A-BALANCED MAP β $\qquad$ $\exists!$ LINEAR MAP γ

α BILINEAR &

$\alpha(v \triangleleft a, w) = \alpha(v, a \triangleright w)$

EXER. 1.16

${}_{B_1}V_A \otimes {}_AW_{B_2} =: V \otimes_A W$

IS A

???

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{R}}$

ALGEBRA

A vs

$m : A \otimes A \to A$

$u : \mathbb{R} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_AV$

$\triangleright : A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft : V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1}V_{B_2}$

$\triangleright : B_1 \otimes V \to V$

$\triangleleft : V \otimes B_2 \to V$

& COMP. AXIOMS

$$\boxed{V \otimes_A W := V_A \otimes {}_AW \equiv \text{VECTOR SPACE}}$$

EXERCISE 1.17

USE UNIVERSAL PROP. FOR PROOF

SHOW: $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ AS ABELIAN GROUPS

$$\frac{a}{b} \otimes_{\mathbb{Z}} \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{d} \otimes_{\mathbb{Z}} \frac{c}{d} \underset{\mathbb{Z}\text{-BALAN}}{\overset{\uparrow}{=}} \frac{a}{b} \cdot \frac{1}{d} \otimes_{\mathbb{Z}} c \underset{\mathbb{Z}\text{-BALAN}}{\overset{\uparrow}{=}} \frac{ac}{bd} \otimes_{\mathbb{Z}} \frac{1}{1}$$

$a, b, c, d \in \mathbb{Z}$

$$\begin{tikzcd}
V \times W \arrow[rr, "\alpha\ \text{A-BALANCED}"] \arrow[dr, "\beta"'] & \arrow[d, phantom, "\circlearrowright"] & V \otimes_A W \arrow[dl, dashed, "\gamma"] \\
& Z \text{ vs} &
\end{tikzcd}$$

$\forall$ A-BALANCED MAP β $\qquad$ $\exists!$ LINEAR MAP γ

α BILINEAR &

$\alpha(v \triangleleft a, w) = \alpha(v, a \triangleright w)$

EXER. 1.16

$${}_{B_1}V_A \otimes {}_AW_{B_2} =: V \underset{A}{\otimes} W$$

IS A

(B_1, B_2)-BIMODULE

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\Bbbk}$

ALGEBRA

A VS

$m : A \otimes A \to A$

$u : \Bbbk \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE ${}_A V$

V VS

$\triangleright : A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE V_A

V VS

$\triangleleft : V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE ${}_{B_1}V_{B_2}$

V VS

$\triangleright : B_1 \otimes V \to V$

$\triangleleft : V \otimes B_2 \to V$

& COMP. AXIOMS

SPECIAL CASE

TAKE ALGEBRA MORPHISM

$$\phi : A \longrightarrow B$$

$$\mathrm{Ind}_A^B(V) := B \otimes_A V$$

START: LEFT A-MODULE

END: LEFT B-MODULE

INDUCTION OF ${}_A V$ TO B ALONG ϕ

EXER. 1.16

$${}_{B_1}V_A \otimes {}_A W_{B_2} =: V \otimes_A W$$

IS A

(B_1, B_2)-BIMODULE

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{R}}$

ALGEBRA

A VS

$m: A \otimes A \to A$

$u: \mathbb{R} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V VS $\quad {}_AV$

$\triangleright: A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V VS $\quad V_A$

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V VS $\quad {}_{B_1}V_{B_2}$

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

SPECIAL CASE

TAKE ALGEBRA MORPHISM

$\phi: A \to B$

$\mathrm{Ind}_A^B(V) := B \otimes_A V$

START: LEFT A-MODULE

END: LEFT B-MODULE

INDUCTION OF ${}_AV$ TO B ALONG ϕ

EXER. 1.16

${}_{B_1}V_A \otimes {}_AW =: V \otimes_A W$

IS A

LEFT B_1-MODULE

(TAKE $B_2 = \mathbb{R} \rightsquigarrow W_{\mathbb{R}} = W_{vs}$)

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{k}}$

ALGEBRA

A vs

$m : A \otimes A \to A$

$u : \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_AV$

$\triangleright : A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft : V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1}V_{B_2}$

$\triangleright : B_1 \otimes V \to V$

$\triangleleft : V \otimes B_2 \to V$

& COMP. AXIOMS

SPECIAL CASE

TAKE ALGEBRA MORPHISM

$$\phi : A \longrightarrow B$$

$$\mathrm{Ind}_A^B(V) := B \otimes_A V$$

START: LEFT A-MODULE

END: LEFT B-MODULE

$B = {}_BB_A$ WITH

$$\begin{cases} \triangleright_B := m_B \\ b \triangleleft_B a := b\,\phi(a) \end{cases}$$

USING m_B

INDUCTION OF ${}_AV$ TO B ALONG ϕ

EXER. 1.16

$${}_{B_1}V_A \otimes {}_AW =: V \otimes_A W$$

IS A

LEFT B_1-MODULE

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

$\otimes := \otimes_{\mathbb{k}}$

ALGEBRA

A vs

$m : A \otimes A \to A$

$u : \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_A V$

$\triangleright : A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft : V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1} V_{B_2}$

$\triangleright : B_1 \otimes V \to V$

$\triangleleft : V \otimes B_2 \to V$

& COMP. AXIOMS

SPECIAL CASE

TAKE ALGEBRA MORPHISM

$$\phi : A \longrightarrow B$$

$$\mathrm{Ind}_A^B(V) := B \otimes_A V$$

START: LEFT A-MODULE

END: LEFT B-MODULE

$B = {}_B B_A$ WITH

$$\begin{cases} \triangleright_B := m_B \\ b \triangleleft_B a := b\,\phi(a) \end{cases}$$

USING m_B

INDUCTION OF ${}_A V$ TO B ALONG ϕ

EXER. 1.16

$${}_{B_1} V_A \otimes {}_A W =: V \otimes_A W$$

IS A

LEFT B_1-MODULE

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

ALGEBRA

A vs

$m : A \otimes A \to A$

$u : \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_AV$

$\triangleright : A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft : V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1}V_{B_2}$

$\triangleright : B_1 \otimes V \to V$

$\triangleleft : V \otimes B_2 \to V$

& COMP. AXIOMS

SPECIAL CASE

TAKE ALGEBRA MORPHISM

$$\phi : A \to B$$

$$\mathrm{Ind}_A^B(V) := B \otimes_A V$$

START: LEFT A-MODULE

END: LEFT B-MODULE

$B = {}_BB_A$ WITH

$$\begin{cases} \triangleright_B := m_B \\ b \triangleleft_B a := b\,\phi(a) \end{cases}$$

USING m_B

INDUCTION OF ${}_AV$ TO B ALONG ϕ

≡ COMPLEXIFICATION ≡

TAKE THE FIELD EXTENSION

$$\mathbb{R} \hookrightarrow \mathbb{C}$$

(ALGEBRA MORPHISM)

$$\mathrm{Ind}_{\mathbb{R}}^{\mathbb{C}}(V) := \mathbb{C} \otimes_{\mathbb{R}} V$$

