MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #5

LAST TIME

OPERATIONS ON
ALGEBRAS
& MODULES:

$\times \quad + \quad \oplus \quad \otimes \quad \mathrm{Hom}$

TOPICS:

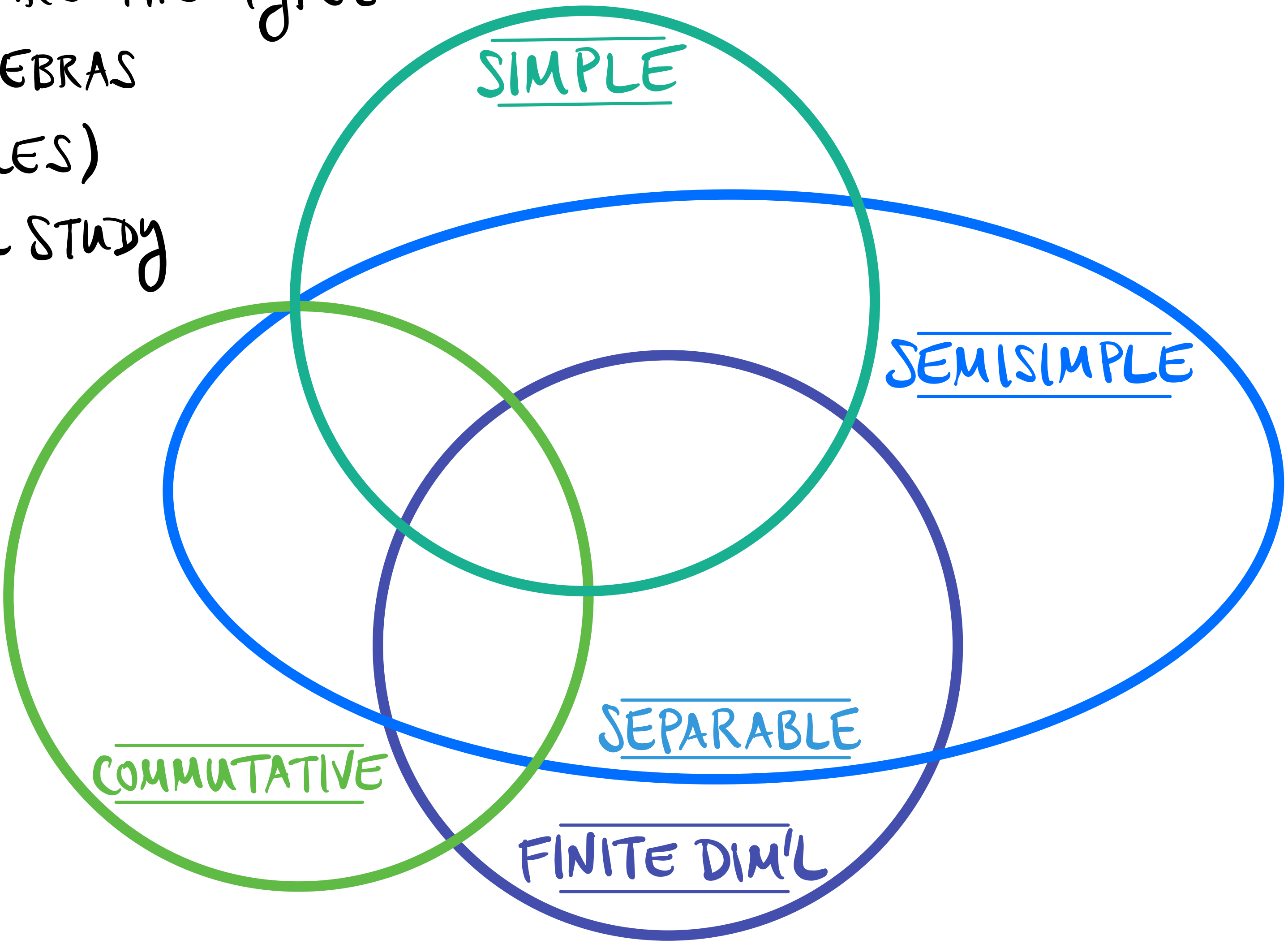
I. SIMPLICITY (§1.5)

II. SEMISIMPLICITY (§1.6)

III. SEPARABILITY (§1.7)

ON THE CLASSIFICATION OF NICE ALGEBRAS —

THESE ARE THE TYPES
OF ALGEBRAS
(& MODULES)
WE WILL STUDY



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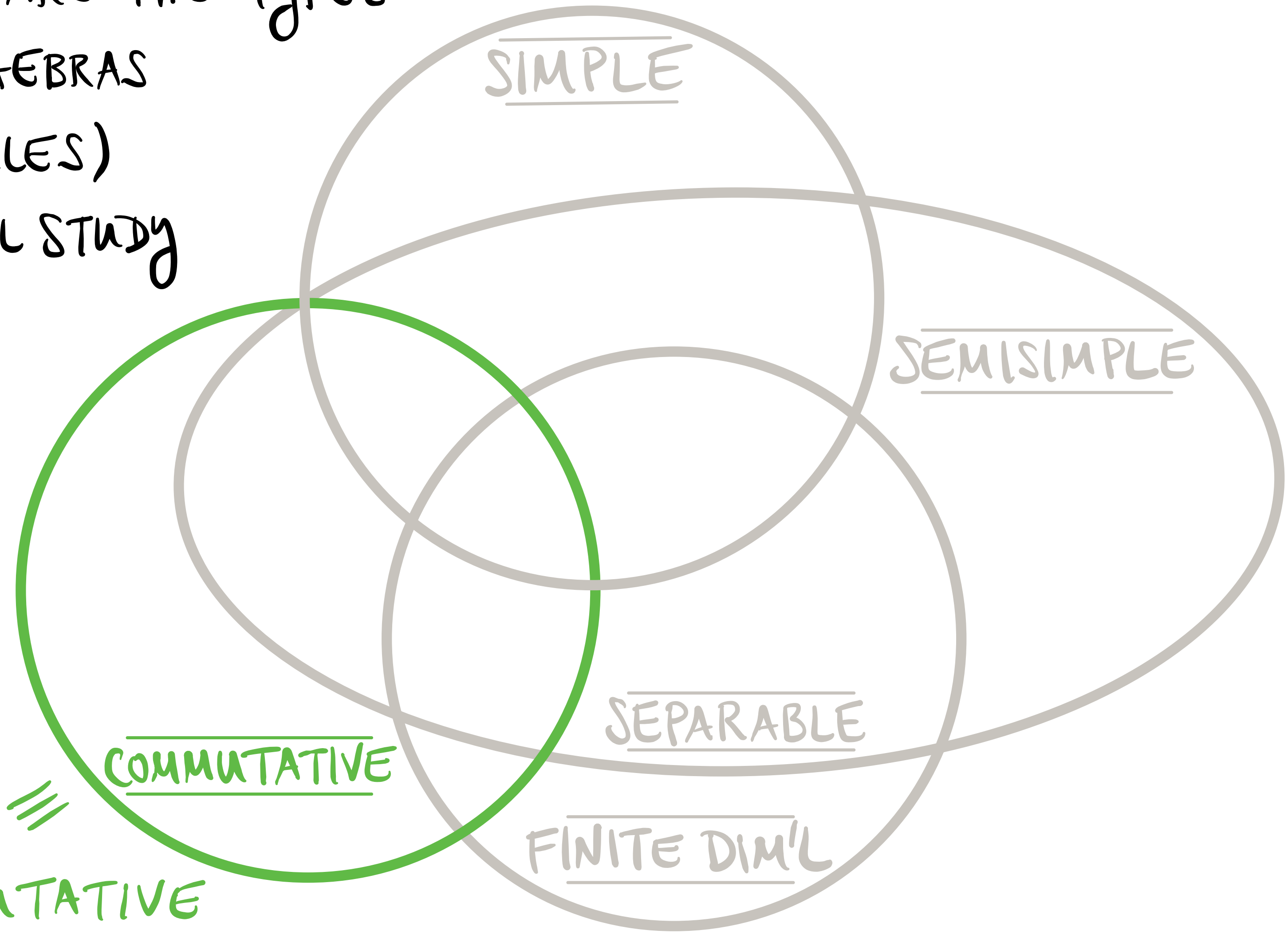

COMMUTATIVE AS A RING

ON THE CLASSIFICATION OF NICE ALGEBRAS —

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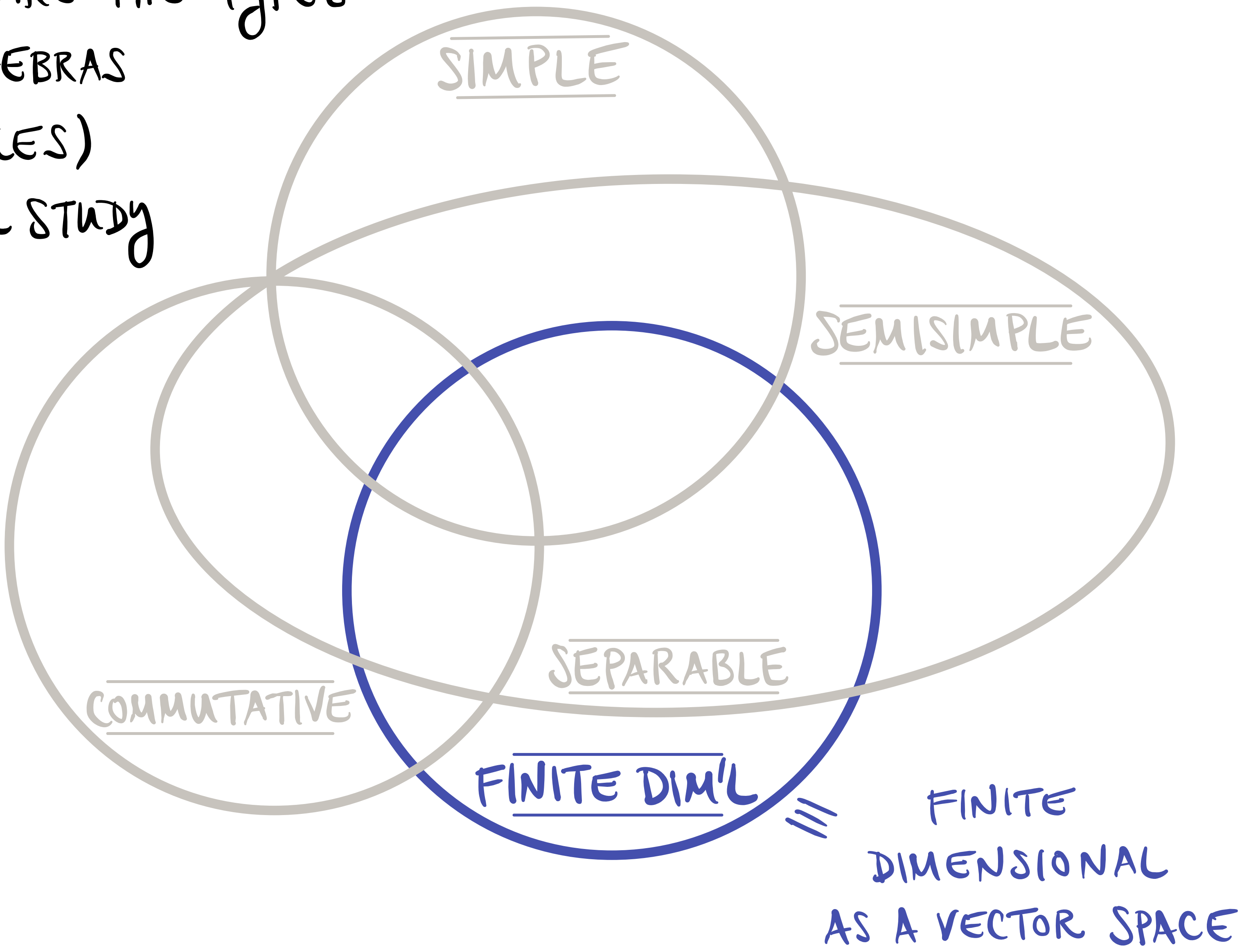
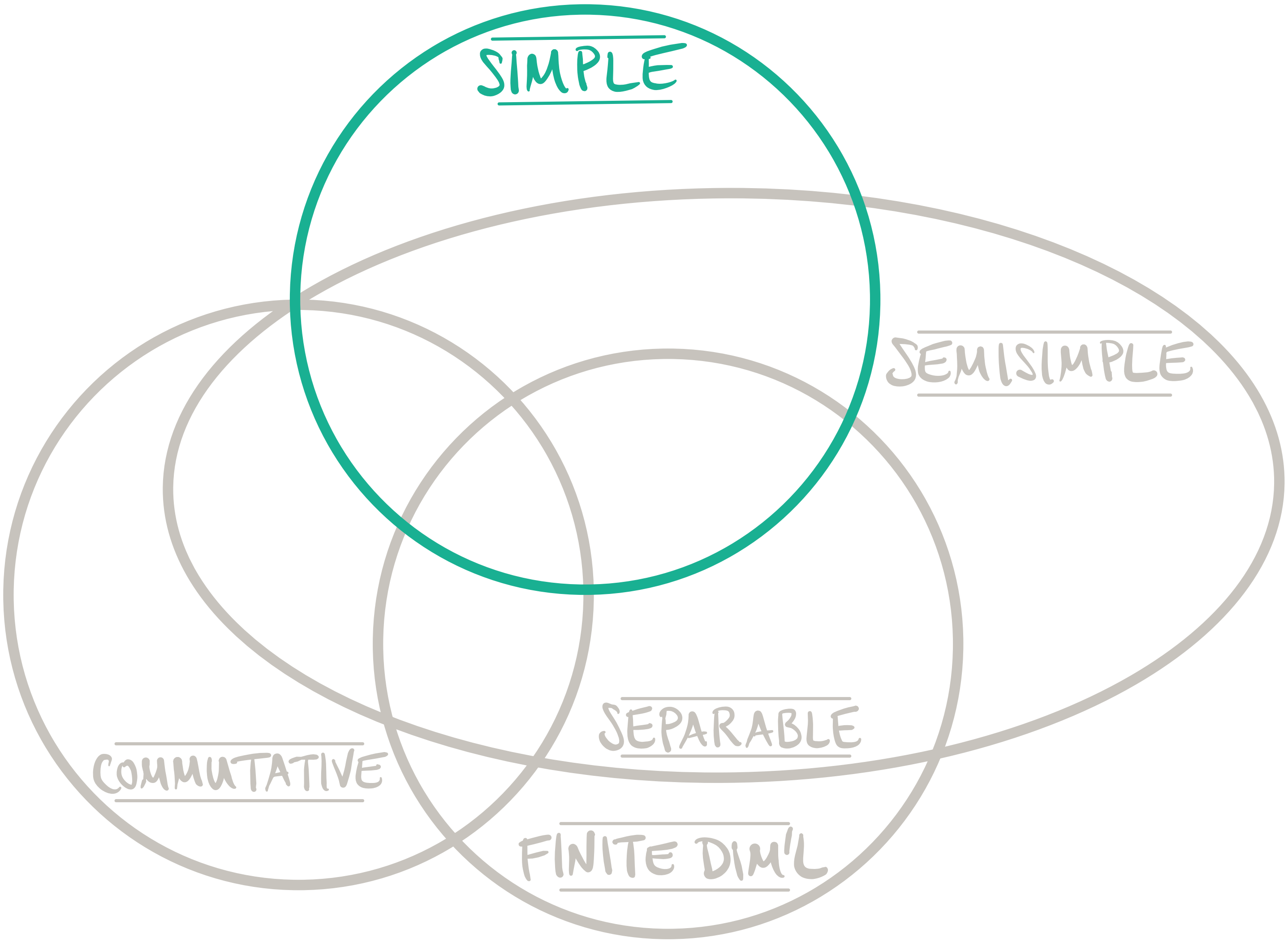


I. SIMPLICITY



I. SIMPLICITY

TAKE A $\Bbbk$-ALGEBRA (A, m, u) & A NONZERO LEFT A-MODULE $(V, \triangleright)$

V IS SIMPLE IF THE ONLY LEFT A-SUBMODULES OF V ARE 0 & V

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V IS **INDECOMPOSABLE** IF $\nexists$ LEFT A-SUBMODULES $\overset{\neq 0}{V_1}$ & $\overset{\neq 0}{V_2}$ OF V SUCH THAT $V \cong V_1 \oplus V_2$

BOTH ARE BUILDING BLOCKS OF MODULE THEORY

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$\leftrightsquigarrow$

$\left(V, \rho_V : A \to \mathrm{End}_{\Bbbk}(V)\right)$

IRREDUCIBLE REP'N

BOTH ARE BUILDING BLOCKS OF MODULE THEORY & REPRESENTATION THEORY

V IS INDECOMPOSABLE IF $\nexists$ LEFT A-SUBMODULES $V_1^{\neq 0}$ & $V_2^{\neq 0}$ OF V SUCH THAT $V \cong V_1 \oplus V_2$

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INDECOMPOSABLE REP'N

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SIMPLICITY ⇒ INDECOMPOSABILITY

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SIMPLICITY $\Rightarrow$ INDECOMPOSABILITY

PF/ IF V IS DECOMPOSABLE, $\exists$ SUBMODULES $V_1^{\neq 0}, V_2^{\neq 0}$ OF V $\ni$. $V \cong V_1 \oplus V_2$.