START: $\mathbb{R}$-VS $\quad$ END: $\mathbb{C}$-VS

EXER. 1.16

$${}_{B_1}V_A \otimes {}_AW =: V \otimes_A W$$

IS A

LEFT B_1-MODULE

II. TENSOR PRODUCT OF ALGEBRAS & MODULES

ALGEBRA

A vs

$m: A \otimes A \to A$

$u: \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_AV$

$\triangleright: A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1}V_{B_2}$

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

SPECIAL CASE

TAKE ALGEBRA MORPHISM

$$\phi: A \longrightarrow B$$

$$\mathrm{Ind}_A^B(V) := B \otimes_A V$$

START: LEFT A-MODULE

END: LEFT B-MODULE

$B = {}_BB_A$ WITH

$$\begin{cases} \triangleright_B := m_B \\ b \triangleleft_B a := b\phi(a) \end{cases}$$

USING m_B

INDUCTION OF ${}_AV$ TO B ALONG ϕ

≡ COMPLEXIFICATION ≡

TAKE THE FIELD EXTENSION

$$\mathbb{R} \hookrightarrow \mathbb{C}$$

$$\mathrm{Ind}_{\mathbb{R}}^{\mathbb{C}}(V) := \mathbb{C} \otimes_{\mathbb{R}} V$$

START: $\mathbb{R}$-VS END: $\mathbb{C}$-VS

E.G. $\mathrm{Ind}_{\mathbb{R}}^{\mathbb{C}}(\mathbb{R}^{\oplus n}) \cong \mathbb{C}^{\oplus n}$

EXER. 1.16

$${}_{B_1}V_A \otimes {}_AW =: V \otimes_A W$$

IS A

LEFT B_1-MODULE

II. TENSOR PRODUCT OF ALGEBRAS & MODULES ✓ READ ABOUT ALGEBRAS CONSTRUCTIONS USING $\otimes_A$

ALGEBRA

A VS

$m: A \otimes A \to A$

$u: \mathbb{R} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V VS $\quad {}_A V$

$\triangleright: A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V VS $\quad V_A$

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V VS $\quad {}_{B_1} V_{B_2}$

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

SPECIAL CASE

TAKE ALGEBRA MORPHISM

$$\phi: A \longrightarrow B$$

$$\mathrm{Ind}_A^B(V) := B \otimes_A V$$

START: LEFT A-MODULE $\qquad$ END: LEFT B-MODULE

$B = {}_B B_A$ WITH

$$\begin{cases} \triangleright_B := m_B \\ b \triangleleft_B a := b\,\phi(a) \end{cases}$$

USING m_B

INDUCTION OF ${}_A V$ TO B ALONG ϕ

≡ COMPLEXIFICATION ≡

TAKE THE FIELD EXTENSION

$$\mathbb{R} \hookrightarrow \mathbb{C}$$

$$\mathrm{Ind}_{\mathbb{R}}^{\mathbb{C}}(V) := \mathbb{C} \otimes_{\mathbb{R}} V$$

START: $\mathbb{R}$-VS $\qquad$ END: $\mathbb{C}$-VS

E.G. $\mathrm{Ind}_{\mathbb{R}}^{\mathbb{C}}(\mathbb{R}^{\oplus n}) \cong \mathbb{C}^{\oplus n}$

EXER. 1.16

$${}_{B_1}V_A \otimes {}_A W =: V \otimes_A W$$

IS A

LEFT B_1-MODULE

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA

A vs

$m : A \otimes A \to A$

$u : \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE $\quad {}_A V$

V vs

$\triangleright : A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE $\quad V_A$

V vs

$\triangleleft : V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE $\quad {}_{B_1}V_{B_2}$

V vs

$\triangleright : B_1 \otimes V \to V$

$\triangleleft : V \otimes B_2 \to V$

& COMP. AXIOMS

$$\mathrm{HOM}_{A\text{-mod}}(\underbrace{V}_{{}_A V_{B_1}}, \underbrace{W}_{{}_A W_{B_2}}) \equiv (B_1, B_2)\text{-BIMODULE}$$

RECALL $\quad \mathrm{HOM}_{\mathbb{k}}(\underbrace{V, W}_{\text{vs}}) \equiv$ VSPACE :

$$(\phi + \phi')(v) := \phi(v) +_W \phi'(v)$$

$$(\lambda \phi)(v) := \lambda *_W \phi(v)$$

$$= \phi(\lambda *_V v)$$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA

A vs

$m : A \otimes A \to A$

$u : \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_A V$

$\triangleright : A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft : V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1}V_{B_2}$

$\triangleright : B_1 \otimes V \to V$

$\triangleleft : V \otimes B_2 \to V$

& COMP. AXIOMS

$$\mathrm{Hom}_{A\text{-mod}}(\underset{{}_A V_{B_1}}{V}, \underset{{}_A W_{B_2}}{W}) \equiv (B_1, B_2)\text{-BIMODULE}$$

$$\mathrm{Hom}_{A\text{-mod}}(V, W) \subseteq \mathrm{Hom}_{\mathbb{k}}(\underset{\text{vs}}{V}, \underset{\text{vs}}{W}) \equiv \text{VSPACE}$$

OF LINEAR MAPS $V \to W$

SUBSPACE OF LINEAR MAPS $\phi : V \to W$ $\ni$.

$$\begin{tikzcd} A \otimes V \arrow[r, "\triangleright_V"] \arrow[d, "\mathrm{id} \otimes \phi"'] \arrow[dr, phantom, "\circlearrowright"] & V \arrow[d, "\phi"] \\ A \otimes W \arrow[r, "\triangleright_W"] & W \end{tikzcd}$$

A-MODULE MAPS

$$(\phi + \phi')(v) := \phi(v) +_W \phi'(v)$$

$$(\lambda \phi)(v) := \lambda *_W \phi(v)$$

$$= \phi(\lambda *_V v)$$

III. Hom and Dual of Algebras & Modules

ALGEBRA
A vs
$m: A \otimes A \to A$
$u: \mathbb{k} \otimes A \to A$
& comp. axioms

LEFT A-MODULE
V vs
$\triangleright: A \otimes V \to V$
& comp. axioms
${}_A V$

RIGHT A-MODULE
V vs
$\triangleleft: V \otimes A \to V$
& comp. axioms
V_A

(B_1, B_2)-BIMODULE
V vs
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& comp. axioms
${}_{B_1}V_{B_2}$

$$\mathrm{Hom}_{A\text{-mod}}(V, W) \equiv (B_1, B_2)\text{-bimodule}$$
$${}_A V_{B_1} \qquad {}_A W_{B_2}$$

$\mathrm{Hom}_{A\text{-mod}}(V, W) \subseteq \mathrm{Hom}_{\mathbb{k}}(V, W) \equiv$ VSPACE OF LINEAR MAPS $V \to W$

vs

SUBSPACE OF A-MODULE MAPS

$$\phi(a \triangleright_V v) = a \triangleright_W \phi(v)$$

$$(\phi + \phi')(v) := \phi(v) +_W \phi'(v)$$

$$(\lambda \phi)(v) := \lambda *_W \phi(v) = \phi(\lambda *_V v)$$

EXERCISE 1.25 — You do!