THEN V IS NOT SIMPLE.

INDEED, V_1 IS A SUBMOD. OF V

$\neq 0$ (BY HYP)

AND $\neq V$ (SINCE $V_2 \neq 0$).

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Ex. THE REGULAR LEFT $\Bbbk[v]$-MODULE $\Bbbk[v]_{reg}$ IS INDECOMPOSABLE

(≡ CHECK ≡)

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BUT $\Bbbk[v]_{reg}$ IS NOT SIMPLE: THE LEFT IDEAL (v) IS A SUBMODULE $\neq 0, \Bbbk[v]$.

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AN IMPORTANT RESULT...

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SCHUR'S LEMMA

TAKE SIMPLE LEFT A-MODULES W, W'.

IF $\phi \in \mathrm{Hom}_{A\text{-Mod}}(W, W')$,

THEN $\phi = 0$ OR ϕ IS AN ISOM.

$\therefore \mathrm{End}_{A\text{-Mod}}(W)$ IS A DIVISION ALG.

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PF/ TAKE $\phi \neq 0 \in \mathrm{Hom}_{A\text{-mod}}(W, W')$.

THEN $\ker(\phi)$ IS A SUBMODULE OF W.

W SIMPLE $\Rightarrow$ $\ker(\phi) = 0$ OR $\ker(\phi) = W$

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W SIMPLE $\Rightarrow$ $\ker(\phi) = 0$ OR $\cancel{\ker(\phi) = W}$

$\therefore \phi$ IS INJECTIVE

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W SIMPLE $\Rightarrow$ $\ker(\phi) = 0$ OR ~~$\ker(\phi) = W$~~

$\therefore$ ϕ IS INJECTIVE

LIKEWISE, W' SIMPLE $\Rightarrow$ $\mathrm{im}(\phi) = W'$. $\therefore$ ϕ IS SURJECTIVE. ///

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V IS SIMPLE

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VERSUS IDEALS OF A ...

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A NONZERO LEFT IDEAL I OF A
IS MINIMAL IF
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$\leftsquigarrow$

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A LEFT IDEAL I OF A IS A LEFT A-MODULE

$$(I \hookrightarrow A \overset{|||}{,} \lambda : A \otimes I \longrightarrow I)$$

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BY DEF'N: A MINIMAL LEFT IDEAL I OF A IS A SIMPLE LEFT A-MODULE

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⇜

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A NONZERO LEFT IDEAL I OF A IS MINIMAL IF THE ONLY LEFT IDEALS OF A CONTAINED IN I ARE 0 & I

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$(I \hookrightarrow A \overset{|||}{,} \lambda : A \otimes I \longrightarrow I)$

IF ${}_A(A_{reg}) = \bigoplus_{i \in I} V_i$, A DIRECT SUM OF LEFT A-MODULES, THEN EACH V_i IS A LEFT IDEAL OF A.

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TAKE A $\Bbbk$-ALGEBRA (A, m, u)

> A IS **SIMPLE** IF THE ONLY (TWO-SIDED) IDEALS OF A ARE 0 & A

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$$A \cong A_1 \times A_2$$

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PF/ IF A IS DECOMPOSABLE, $\exists$ ALGEBRAS $A_1^{\neq 0}, A_2^{\neq 0}$ $\ni$. $A \cong A_1 \times A_2$. THEN A IS NOT SIMPLE.

INDEED, $(A_1)_{reg} \times 0_{A_2}$ IS AN IDEAL OF A.

$\neq 0$ (BY HYP)

AND $\neq A$ (SINCE $A_2 \neq 0$).

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SIMPLICITY $\not\Leftarrow$ INDECOMPOSABILITY

Ex. $\Bbbk[v]$ IS AN INDECOMP. ALGEBRA.

($Z(\Bbbk[v]) = \Bbbk[v]$ & THE ONLY CENTRAL IDEMPOTENTS OF $\Bbbk[v]$ ARE $0_{\Bbbk[v]}$ & $1_{\Bbbk[v]}$)

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SIMPLICITY $\not\Leftarrow$ INDECOMPOSABILITY

Ex. $\Bbbk[v]$ IS AN INDECOMP. ALGEBRA.

BUT

$\Bbbk[v]$ IS NOT A SIMPLE ALG.

((v) IS AN IDEAL OF $\Bbbk[v]$ $\neq 0, \neq \Bbbk[v]$)

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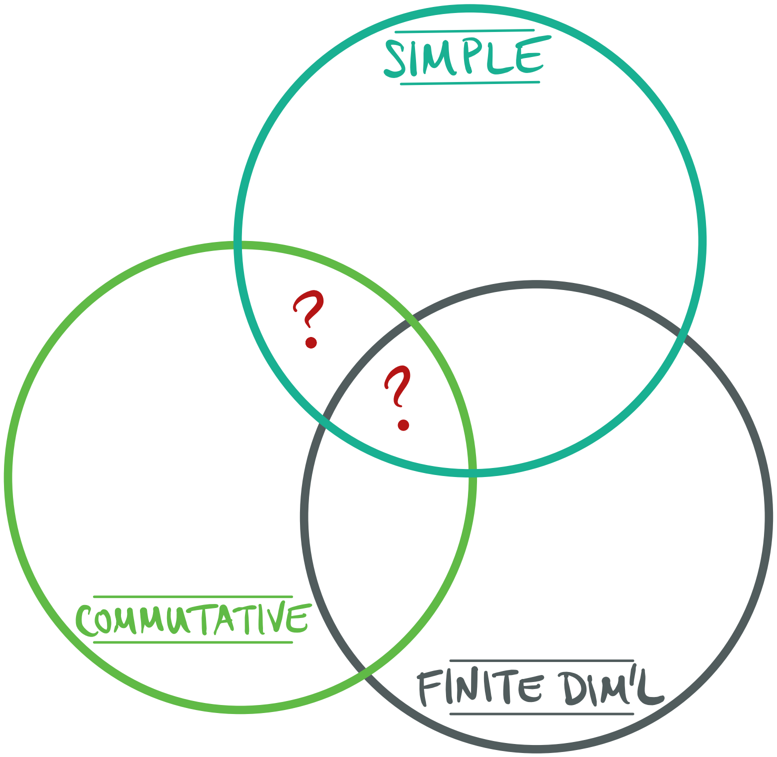

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A IS SIMPLE IF THE ONLY (TWO-SIDED) IDEALS OF A ARE 0 & A

TAKE A COMMUTATIVE ALG. $C \neq 0$.

THEN: C IS SIMPLE

$$\Updownarrow$$

C IS A FIELD.

IN THIS CASE: $\dim_{\Bbbk} C < \infty$

$$\Updownarrow$$

$C = \Bbbk$.

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PF/ (*) $\Uparrow$ ✓ FIELDS ARE COMMUTATIVE AND SIMPLE.

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PF/ (*) $\Uparrow$ ✓ FIELDS ARE COMMUTATIVE AND SIMPLE.

(*) $\Downarrow$ TAKE $x^{\neq 0} \in C$.

THE (NONZERO) IDEAL (x) OF C EQUALS C SINCE C IS SIMPLE.

$\therefore \exists y \in C$ s.t. $xy = 1_C$.

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THEN: $(*)$

$$\begin{gathered} C \text{ IS SIMPLE} \\ \Updownarrow \\ C \text{ IS A FIELD.} \end{gathered}$$

IN THIS CASE: $(**)$

$$\begin{gathered} \dim_{\Bbbk} C < \infty \\ \Updownarrow \\ C = \Bbbk . \end{gathered}$$

PF/ $(*)$ $\Uparrow$ ✓ FIELDS ARE COMMUTATIVE AND SIMPLE.

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THE (NONZERO) IDEAL (x) OF C EQUALS C SINCE C IS SIMPLE.

$\therefore \exists y \in C$ s.t. $xy = 1_C$. $\therefore$ x IS INVERTIBLE

$$\underset{\substack{\| \\ yx}}{xy}$$

C COMMUTATIVE

$\therefore$ C IS A FIELD. /// $(*)$

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PF/ (**) $\Uparrow$ ✓ $\Bbbk$ IS A FINITE DIM'L (SIMPLE) ALGEBRA.

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(**) $\Downarrow$ IF $\dim_{\Bbbk} C < \infty$, THEN C IS A FINITE FIELD EXT'N OF $\Bbbk$

$\Downarrow$

ALGEBRAIC

I. SIMPLICITY

TAKE A $\Bbbk$-ALGEBRA (A, m, u)

A IS **SIMPLE** IF THE ONLY (TWO-SIDED) IDEALS OF A ARE 0 & A

TAKE A **COMMUTATIVE** ALG. $C^{\neq 0}$.

THEN: (*) C IS SIMPLE $\Updownarrow$ C IS A FIELD.

IN THIS CASE: (**) $\dim_{\Bbbk} C < \infty$ $\Updownarrow$ $C = \Bbbk$.

PF/ (**) $\Uparrow$ ✓ $\Bbbk$ IS A FINITE DIM'L (SIMPLE) ALGEBRA.

(**) $\Downarrow$ IF $\dim_{\Bbbk} C < \infty$, THEN C IS A FINITE FIELD EXT'N OF $\Bbbk$
$\Downarrow$
ALGEBRAIC

$\Bbbk$ ALGEBRAICALLY CLOSED (BY ASSUMPTION) $\Rightarrow$ $C \cong \Bbbk$. /// (**)

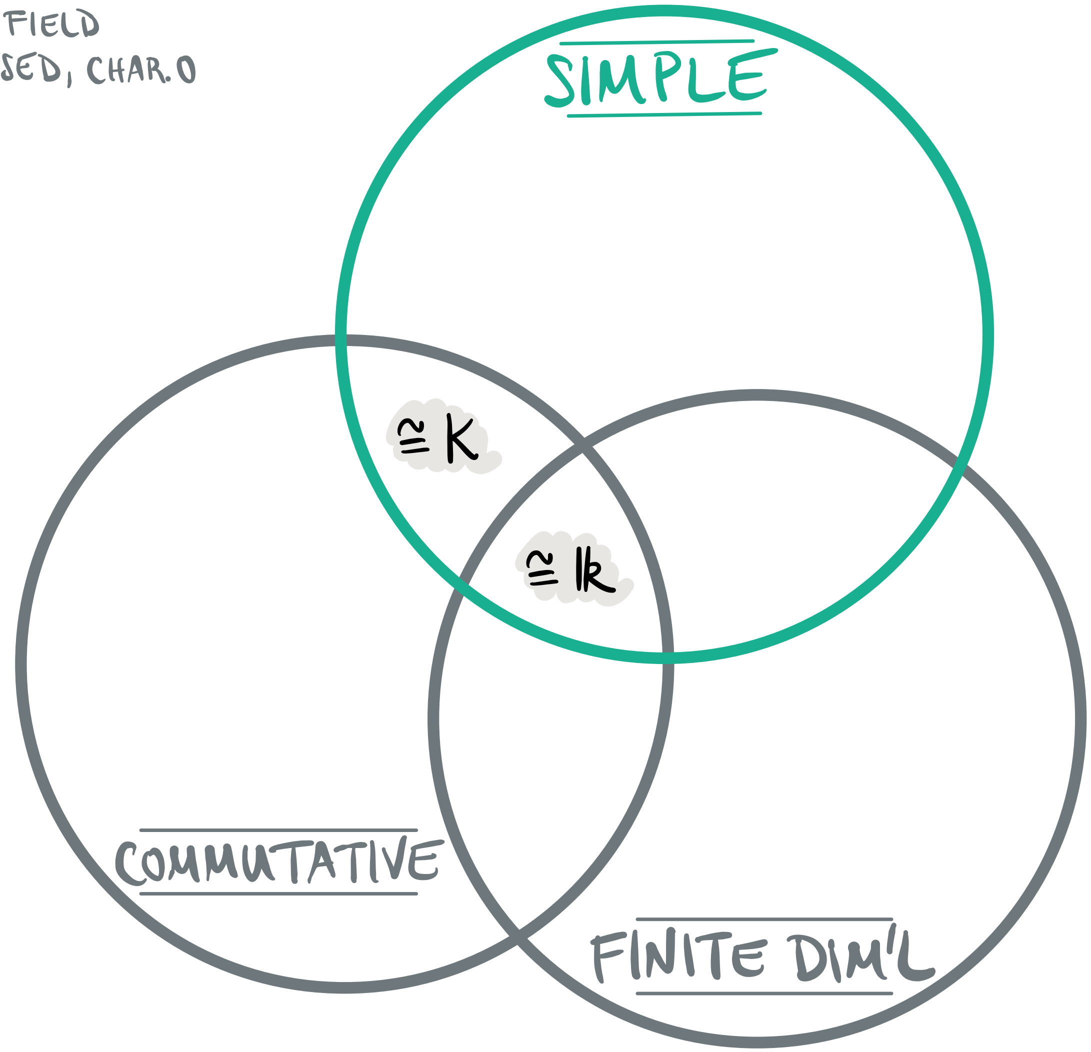
I. SIMPLICITY

$$\Bbbk \begin{cases} \text{GROUND FIELD} \\ \text{ALG. CLOSED, CHAR. 0} \end{cases}$$

Ex. $\Bbbk = \mathbb{C}$

K FIELD EXT'N OF $\Bbbk$

Ex. $K = \text{Frac}(\Bbbk[x])$



I. SIMPLICITY

n POSITIVE INTEGER

$\Bbbk$ $\begin{cases} \text{GROUND FIELD} \\ \text{ALG. CLOSED, CHAR. 0} \end{cases}$