$$(b_1 \triangleright \phi)(v) := ??$$

$$(\phi \triangleleft b_2)(v) := ??$$

MAKES THE ABOVE HOLD

FOR $b_1 \in B_1$, $b_2 \in B_2$, $\phi \in \mathrm{Hom}_{A\text{-mod}}(V, W)$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA

A vs

$m: A \otimes A \to A$

$u: \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_A V$

$\triangleright: A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1} V_{B_2}$

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

$$\mathrm{Hom}_{A\text{-mod}}(\underset{{}_A V_{B_1}}{V}, \underset{{}_A W_{B_2}}{W}) \equiv (B_1, B_2)\text{-BIMODULE}$$

$$\mathrm{Hom}_{A\text{-mod}}(V, W) \subseteq \mathrm{Hom}_{\mathbb{k}}(\underset{vs}{V}, \underset{vs}{W}) \equiv \text{VSPACE OF LINEAR MAPS } V \to W$$

SUBSPACE OF A-MODULE MAPS

$$\phi(a \triangleright_V v) = a \triangleright_W \phi(v)$$

$$(\phi + \phi')(v) := \phi(v) +_W \phi'(v)$$

$$(\lambda \phi)(v) := \lambda *_W \phi(v) = \phi(\lambda *_V v)$$

EXERCISE 1.25 $\quad$ ≡ CHECK THE REST ≡

$$(b_1 \triangleright \phi)(v) := \phi(v \triangleleft_V b_1)$$

$$(\phi \triangleleft b_2)(v) := \phi(v) \triangleleft_W b_2$$

MAKES THE ABOVE HOLD

FOR $b_1 \in B_1$, $b_2 \in B_2$, $\phi \in \mathrm{Hom}_{A\text{-mod}}(V, W)$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m: A \otimes A \to A$
$u: \mathbb{k} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V vs $\quad {}_AV$
$\triangleright: A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE
V vs $\quad V_A$
$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V vs $\quad {}_{B_1}V_{B_2}$
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

$$\mathrm{HOM}_{A\text{-mod}}(\underset{{}_AV_{B_1}}{V}, \underset{{}_AW_{B_2}}{W}) \equiv (B_1, B_2)\text{-BIMODULE}$$

Ex. $V^* := \mathrm{HOM}_{\mathbb{k}}(V, \mathbb{k})$

$V \equiv$ RIGHT B_1-MODULE

$\rightsquigarrow V^* \equiv$ LEFT B_1-MODULE

III. Hom and dual of algebras & modules

Algebra

A vs

$m : A \otimes A \to A$

$u : \Bbbk \otimes A \to A$

& comp. axioms

Left A-module

V vs $\quad {}_A V$

$\triangleright : A \otimes V \to V$

& comp. axioms

Right A-module

V vs $\quad V_A$

$\triangleleft : V \otimes A \to V$

& comp. axioms

(B_1, B_2)-bimodule

V vs $\quad {}_{B_1}V_{B_2}$

$\triangleright : B_1 \otimes V \to V$

$\triangleleft : V \otimes B_2 \to V$

& comp. axioms

$$\mathrm{Hom}_{A\text{-mod}}(\underbrace{V}_{{}_A V_{B_1}}, \underbrace{W}_{{}_A W_{B_2}}) \equiv (B_1, B_2)\text{-BIMODULE}$$

Ex. $V^* := \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$

$V \equiv$ RIGHT B_1-MODULE

$\rightsquigarrow V^* \equiv$ LEFT B_1-MODULE

Restriction of ${}_B V$ to A along

$\phi : A \to B$

ALG. MAP

$\mathrm{Res}^B_A(\underbrace{V}_{\text{LEFT } B\text{-MOD}}) := \underbrace{??}_{\text{LEFT } A\text{-MOD}}$

Coinduction of ${}_A V$ to B along

$\phi : A \to B$

ALG. MAP

$\mathrm{Coind}^B_A(\underbrace{V}_{\text{LEFT } A\text{-MOD}}) := \underbrace{??}_{\text{LEFT } B\text{-MOD}}$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA

A VS

$m: A \otimes A \to A$

$u: \Bbbk \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE $\;{}_AV$

V VS

$\triangleright: A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE $\;V_A$

V VS

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE $\;{}_{B_1}V_{B_2}$

V VS

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

$$\mathrm{Hom}_{A\text{-mod}}(\underset{{}_AV_{B_1}}{V}, \underset{{}_AW_{B_2}}{W}) \equiv (B_1, B_2)\text{-BIMODULE}$$

Ex. $V^* := \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$

$V \equiv$ RIGHT B_1-MODULE

$\rightsquigarrow V^* \equiv$ LEFT B_1-MODULE

RESTRICTION OF ${}_BV$ TO A ALONG

$\phi: A \longrightarrow B$

ALG. MAP

$$\mathrm{Res}_A^B(V) := \mathrm{Hom}_{B\text{-mod}}(B, V)$$

$B = {}_BB_A$ LEFT A-MODULE

COINDUCTION OF ${}_AV$ TO B ALONG

$\phi: A \longrightarrow B$

ALG. MAP

$$\mathrm{Coind}_A^B(V) := \mathrm{Hom}_{A\text{-mod}}(B, V)$$

$B = {}_AB_B$ LEFT B-MODULE

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA

A vs

$m: A \otimes A \to A$

$u: \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_A V$

$\triangleright: A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1} V_{B_2}$

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

$$\mathrm{HOM}_{A\text{-mod}}(\underset{{}_A V_{B_1}}{V}, \underset{{}_A W_{B_2}}{W}) \equiv (B_1, B_2)\text{-BIMODULE}$$

LIKEWISE:

$$\mathrm{HOM}_{\text{mod-}A}(\underset{{}_{B_1} V_A}{V}, \underset{{}_{B_2} W_A}{W}) \equiv (B_2, B_1)\text{-BIMODULE}$$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA

A vs

$m : A \otimes A \to A$

$u : \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_A V$

$\triangleright : A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft : V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1} V_{B_2}$

$\triangleright : B_1 \otimes V \to V$

$\triangleleft : V \otimes B_2 \to V$

& COMP. AXIOMS

$$\mathrm{HOM}_{A\text{-mod}}(\underset{{}_A V_{B_1}}{V}, \underset{{}_A W_{B_2}}{W}) \equiv (B_1, B_2)\text{-BIMODULE}$$

LIKEWISE:

$$\mathrm{HOM}_{\text{mod-}A}(\underset{{}_{B_1} V_A}{V}, \underset{{}_{B_2} W_A}{W}) \equiv (B_2, B_1)\text{-BIMODULE}$$

RECALL:

$$\underset{{}_{B_1} V_A}{V} \otimes_A \underset{{}_A W_{B_2}}{W} \equiv (B_1, B_2)\text{-BIMODULE}$$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m: A \otimes A \to A$
$u: \mathbb{k} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V vs $\quad {}_AV$
$\triangleright: A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE
V vs $\quad V_A$
$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V vs $\quad {}_{B_1}V_{B_2}$
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

$$\mathrm{Hom}_{A\text{-mod}}(\underset{{}_AV_{B_1}}{V}, \underset{{}_AW_{B_2}}{W}) \equiv (B_1, B_2)\text{-BIMODULE}$$

RECALL:

$$\underset{{}_{B_1}V_A}{V} \otimes_A \underset{{}_AW_{B_2}}{W} \equiv (B_1, B_2)\text{-BIMODULE}$$

TENSOR ADJUNCTION FOR VECTOR SPACES:

$$\boxed{\mathrm{Hom}_{\mathbb{k}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(V, \mathrm{Hom}_{\mathbb{k}}(U, W))}$$

ALSO HAVE:

$$\mathrm{Hom}_{\mathbb{k}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(U, \mathrm{Hom}_{\mathbb{k}}(V, W))$$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA

A vs

$m: A \otimes A \to A$

$u: \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_A V$

$\triangleright: A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1}V_{B_2}$

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

$$\mathrm{Hom}_{A\text{-mod}}(\underset{{}_A V_{B_1}}{V}, \underset{{}_A W_{B_2}}{W}) \equiv (B_1, B_2)\text{-BIMODULE}$$

RECALL:

$$\underset{{}_{B_1} V_A}{V} \otimes_A \underset{{}_A W_{B_2}}{W} \equiv (B_1, B_2)\text{-BIMODULE}$$

TENSOR ADJUNCTION FOR VECTOR SPACES:

$$\mathrm{Hom}_{\mathbb{k}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(V, \mathrm{Hom}_{\mathbb{k}}(U, W))$$

EXER. 1.25 DERIVE TENSOR-HOM ADJUNCTION FOR BIMODULES

you do!

$$\mathrm{Hom}_{??}(U \otimes_{??} V, W) \cong \mathrm{Hom}_{??}(V, \mathrm{Hom}_{??}(U, W))$$

$$U = {}_{??}U_{??} \qquad V = {}_{??}V_{??} \qquad W = {}_{??}W_{??}$$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA

A vs

$m: A \otimes A \to A$

$u: \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_AV$

$\triangleright: A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1}V_{B_2}$

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

$$\boxed{\mathrm{Hom}_{A\text{-mod}}(\underset{{}_AV_{B_1}}{V}, \underset{{}_AW_{B_2}}{W}) \equiv (B_1, B_2)\text{-BIMODULE}}$$

RECALL:

$$\boxed{\underset{{}_{B_1}V_A}{V} \otimes_A \underset{{}_AW_{B_2}}{W} \equiv (B_1, B_2)\text{-BIMODULE}}$$

TENSOR ADJUNCTION FOR VECTOR SPACES:

$$\boxed{\mathrm{Hom}_{\mathbb{k}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(V, \mathrm{Hom}_{\mathbb{k}}(U, W))}$$

EXER. 1.25 DERIVE TENSOR-HOM ADJUNCTION FOR BIMODULES

≡ HINTS ≡

$$\mathrm{Hom}_{B_1\text{-mod}}(U \otimes_A V, W) \cong \mathrm{Hom}_{??}(V, \mathrm{Hom}_{??}(U, W))$$

$$U = {}_{B_1}U_A \qquad V = {}_AV_{??} \qquad W = {}_{??}W_{??}$$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m: A \otimes A \to A$
$u: \mathbb{k} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V vs $\quad {}_AV$
$\triangleright : A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE
V vs $\quad V_A$
$\triangleleft : V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V vs $\quad {}_{B_1}V_{B_2}$
$\triangleright : B_1 \otimes V \to V$
$\triangleleft : V \otimes B_2 \to V$
& COMP. AXIOMS

$$\mathrm{Hom}_{A\text{-mod}}(V, W) \equiv (B_1, B_2)\text{-BIMODULE}$$
$${}_AV_{B_1} \qquad {}_AW_{B_2}$$

RECALL:

$$V \otimes_A W \equiv (B_1, B_2)\text{-BIMODULE}$$
$${}_{B_1}V_A \qquad {}_AW_{B_2}$$

TENSOR ADJUNCTION FOR VECTOR SPACES:

$$\mathrm{Hom}_{\mathbb{k}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(V, \mathrm{Hom}_{\mathbb{k}}(U, W))$$

EXER. 1.25 DERIVE TENSOR-HOM ADJUNCTION FOR BIMODULES

≡ HINTS ≡

$$\mathrm{Hom}_{B_1\text{-mod}}(U \otimes_A V, W) \cong \mathrm{Hom}_{A\text{-mod}}(V, \mathrm{Hom}_{??}(U, W))$$

$$U = {}_{B_1}U_A \qquad V = {}_AV_{??} \qquad W = {}_{B_1}W_{??}$$

III. Hom and Dual of Algebras & Modules

ALGEBRA
A vs
$m: A\otimes A \to A$
$u: \Bbbk \otimes A \to A$
& comp. axioms

LEFT A-MODULE
V vs $\quad {}_AV$
$\triangleright: A\otimes V \to V$
& comp. axioms

RIGHT A-MODULE
V vs $\quad V_A$
$\triangleleft: V\otimes A \to V$
& comp. axioms

(B_1,B_2)-BIMODULE
V vs $\quad {}_{B_1}V_{B_2}$
$\triangleright: B_1\otimes V \to V$
$\triangleleft: V\otimes B_2 \to V$
& comp. axioms

$$\underset{\substack{\uparrow\\ {}_AV_{B_1}}}{\mathrm{Hom}_{A\text{-mod}}(V},\underset{\substack{\uparrow\\ {}_AW_{B_2}}}{W)} \;\equiv\; (B_1,B_2)\text{-BIMODULE}$$

RECALL:

$$\underset{\substack{\uparrow\\ {}_{B_1}V_A}}{V} \otimes_A \underset{\substack{\uparrow\\ {}_AW_{B_2}}}{W} \;\equiv\; (B_1,B_2)\text{-BIMODULE}$$

TENSOR ADJUNCTION FOR VECTOR SPACES:

$$\boxed{\mathrm{Hom}_{\Bbbk}(U\otimes V, W) \cong \mathrm{Hom}_{\Bbbk}(V, \mathrm{Hom}_{\Bbbk}(U,W))}$$

EXER. 1.25 DERIVE TENSOR-HOM ADJUNCTION FOR BIMODULES

$\equiv$ CHECK THE REST $\equiv$ AS B_2-BIMODS

$$\mathrm{Hom}_{B_1\text{-mod}}(U\otimes_A V, W) \cong \mathrm{Hom}_{A\text{-mod}}(V, \mathrm{Hom}_{B_1\text{-mod}}(U,W))$$

$$U = {}_{B_1}U_A \qquad V = {}_AV_{B_2} \qquad W = {}_{B_1}W_{B_2}$$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m : A \otimes A \to A$
$u : \mathbb{k} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V vs $\quad {}_AV$
$\triangleright : A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE
V vs $\quad V_A$
$\triangleleft : V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V vs $\quad {}_{B_1}V_{B_2}$
$\triangleright : B_1 \otimes V \to V$
$\triangleleft : V \otimes B_2 \to V$
& COMP. AXIOMS