Ex. $\Bbbk = \mathbb{C}$

ALSO HAVE ⤵

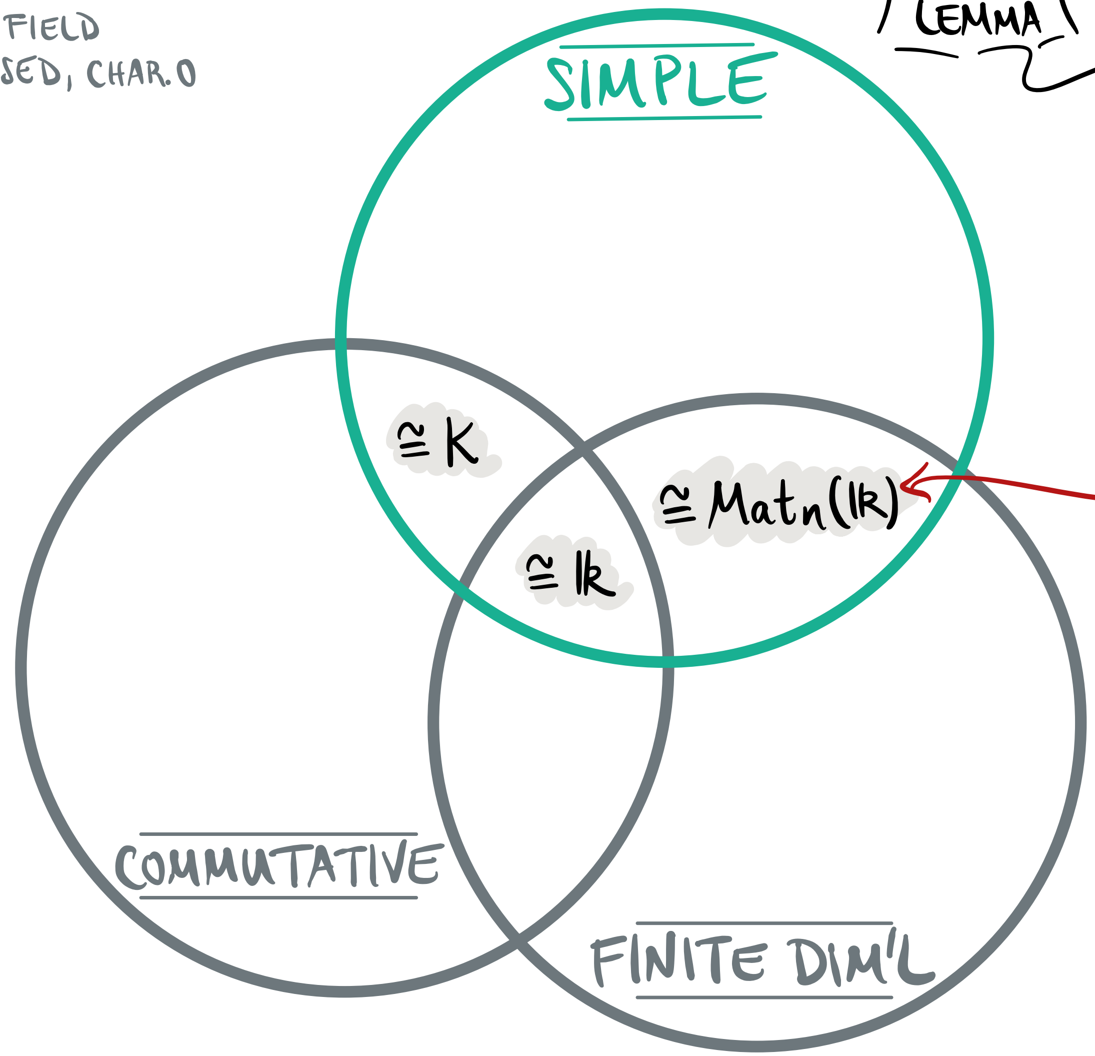
SCHUR'S LEMMA

D DIVISION ALGEBRA $/\Bbbk$

Ex. $D = \mathrm{End}_{A\text{-mod}}(\underset{\text{SIMPLE } A\text{-MODULE}}{W})$

K FIELD EXT'N OF $\Bbbk$

Ex. $K = \mathrm{Frac}(\Bbbk[x])$



WILL SHOW

I. SIMPLICITY

TAKE A $\Bbbk$-ALG A

A IS SIMPLE IF THE ONLY IDEALS OF A ARE 0 & A

Ex. $Mat_n(\Bbbk)$ IS SIMPLE.

I. SIMPLICITY

TAKE A $\Bbbk$-ALG A

A IS **SIMPLE**

IF THE ONLY

IDEALS OF A

ARE 0 & A

Ex. $\mathrm{Mat}_n(\Bbbk)$ IS SIMPLE.

TAKE $I^{\neq 0}$ IDEAL OF $\mathrm{Mat}_n(\Bbbk)$.

THEN $\exists\, (c_{i,j}) \in I$ WITH $c_{p,q}^{\neq 0}$ FOR SOME p,q.

GET $\underset{\substack{\uparrow \\ \text{ELEMENTARY} \\ \text{MATRIX}}}{E_{k,l}} = \dfrac{1}{c_{p,q}} E_{k,p} (c_{i,j}) E_{q,l} \in I.$

$\forall k,l$

$\therefore\ \mathrm{Mat}_n(\Bbbk) \subseteq I \qquad \therefore\ \mathrm{Mat}_n(\Bbbk) = I$

I. SIMPLICITY

TAKE A $\Bbbk$-ALG A

A IS SIMPLE IF THE ONLY IDEALS OF A ARE 0 & A

LEMMA:

IF D IS A DIVISION ALG WITH $\dim_{\Bbbk} D < \infty$, THEN $D \cong \Bbbk$.

Ex. $\mathrm{Mat}_n(\Bbbk)$ IS SIMPLE.

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ELEMENTARY MATRIX

$\forall k,\ell$

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$\dim_{\Bbbk} D < \infty$,
THEN $D \cong \Bbbk$.

Ex. $\mathrm{Mat}_n(\Bbbk)$ IS SIMPLE.

PF/ TAKE $x \in D$.

$\dim_{\Bbbk} D < \infty \Rightarrow \exists$ MINIMAL $n \in \mathbb{N} \ni.$

$1_D = x^0, x, x^2, \dots, x^n$

ARE LINEARLY DEPENDENT

$$\therefore \exists \lambda_i \in \Bbbk \ni. \; p(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \dots + x^n = 0$$

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$\exists$ ROOT λ OF $p(x)$ BECAUSE $\Bbbk$ IS ALG. CLOSED.

$\therefore$ $p(x) = (x-\lambda)\, q(x) = 0$

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DIVISION ALGS ARE DOMAINS $\Rightarrow x - \lambda = 0$.

$\therefore x = \lambda \in \Bbbk$ ///

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WILL SHOW THE CONVERSE

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CARTAN–WEDDERBURN THEOREM

ASSUME A IS FINITE DIMENSIONAL.

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$\therefore$ $I = Ax$ FOR SOME $\overset{\neq 0}{x} \in A$.

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NEXT, A SIMPLE $\Rightarrow$ $AxA = A$ AS A-BIMODULES.

$\therefore$ ${}_A(A_{\mathrm{reg}}) = \sum_{a \in A} \underbrace{Axa}$ LEFT IDEAL OF A. ALSO HAVE:

$$\begin{aligned} I = Ax &\twoheadrightarrow Axa \\ x &\longmapsto xa \end{aligned}$$

I SIMPLE LEFT A-MOD. $\Rightarrow$ $I = 0$ OR $I = Axa$ FOR SOME $a \in A$.

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I SIMPLE LEFT A-MOD. $\Rightarrow$ $I = 0$ OR $I = Axa$ FOR SOME $a \in A$.

$\therefore$ ${}_A(A_{reg}) = \sum_{a \in S} I$ FOR SOME SUBSET S OF A.

$\cong$ AS LEFT A-MODS $I^{\oplus n}$ FOR SOME $n \in \mathbb{N}$ (≡CHECK≡)

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NOW AS ALGEBRAS: $A^{op} \cong \mathrm{End}_{A\text{-Mod}}({}_A(A_{reg}))$

EXERCISE 1.26
(SEE LECTURE #4)

I. SIMPLICITY

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\end{aligned}
$$

DIV. ALG. BY SCHUR'S LEMMA

& FINITE DIM SINCE A IS FINITE DIM

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EXERCISE 1.33

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$\therefore A \cong \mathrm{Mat}_n(\Bbbk)$ ///

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NEED THIS

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EXAMPLE ∃ INFINITE DIM. SIMPLE ALG.

TAKE THE **WEYL ALGEBRA**

$$A_n(\Bbbk) := \frac{\Bbbk\langle v_1, \dots, v_n, w_1, \dots, w_n \rangle}{\begin{pmatrix} w_i v_j - v_j w_i - \delta_{i,j} 1 \\ v_i v_j - v_j v_i \\ w_i w_j - w_j w_i \end{pmatrix}_{1 \le i,j \le n}}$$

$A_n(\Bbbk)$ IS INFINITE DIM. WITH VS BASIS:

$$\{v_1^{i_1} \cdots v_n^{i_n} w_1^{j_1} \cdots w_n^{j_n}\}_{i_1, \dots, i_n, j_1, \dots, j_n \in \mathbb{N}}$$

$A_n(\Bbbk)$ IS SIMPLE [HIRSCH]

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EXAMPLE $\exists$ INFINITE DIM. SIMPLE ALG.