EXER. 1.25 HAVE ANOTHER OF VERSION OF: BIMODULE TENSOR-HOM ADJUNCTION

ALSO HAVE:

$$\mathrm{Hom}_{\mathbb{k}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(U, \mathrm{Hom}_{\mathbb{k}}(V, W))$$

TENSOR ADJUNCTION FOR VECTOR SPACES:

$$\boxed{\mathrm{Hom}_{\mathbb{k}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(V, \mathrm{Hom}_{\mathbb{k}}(U, W))}$$

EXER. 1.25 DERIVE TENSOR-HOM ADJUNCTION FOR BIMODULES

$$\mathrm{Hom}_{B_1\text{-mod}}(U \otimes_A V, W) \overset{\text{AS } B_2\text{-BIMODS}}{\cong} \mathrm{Hom}_{A\text{-mod}}(V, \mathrm{Hom}_{B_1\text{-mod}}(U, W))$$

$U = {}_{B_1}U_A \qquad V = {}_AV_{B_2} \qquad W = {}_{B_1}W_{B_2}$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m: A \otimes A \to A$
$u: \Bbbk \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V vs $\quad {}_AV$
$\triangleright: A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE
V vs $\quad V_A$
$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V vs $\quad {}_{B_1}V_{B_2}$
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

<u>EXER. 1.25</u> HAVE ANOTHER OF VERSION OF: BIMODULE TENSOR-HOM ADJUNCTION

WANT TO TRY IT???

ALSO HAVE:

$$\mathrm{Hom}_{\Bbbk}(U \otimes V, W) \cong \mathrm{Hom}_{\Bbbk}(U, \mathrm{Hom}_{\Bbbk}(V, W))$$

TENSOR ADJUNCTION FOR VECTOR SPACES:

$$\boxed{\mathrm{Hom}_{\Bbbk}(U \otimes V, W) \cong \mathrm{Hom}_{\Bbbk}(V, \mathrm{Hom}_{\Bbbk}(U, W))}$$

<u>EXER. 1.25</u> DERIVE TENSOR-HOM ADJUNCTION FOR BIMODULES

$$\mathrm{Hom}_{B_1\text{-mod}}(U \otimes_A V, W) \overset{\text{AS } B_2\text{-BIMODS}}{\cong} \mathrm{Hom}_{A\text{-mod}}(V, \mathrm{Hom}_{B_1\text{-mod}}(U, W))$$

$$U = {}_{B_1}U_A \qquad V = {}_AV_{B_2} \qquad W = {}_{B_1}W_{B_2}$$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m: A \otimes A \to A$
$u: \mathbb{k} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V vs $\quad {}_AV$
$\triangleright: A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE
V vs $\quad V_A$
$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V vs $\quad {}_{B_1}V_{B_2}$
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

EXER. 1.25 DERIVE ANOTHER TENSOR-HOM ADJUNCTION FOR BIMODULES

$$\mathrm{Hom}_{\text{mod-}B_2}(U \otimes_A V, W) \cong \mathrm{Hom}_{??}(U, \mathrm{Hom}_{??}(V, W))$$

$$U = {}_{?}U_{?} \qquad V = {}_AV_{B_2} \qquad W = {}_{?}W_{B_2}$$

$$\mathrm{Hom}_{\mathbb{k}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(U, \mathrm{Hom}_{\mathbb{k}}(V, W))$$

$$\mathrm{Hom}_{\text{mod-}A}(V, W) \equiv (B_2, B_1)\text{-BIMODULE}$$

$${}_{B_1}V_A \qquad {}_{B_2}W_A$$

EXER. 1.25 DERIVE TENSOR-HOM ADJUNCTION FOR BIMODULES

AS B_2-BIMODS

$$\mathrm{Hom}_{B_1\text{-mod}}(U \otimes_A V, W) \cong \mathrm{Hom}_{A\text{-mod}}(V, \mathrm{Hom}_{B_1\text{-mod}}(U, W))$$

$$U = {}_{B_1}U_A \qquad V = {}_AV_{B_2} \qquad W = {}_{B_1}W_{B_2}$$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA

A VS

$m: A \otimes A \to A$

$u: \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V VS $\quad {}_A V$

$\triangleright : A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V VS $\quad V_A$

$\triangleleft : V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V VS $\quad {}_{B_1} V_{B_2}$

$\triangleright : B_1 \otimes V \to V$

$\triangleleft : V \otimes B_2 \to V$

& COMP. AXIOMS

EXER. 1.25 DERIVE ANOTHER TENSOR-HOM ADJUNCTION FOR BIMODULES

$$\mathrm{Hom}_{\text{mod-}B_2}(U \otimes_A V, W) \cong \mathrm{Hom}_{\text{mod-}A}(U, \mathrm{Hom}_{??}(V, W))$$

$$U = {}_{?}U_A \qquad V = {}_A V_{B_2} \qquad W = {}_{?}W_{B_2}$$

$$\mathrm{Hom}_{\mathbb{k}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(U, \mathrm{Hom}_{\mathbb{k}}(V, W))$$

$$\mathrm{Hom}_{\text{mod-}A}(\underset{{}_{B_1}V_A}{V}, \underset{{}_{B_2}W_A}{W}) \equiv (B_2, B_1)\text{-BIMODULE}$$

EXER. 1.25 DERIVE TENSOR-HOM ADJUNCTION FOR BIMODULES

AS B_2-BIMODS

$$\mathrm{Hom}_{B_1\text{-mod}}(U \otimes_A V, W) \cong \mathrm{Hom}_{A\text{-mod}}(V, \mathrm{Hom}_{B_1\text{-mod}}(U, W))$$

$$U = {}_{B_1}U_A \qquad V = {}_A V_{B_2} \qquad W = {}_{B_1}W_{B_2}$$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m: A \otimes A \to A$
$u: \mathbb{k} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE ${}_AV$
V vs
$\triangleright: A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE V_A
V vs
$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE ${}_{B_1}V_{B_2}$
V vs
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

EXER. 1.25 DERIVE ANOTHER TENSOR-HOM ADJUNCTION FOR BIMODULES

$$\mathrm{Hom}_{\mathrm{Mod}\text{-}B_2}(U \otimes_A V, W) \cong \mathrm{Hom}_{\mathrm{Mod}\text{-}A}(U, \mathrm{Hom}_{\mathrm{Mod}\text{-}B_2}(V, W))$$

$$U = {}_{B_1}U_A \qquad V = {}_AV_{B_2} \qquad W = {}_{B_1}W_{B_2}$$

$$\mathrm{Hom}_{\mathbb{k}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(U, \mathrm{Hom}_{\mathbb{k}}(V, W))$$

$$\mathrm{Hom}_{\mathrm{Mod}\text{-}A}(\underset{{}_{B_1}V_A}{V}, \underset{{}_{B_2}W_A}{W}) \equiv (B_2, B_1)\text{-BIMODULE}$$