TAKE THE FIRST WEYL ALGEBRA

$$A_1(\Bbbk) := \Bbbk\langle v, w\rangle \big/ (wv - vw - 1)$$

ALG. OF DIFF'L OPERATORS ON $\Bbbk[v]$

$A_1(\Bbbk)$ IS INF. DIM. WITH VS BASIS:

$$\{v^i w^j\}_{i,j \in \mathbb{N}}$$

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$$A_1(\Bbbk) := \Bbbk\langle v, \underset{\frac{d}{dv}}{\underset{\|}{w}} \rangle \Big/ (wv - vw - 1)$$

ALG. OF DIFF'L OPERATORS ON $\Bbbk[v]$

FOR $f(v) \in \Bbbk[v]$:

$$\left(\frac{d}{dv} v\right)(f(v)) = \frac{d}{dv}(v f(v)) = v\left(\frac{d}{dv} f(v)\right) + \left(\frac{d}{dv} v\right) f(v)$$

$$\|$$

$$\left(v \frac{d}{dv} + id_{\Bbbk[v]}\right)(f(v))$$

$A_1(\Bbbk)$ IS INF. DIM. WITH VS BASIS:

$$\{v^i w^j\}_{i,j \in \mathbb{N}}$$

$A_1(\Bbbk)$ IS SIMPLE [HIRSCH]

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$A_n(\Bbbk)$ IS SIMPLE [HIRSCH]

I. SIMPLICITY

n POSITIVE INTEGER

$\mathbb{k}$ { GROUND FIELD
ALG. CLOSED, CHAR. 0

Ex. $\mathbb{k} = \mathbb{C}$

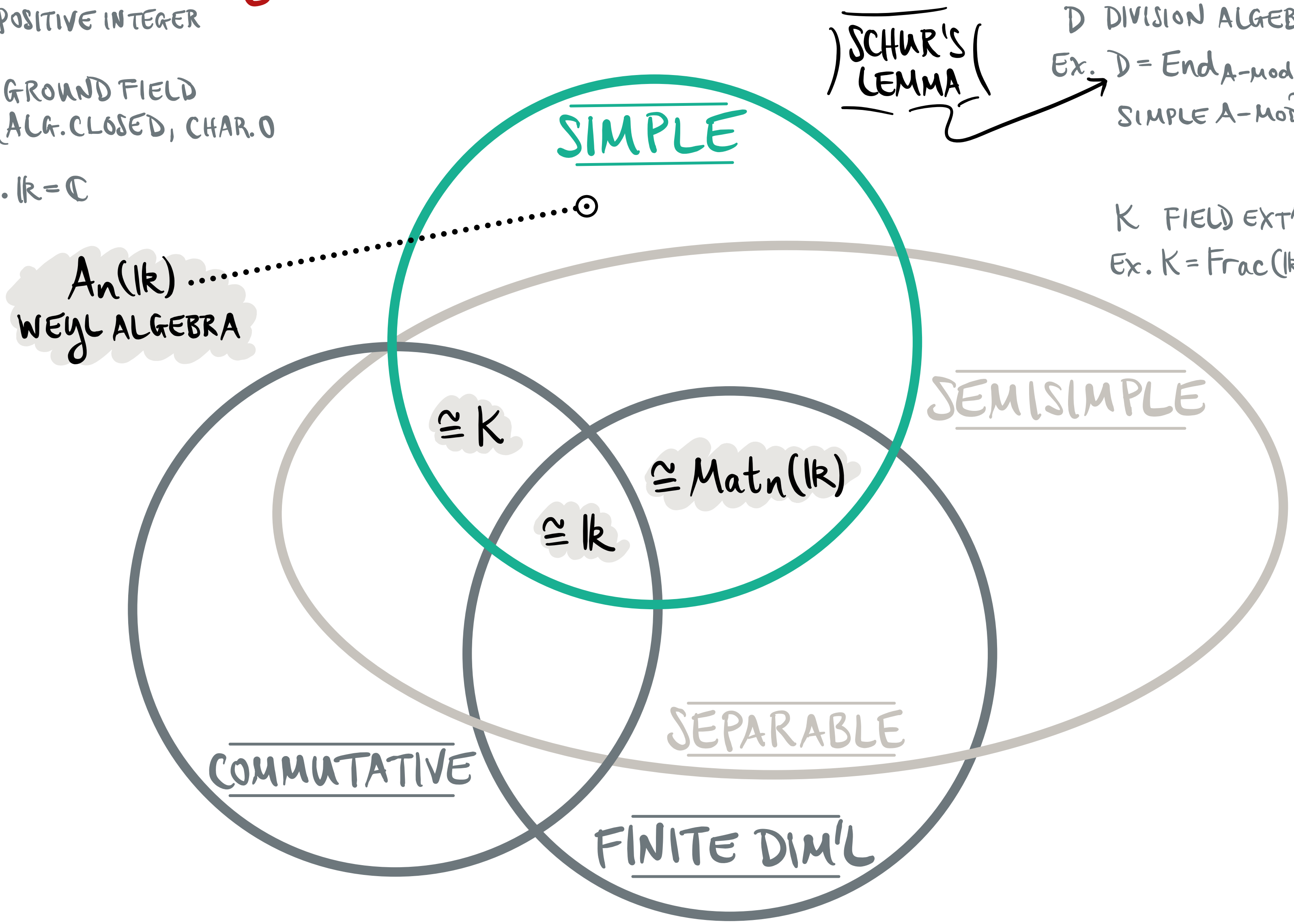
D DIVISION ALGEBRA $/\mathbb{k}$

Ex. $D = \text{End}_{A\text{-mod}}(W)$

SIMPLE A-MODULE

K FIELD EXT'N OF $\mathbb{k}$

Ex. $K = \text{Frac}(\mathbb{k}[x])$



II. SEMISIMPLICITY

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ALG. CLOSED, CHAR. 0

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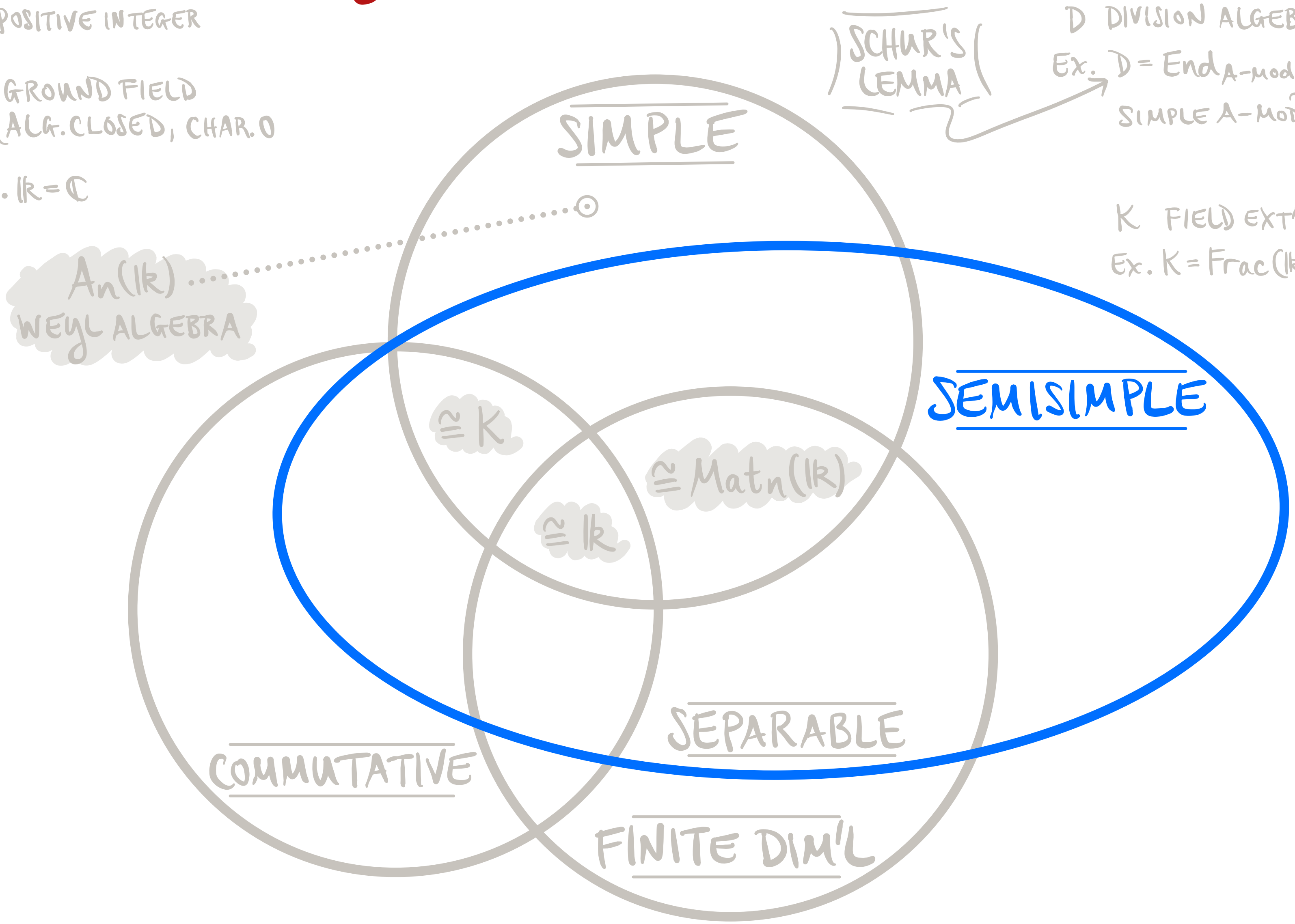