EXER. 1.25 DERIVE TENSOR-HOM ADJUNCTION FOR BIMODULES

AS B_2-BIMODS

$$\mathrm{Hom}_{B_1\text{-}\mathrm{Mod}}(U \otimes_A V, W) \cong \mathrm{Hom}_{A\text{-}\mathrm{Mod}}(V, \mathrm{Hom}_{B_1\text{-}\mathrm{Mod}}(U, W))$$

$$U = {}_{B_1}U_A \qquad V = {}_AV_{B_2} \qquad W = {}_{B_1}W_{B_2}$$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m: A \otimes A \to A$
$u: \mathbb{k} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V vs $\quad {}_A V$
$\triangleright: A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE
V vs $\quad V_A$
$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V vs $\quad {}_{B_1}V_{B_2}$
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

EXER. 1.25 DERIVE ANOTHER TENSOR-HOM ADJUNCTION FOR BIMODULES

= CHECK THE REST =

$$\mathrm{Hom}_{\text{Mod-}B_2}(U \otimes_A V, W) \underset{\text{AS } B_1\text{-BIMODS}}{\cong} \mathrm{Hom}_{\text{Mod-}A}(U, \mathrm{Hom}_{\text{Mod-}B_2}(V, W))$$

$$U = {}_{B_1}U_A \qquad V = {}_A V_{B_2} \qquad W = {}_{B_1}W_{B_2}$$

$$\mathrm{Hom}_{\mathbb{k}}(U \otimes V, W) \cong \mathrm{Hom}_{\mathbb{k}}(U, \mathrm{Hom}_{\mathbb{k}}(V, W))$$

$$\mathrm{Hom}_{\text{Mod-}A}(\underset{{}_{B_1}V_A}{V}, \underset{{}_{B_2}W_A}{W}) \equiv (B_2, B_1)\text{-BIMODULE}$$

EXER. 1.25 DERIVE TENSOR-HOM ADJUNCTION FOR BIMODULES

$$\mathrm{Hom}_{B_1\text{-Mod}}(U \otimes_A V, W) \underset{\text{AS } B_2\text{-BIMODS}}{\cong} \mathrm{Hom}_{A\text{-Mod}}(V, \mathrm{Hom}_{B_1\text{-Mod}}(U, W))$$

$$U = {}_{B_1}U_A \qquad V = {}_A V_{B_2} \qquad W = {}_{B_1}W_{B_2}$$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA

A vs

$m: A \otimes A \to A$

$u: \mathbb{k} \otimes A \to A$

& comp. axioms

LEFT A-MODULE $_AV$

V vs

$\triangleright: A \otimes V \to V$

& comp. axioms

RIGHT A-MODULE V_A

V vs

$\triangleleft: V \otimes A \to V$

& comp. axioms

(B_1, B_2)-BIMODULE $_{B_1}V_{B_2}$

V vs

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& comp. axioms

ONE BIMODULE TENSOR-HOM ADJUNCTION

$$\mathrm{Hom}_{B_1\text{-mod}}(U \otimes_A V, W) \underset{\text{as } B_2\text{-bimods}}{\cong} \mathrm{Hom}_{A\text{-mod}}(V, \mathrm{Hom}_{B_1\text{-mod}}(U, W))$$

$$U = {}_{B_1}U_A \qquad V = {}_AV_{B_2} \qquad W = {}_{B_1}W_{B_2}$$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA

A vs

$m: A \otimes A \to A$

$u: \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V vs $\quad {}_A V$

$\triangleright: A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V vs $\quad {}_{B_1} V_{B_2}$

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

ONE BIMODULE TENSOR-HOM ADJUNCTION

$$\mathrm{Hom}_{B_1\text{-mod}}(U \otimes_A V, W) \cong \mathrm{Hom}_{A\text{-mod}}(V, \mathrm{Hom}_{B_1\text{-mod}}(U, W))$$

AS ~~B_2-BIMODS~~ VS

TAKE $B_2 = \mathbb{k}$ $\quad U = {}_{B_1}U_A \quad V = {}_A V_{\cancel{B_2}} \quad W = {}_{B_1} W_{\cancel{B_2}}$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA

A VS

$m: A \otimes A \to A$

$u: \Bbbk \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE

V VS $\quad {}_A V$

$\triangleright: A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V VS $\quad V_A$

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE

V VS $\quad {}_{B_1} V_{B_2}$

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

ONE BIMODULE TENSOR-HOM ADJUNCTION

$$\mathrm{Hom}_{B_1\text{-mod}}(U \otimes_A V, W) \underset{\text{AS } \Bbbk\text{-VS}}{\uparrow\cong} \mathrm{Hom}_{A\text{-mod}}(V, \mathrm{Hom}_{B_1\text{-mod}}(U, W))$$

$$U = {}_{B_1}U_A \qquad V = {}_A V \qquad W = {}_{B_1} W$$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA
A VS
$m: A \otimes A \to A$
$u: \Bbbk \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V VS $\quad {}_AV$
$\triangleright: A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE
V VS $\quad V_A$
$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V VS $\quad {}_{B_1}V_{B_2}$
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

ASSUME $\exists$ ALGEBRA MAP $\phi: A \to B$

ONE BIMODULE TENSOR-HOM ADJUNCTION

$$\mathrm{Hom}_{B_1\text{-mod}}(U \otimes_A V, W) \cong \mathrm{Hom}_{A\text{-mod}}(V, \mathrm{Hom}_{B_1\text{-mod}}(U, W))$$

AS $\Bbbk$-VS

TAKE $B_1 = B$ $\quad U = {}_{B_1}U_A \quad V = {}_AV \quad W = {}_{B_1}W$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA

A vs

$m: A \otimes A \to A$

$u: \Bbbk \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE $\quad {}_A V$

V vs

$\triangleright: A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE $\quad V_A$

V vs

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE $\quad {}_{B_1}V_{B_2}$

V vs

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

ASSUME $\exists$ ALGEBRA MAP $\phi: A \to B$

ONE BIMODULE TENSOR-HOM ADJUNCTION

$$\mathrm{Hom}_{B\text{-mod}}(U \otimes_A V, W) \underset{\text{AS } \Bbbk\text{-VS}}{\cong} \mathrm{Hom}_{A\text{-mod}}(V, \mathrm{Hom}_{B\text{-mod}}(U, W))$$

$U = {}_B U_A \qquad V = {}_A V \qquad W = {}_B W$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA
A VS
$m : A \otimes A \to A$
$u : \Bbbk \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE $\quad {}_AV$
V VS
$\triangleright : A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE $\quad V_A$
V VS
$\triangleleft : V \otimes A \to V$
& COMP. AXIOMS

(B_1,B_2)-BIMODULE $\quad {}_{B_1}V_{B_2}$
V VS
$\triangleright : B_1 \otimes V \to V$
$\triangleleft : V \otimes B_2 \to V$
& COMP. AXIOMS

ASSUME $\exists$ ALGEBRA MAP $\phi : A \to B$

GET $B = {}_BB_A$

ONE BIMODULE TENSOR-HOM ADJUNCTION

$$\mathrm{Hom}_{B\text{-mod}}(U \otimes_A V, W) \underset{\text{AS } \Bbbk\text{-VS}}{\cong} \mathrm{Hom}_{A\text{-mod}}(V, \mathrm{Hom}_{B\text{-mod}}(U, W))$$