II. SEMISIMPLICITY

TAKE A $\Bbbk$-ALGEBRA (A, m, u)

A LEFT A-MODULE V
IS SEMISIMPLE IF
$V = \bigoplus_i$ (SIMPLE LEFT A-MODS)

A IS SEMISIMPLE
IF $_A(A_{reg})$ IS A
SEMISIMPLE MODULE

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<u>**LEMMA**</u> $\Updownarrow$

EACH LEFT A-MODULE
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SEMISIMPLICITY IS
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A IS SEMISIMPLE IF $_A(A_{reg})$ IS A SEMISIMPLE MODULE

LEMMA $\Updownarrow$

EACH LEFT A-MODULE IS SEMISIMPLE

SEMISIMPLICITY IS MODULE-THEORETIC

PF/ $\Uparrow$ ✓

$\Downarrow$: TAKE A LEFT A-MODULE M WITH GENERATORS $\{m_i\}_{i\in I}$.

GET $A^{\oplus I} \twoheadrightarrow M$ MOD. MAP. &
$(a_i)_{i\in I} \mapsto \sum_{i\in I}(a_i \triangleright m_i)$

HOM. IMAGE OF A SS MOD IS SS. ///

II. SEMISIMPLICITY

TAKE A $\Bbbk$-ALGEBRA (A, m, u)

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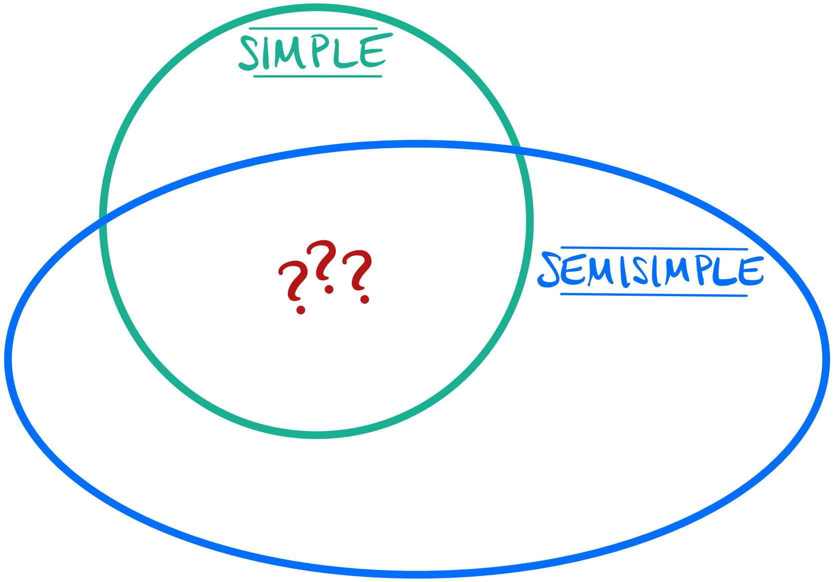

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II. SEMISIMPLICITY

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A LEFT A-MODULE V IS **SEMISIMPLE** IF

$$V = \bigoplus_i (\text{SIMPLE LEFT } A\text{-MODS})$$

A IS **SEMISIMPLE** IF ${}_A(A_{reg})$ IS A SEMISIMPLE MODULE

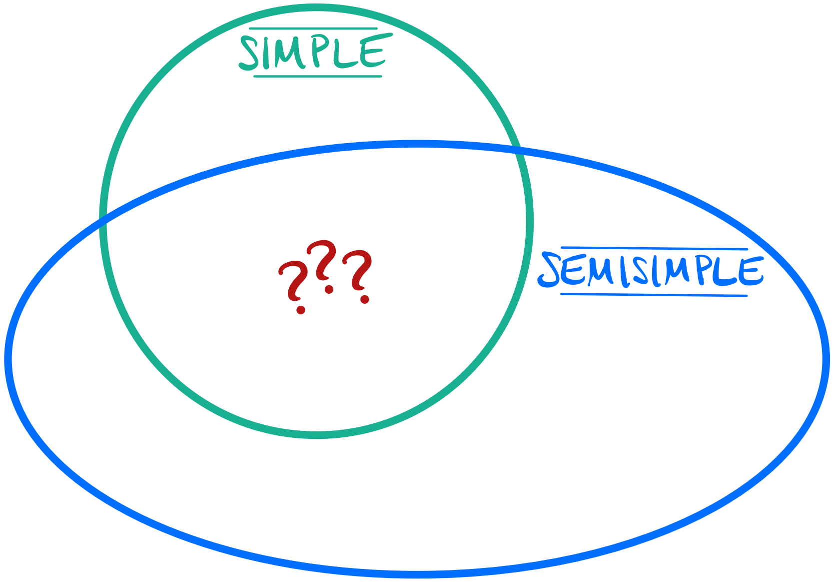
A IS SIMPLE & SEMISIMPLE

$$\Updownarrow$$

$$A \cong \mathrm{Mat}_n(D)$$

POS. INTEGER

DIVISION ALG.



II. SEMISIMPLICITY

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PF/ $\Uparrow$ $\mathrm{Mat}_n(D)$ IS SIMPLE & SS

EXERCISE 1.33

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A LEFT A-MODULE V IS **SEMISIMPLE** IF

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PF/ $\Uparrow$ $\mathrm{Mat}_n(D)$ IS SIMPLE & SS

EXERCISE 1.33

$\Downarrow$: A SS $\Rightarrow$ ${}_A(A_{reg}) \cong \bigoplus_{j=1}^{r} V_j^{\oplus n_j}$

PAIRWISE NON-ISOM SIMPLE A-MODULE

OR

MINIMAL LEFT IDEALS

II. SEMISIMPLICITY

TAKE A $\Bbbk$-ALGEBRA (A, m, u)

A LEFT A-MODULE V IS **SEMISIMPLE** IF

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OR

MINIMAL LEFT IDEALS

NOW AS ALGEBRAS:

$$\begin{aligned} A^{op} &\cong \operatorname{End}_{A\text{-Mod}}({}_A(A_{reg})) \\ &\cong \operatorname{End}_{A\text{-mod}}\left(\bigoplus_{j=1}^{r} V_j^{\oplus n_j}\right) \\ &\cong \prod_{j=1}^{n} \operatorname{Mat}_{n_j}(\operatorname{End}_{A\text{-mod}}(V_j)) \end{aligned}$$

II. SEMISIMPLICITY

TAKE A $\Bbbk$-ALGEBRA (A, m, u)

A LEFT A-MODULE V IS **SEMISIMPLE** IF

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A IS **SEMISIMPLE** IF ${}_A(A_{reg})$ IS A SEMISIMPLE MODULE

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EXERCISE 1.26

II. SEMISIMPLICITY

TAKE A $\Bbbk$-ALGEBRA (A, m, u)

A LEFT A-MODULE V IS **SEMISIMPLE** IF

$$V = \oplus_i \left(\text{SIMPLE LEFT A-MODS}\right)$$

A IS **SEMISIMPLE** IF ${}_A(A_{reg})$ IS A SEMISIMPLE MODULE

A IS SIMPLE & SEMISIMPLE

$$\Updownarrow$$

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DIVISION ALG.