$U = {}_BU_A \qquad V = {}_AV \qquad W = {}_BW$

TAKE $U = B$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA

A vs

$\mu : A \otimes A \to A$

$u : \Bbbk \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE ${}_A V$

V vs

$\triangleright : A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE V_A

V vs

$\triangleleft : V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE ${}_{B_1} V_{B_2}$

V vs

$\triangleright : B_1 \otimes V \to V$

$\triangleleft : V \otimes B_2 \to V$

& COMP. AXIOMS

ASSUME $\exists$ ALGEBRA MAP $\phi : A \to B$

GET $B = {}_B B_A$

ONE BIMODULE TENSOR-HOM ADJUNCTION

$$\mathrm{Hom}_{B\text{-mod}}(B \otimes_A V, W) \underset{\text{AS } \Bbbk\text{-VS}}{\cong} \mathrm{Hom}_{A\text{-mod}}(V, \mathrm{Hom}_{B\text{-mod}}(B, W))$$

$B = {}_B B_A \qquad V = {}_A V \qquad W = {}_B W$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m: A \otimes A \to A$
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& COMP. AXIOMS

LEFT A-MODULE ${}_A V$
V vs
$\triangleright: A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE V_A
V vs
$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE ${}_{B_1}V_{B_2}$
V vs
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

ASSUME $\exists$ ALGEBRA MAP $\phi: A \to B$
GET $B = {}_B B_A$

INDUCTION OF ${}_A V$ TO B ALONG

$$\underset{\text{ALG. MAP}}{\phi: A \longrightarrow B}$$

$$\mathrm{Ind}_A^B(V) := B \otimes_A V$$

$B = {}_B B_A$ LEFT B-MODULE

RESTRICTION OF ${}_B V$ TO A ALONG

$$\underset{\text{ALG. MAP}}{\phi: A \longrightarrow B}$$

$$\mathrm{Res}_A^B(V) := \mathrm{Hom}_{B\text{-mod}}(B, V)$$

$B = {}_B B_A$ LEFT A-MODULE

ONE BIMODULE TENSOR-HOM ADJUNCTION

$$\mathrm{Hom}_{B\text{-mod}}(B \otimes_A V, W) \underset{\text{AS } \Bbbk\text{-VS}}{\cong} \mathrm{Hom}_{A\text{-mod}}(V, \mathrm{Hom}_{B\text{-mod}}(B, W))$$

$B = {}_B B_A \qquad V = {}_A V \qquad W = {}_B W$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA

A vs

$m: A \otimes A \to A$

$u: \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

LEFT A-MODULE $\quad {}_A V$

V vs

$\triangleright: A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE $\quad V_A$

V vs

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1, B_2)-BIMODULE $\quad {}_{B_1}V_{B_2}$

V vs

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

ASSUME $\exists$ ALGEBRA MAP $\phi: A \to B$

GET $B = {}_B B_A$

INDUCTION OF ${}_A V$ TO B ALONG

$\phi: A \longrightarrow B$

ALG. MAP

$$\mathrm{Ind}_A^B(V) := B \otimes_A V$$

$B = {}_B B_A$ LEFT B-MODULE

RESTRICTION OF ${}_B V$ TO A ALONG

$\phi: A \longrightarrow B$

ALG. MAP

$$\mathrm{Res}_A^B(V) := \mathrm{Hom}_{B\text{-mod}}(B, V)$$

$B = {}_B B_A$ LEFT A-MODULE

$\equiv$ FROBENIUS RECIPROCITY $\equiv$

$$\mathrm{Hom}_{B\text{-mod}}(\mathrm{Ind}_A^B(V), W) \underset{\text{AS } \mathbb{k}\text{-VS}}{\cong} \mathrm{Hom}_{A\text{-mod}}(V, \mathrm{Res}_A^B(W))$$

$B = {}_B B_A \qquad V = {}_A V \qquad W = {}_B W$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA
A VS
$m: A \otimes A \to A$
$u: \Bbbk \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V VS $\quad {}_A V$
$\triangleright: A \otimes V \to V$
& COMP. AXIOMS

RIGHT A-MODULE
V VS $\quad V_A$
$\triangleleft: V \otimes A \to V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V VS $\quad {}_{B_1}V_{B_2}$
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

ORIGINAL VERSION OF FROBENIUS RECIPROCITY WAS FOR GROUP REPRESENTATIONS

TAKE $H \leq G$ SUBGROUP

V A REP'N OF H

W A REP'N OF G

$\mathrm{Hom}_G(W, W')$ MAPS OF G-REP'NS

$\cong$

$\Bbbk$-LINEAR MAPS $W \to W'$ THAT INTERTWINES WITH G-ACTION

≡ FROBENIUS RECIPROCITY ≡

$$\mathrm{Hom}_{B\text{-mod}}\left(\mathrm{Ind}_A^B(V), W\right) \underset{\text{AS } \Bbbk\text{-VS}}{\cong} \mathrm{Hom}_{A\text{-mod}}\left(V, \mathrm{Res}_A^B(W)\right)$$

$$B = {}_B B_A \qquad V = {}_A V \qquad W = {}_B W$$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m: A \otimes A \to A$
$u: \mathbb{k} \otimes A \to A$
& comp. axioms

LEFT A-MODULE
V vs $\quad {}_AV$
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(B_1, B_2)-BIMODULE
V vs $\quad {}_{B_1}V_{B_2}$
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& comp. axioms

ORIGINAL VERSION OF FROBENIUS RECIPROCITY WAS FOR GROUP REPRESENTATIONS

TAKE $H \leq G$ subgroup

V a rep'n of H

W a rep'n of G

GET:

$$\boxed{\begin{array}{c} \mathrm{Hom}_G(\mathrm{Ind}_H^G(V), W) \\ \text{SII as } \mathbb{k}\text{-vs} \\ \mathrm{Hom}_H(V, \mathrm{Res}_H^G(W)) \end{array}}$$

≡ FROBENIUS RECIPROCITY ≡

$$\mathrm{Hom}_{B\text{-mod}}(\mathrm{Ind}_A^B(V), W) \underset{\text{as } \mathbb{k}\text{-vs}}{\cong} \mathrm{Hom}_{A\text{-mod}}(V, \mathrm{Res}_A^B(W))$$

$B = {}_BB_A \qquad V = {}_AV \qquad W = {}_BW$

III. HOM AND DUAL OF ALGEBRAS & MODULES

WILL ONLY DISCUSS ONE CONSTRUCTION

ALGEBRA
A vs
$m: A \otimes A \longrightarrow A$
$u: \Bbbk \otimes A \longrightarrow A$
& COMP. AXIOMS

LEFT A-MODULE
V vs
${}_A V$
$\triangleright: A \otimes V \longrightarrow V$
& COMP. AXIOMS

RIGHT A-MODULE
V vs
V_A
$\triangleleft: V \otimes A \longrightarrow V$
& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V vs
${}_{B_1} V_{B_2}$
$\triangleright: B_1 \otimes V \longrightarrow V$
$\triangleleft: V \otimes B_2 \longrightarrow V$
& COMP. AXIOMS

III. HOM AND DUAL OF ALGEBRAS & MODULES

WILL ONLY DISCUSS ONE CONSTRUCTION

ALGEBRA

A vs

$m: A \otimes A \to A$

$u: \mathbb{k} \otimes A \to A$

& COMP. AXIOMS

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V vs $\quad {}_AV$

$\triangleright: A \otimes V \to V$

& COMP. AXIOMS

RIGHT A-MODULE

V vs $\quad V_A$

$\triangleleft: V \otimes A \to V$

& COMP. AXIOMS

(B_1,B_2)-BIMODULE

V vs $\quad {}_{B_1}V_{B_2}$

$\triangleright: B_1 \otimes V \to V$

$\triangleleft: V \otimes B_2 \to V$

& COMP. AXIOMS

$$\mathrm{HOM}_{A\text{-mod}}(V, W) \equiv (B_1, B_2)\text{-BIMODULE}$$

${}_AV_{B_1} \qquad {}_AW_{B_2}$

$$\mathrm{HOM}_{\text{mod-}A}(V, W) \equiv (B_2, B_1)\text{-BIMODULE}$$

${}_{B_1}V_A \qquad {}_{B_2}W_A$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m: A \otimes A \to A$
$u: \mathbb{k} \otimes A \to A$
& COMP. AXIOMS

LEFT A-MODULE
V vs $\quad {}_AV$
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& COMP. AXIOMS

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& COMP. AXIOMS

(B_1, B_2)-BIMODULE
V vs $\quad {}_{B_1}V_{B_2}$
$\triangleright: B_1 \otimes V \to V$
$\triangleleft: V \otimes B_2 \to V$
& COMP. AXIOMS

$$\mathrm{Hom}_{A\text{-mod}}(\underset{{}_AV_{B_1}}{V}, \underset{{}_AW_{B_2}}{W}) \equiv (B_1, B_2)\text{-BIMODULE}$$

$$\mathrm{Hom}_{\text{mod-}A}(\underset{{}_{B_1}V_A}{V}, \underset{{}_{B_2}W_A}{W}) \equiv (B_2, B_1)\text{-BIMODULE}$$

THESE ARE ALGEBRAS:

$\mathbb{k}$-VS ✓ MULTIPLICATION = COMPOSITION UNIT $= \mathrm{Id}_A$

$$\mathrm{End}_{A\text{-mod}}({}_A(A_{reg})_A) \qquad \mathrm{End}_{\text{mod-}A}({}_A(A_{reg})_A)$$

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m: A \otimes A \to A$
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V vs $\quad {}_{B_1}V_{B_2}$
$\triangleright: B_1 \otimes V \to V$
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& COMP. AXIOMS

$$\mathrm{Hom}_{A\text{-mod}}(V, W) \equiv (B_1, B_2)\text{-BIMODULE}$$
${}_AV_{B_1}$ ${}_AW_{B_2}$

$$\mathrm{Hom}_{\text{mod-}A}(V, W) \equiv (B_2, B_1)\text{-BIMODULE}$$
${}_{B_1}V_A$ ${}_{B_2}W_A$

THESE ARE ALGEBRAS :

$\mathbb{k}$-VS ✓ MULTIPLICATION = COMPOSITION UNIT $= \mathrm{Id}_A$

$$\mathrm{End}_{A\text{-mod}}({}_A(A_{reg})_A)$$
$\cong$
A^{op} OPPOSITE ALG.
$m_{A^{op}}(a \otimes b) = ba$

EXER 1.26

$$\mathrm{End}_{\text{mod-}A}({}_A(A_{reg})_A)$$
$\cong$
A

EXER 1.26

III. HOM AND DUAL OF ALGEBRAS & MODULES

ALGEBRA
A vs
$m: A \otimes A \to A$
$u: \Bbbk \otimes A \to A$
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$\triangleright: B_1 \otimes V \to V$
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& COMP. AXIOMS

$$\mathrm{Hom}_{A\text{-mod}}(V, W) \equiv (B_1, B_2)\text{-BIMODULE}$$
$${}_A V_{B_1} \qquad {}_A W_{B_2}$$

$$\mathrm{Hom}_{\text{mod-}A}(V, W) \equiv (B_2, B_1)\text{-BIMODULE}$$
$${}_{B_1} V_A \qquad {}_{B_2} W_A$$

THESE ARE ALGEBRAS:

$\Bbbk$-VS ✓ MULTIPLICATION = COMPOSITION UNIT $= \mathrm{Id}_A$

$$\mathrm{End}_{A\text{-mod}}({}_A(A_{reg})_A)$$
$$\cong$$
$$A^{op}$$

youdo!

EXER 1.26

$$\mathrm{End}_{\text{mod-}A}({}_A(A_{reg})_A)$$
$$\cong$$
$$A$$

$f: A \to A \longmapsto f(1_A)$

$f: A \to A$, $b \mapsto ab$ $\longleftarrow a$

EXER 1.26 ≡ HINT ≡

MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #4

TOPICS:

I. DIRECT PRODUCT, SUM, DIRECT SUM OF ALGEBRAS & MODULES (§1.4.1)

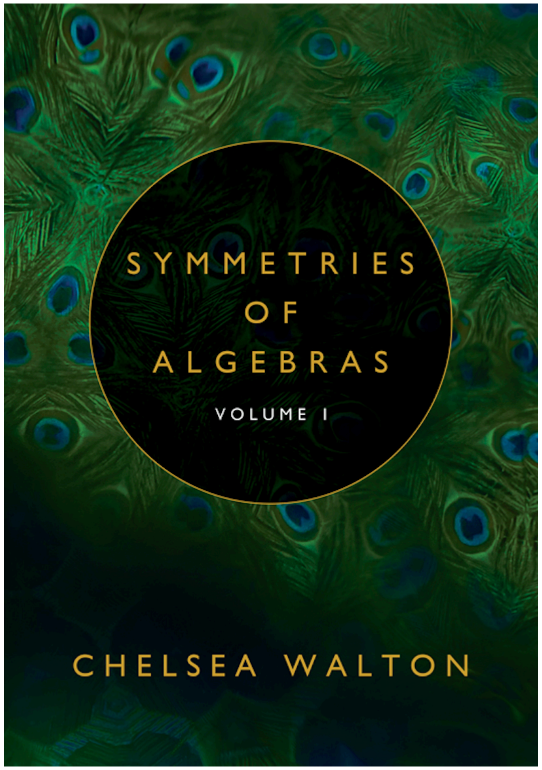
II. TENSOR PRODUCT OF ALGEBRAS & MODULES (§§1.4.2, 1.4.4)

III. HOM AND DUAL OF ALGEBRAS & MODULES (§§1.4.3, 1.4.4)

NEXT TIME: CLASSIFYING NICE ALGEBRAS

Enjoy this lecture?
You'll enjoy the textbook!

C. Walton's "Symmetries of Algebras, Volume 1" (2024)



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Also on Amazon
&
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Lecture #4 keywords: Bimodule Tensor-Hom adjunction, coinduction, direct product/direct sum of algebras, direct product/direct sum of modules, Frobenius Reciprocity, induction, restriction, tensor product of modules