PF/ $\Uparrow$ $\mathrm{Mat}_n(D)$ IS SIMPLE & SS

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EXERCISE 1.26

$$A \text{ SIMPLE} \Longrightarrow A \cong \mathrm{Mat}_n\big(\overbrace{\mathrm{End}_{A\text{-Mod}}(V)}^{\text{DIV. ALG BY SCHUR}}\big) ///$$

II. SEMISIMPLICITY

n POSITIVE INTEGER

$\Bbbk$ $\begin{cases} \text{GROUND FIELD} \\ \text{ALG. CLOSED, CHAR. 0} \end{cases}$

Ex. $\Bbbk = \mathbb{C}$

SCHUR'S LEMMA

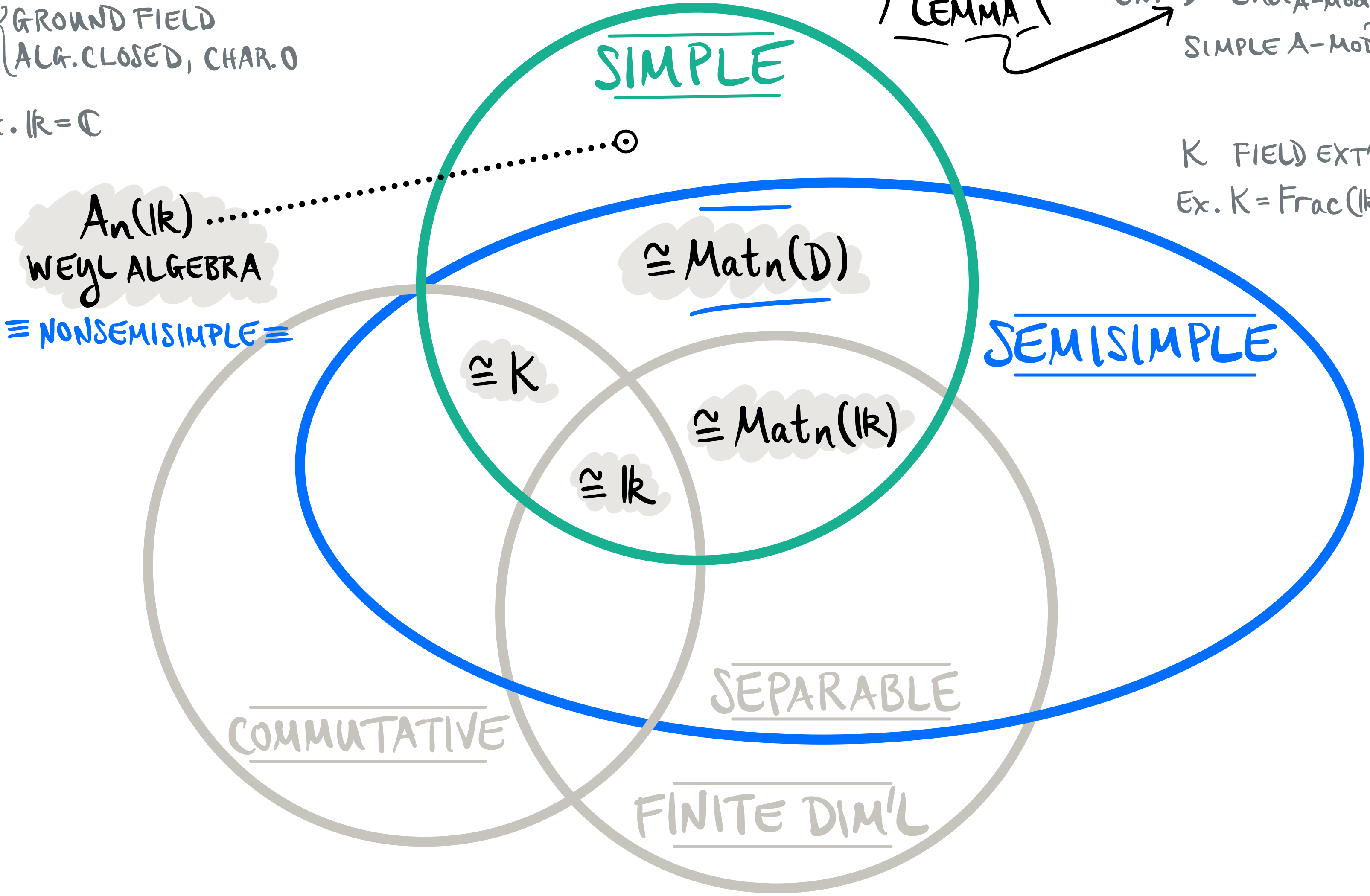
D DIVISION ALGEBRA $/\Bbbk$

Ex. $D = \mathrm{End}_{A\text{-}mod}(W)$

SIMPLE A-MODULE

K FIELD EXT'N OF $\Bbbk$

Ex. $K = \mathrm{Frac}(\Bbbk[x])$



II. SEMISIMPLICITY

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A IS SEMISIMPLE IF ${}_A(A_{reg})$ IS A SEMISIMPLE MODULE

A IS SIMPLE & SEMISIMPLE

$$\Updownarrow$$

$$A \cong \text{Mat}_n(D)$$

POS. INTEGER

DIVISION ALG.

PF/ $\Uparrow$ $\text{Mat}_n(D)$ IS SIMPLE & SS

EXERCISE 1.33

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EXERCISE 1.26

A SIMPLE $\Longrightarrow$ $A \cong \text{Mat}_n(\overline{\text{End}_{A\text{-mod}}(V)})$

DIV. ALG BY SCHUR

II. SEMISIMPLICITY

TAKE A $\Bbbk$-ALGEBRA (A, m, u)

A LEFT A-MODULE V IS **SEMISIMPLE** IF $V = \bigoplus_i$ (SIMPLE LEFT A-MODS)

A IS **SEMISIMPLE** IF ${}_A(A_{reg})$ IS A SEMISIMPLE MODULE

A IS SIMPLE & SEMISIMPLE

$$\Updownarrow$$

$$A \cong \mathrm{Mat}_n(D)$$

POS. INTEGER

DIVISION ALG.

ARTIN–WEDDERBURN THEOREM

A IS SEMISIMPLE

$$\Updownarrow$$

$$A \cong \prod_{i=1}^{r} \mathrm{Mat}_{n_i}(D_i)$$

UNIQUE CHOICE OF

POS. INTEGER

DIV. ALG.

PF/ $\Uparrow$ $\mathrm{Mat}_n(D)$ IS SIMPLE & SS

EXERCISE 1.33

$\Downarrow$: A SS $\Rightarrow$ ${}_A(A_{reg}) \cong \bigoplus_{j=1}^{r} V_j^{\oplus n_j}$

PAIRWISE NON-ISOM SIMPLE A-MODULE OR MINIMAL LEFT IDEALS

NOW AS ALGEBRAS:

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IF $_A(A_{reg})$ IS A
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A IS SIMPLE & SEMISIMPLE

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UNIQUE CHOICE OF

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≡ READ THIS ≡

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DIV. ALG BY SCHUR

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UNIQUE CHOICE OF

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DIV. ALG.

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DIVISION ALG.

ARTIN–WEDDERBURN THEOREM

A IS SEMISIMPLE

$$\Updownarrow$$

$$A \cong \prod_{i=1}^{r} \mathrm{Mat}_{n_i}(D_i)$$

UNIQUE CHOICE OF

POS. INTEGER

DIV. ALG.

IN THIS CASE:

A FINITE DIM $\Leftrightarrow$ $A \cong \prod_{i=1}^{r} \mathrm{Mat}_{n_i}(\Bbbk)$

PF/ $\Uparrow$ $\prod_{i=1}^{r} \mathrm{Mat}_{n_i}(D_i)$ IS SEMISIMPLE

$\Downarrow$: A SS $\Rightarrow$ ${}_A(A_{reg}) \cong \bigoplus_{j=1}^{r} V_j^{\oplus n_j}$

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$\equiv$

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A IS SIMPLE &
SEMISIMPLE

$$\Updownarrow$$

$$A \cong \mathrm{Mat}_n(D)$$

POS. INTEGER
DIVISION ALG.

ARTIN–WEDDERBURN THEOREM

A IS SEMISIMPLE

$$\Updownarrow$$

$$A \cong \prod_{i=1}^{r} \mathrm{Mat}_{n_i}(D_i)$$

UNIQUE CHOICE OF
POS. INTEGER
DIV. ALG.

IN THIS CASE:

A FINITE DIM $\Leftrightarrow$ $A \cong \prod_{i=1}^{r} \mathrm{Mat}_{n_i}(\Bbbk)$

LEMMA:

IF D IS A
DIVISION ALG
WITH
$\dim_{\Bbbk} D < \infty$,
THEN $D \cong \Bbbk$.

II. SEMISIMPLICITY

n, n_i POSITIVE INTEGERS

$\Bbbk$ $\begin{cases}\text{GROUND FIELD} \\ \text{ALG. CLOSED, CHAR. 0}\end{cases}$

Ex. $\Bbbk = \mathbb{C}$

SCHUR'S LEMMA

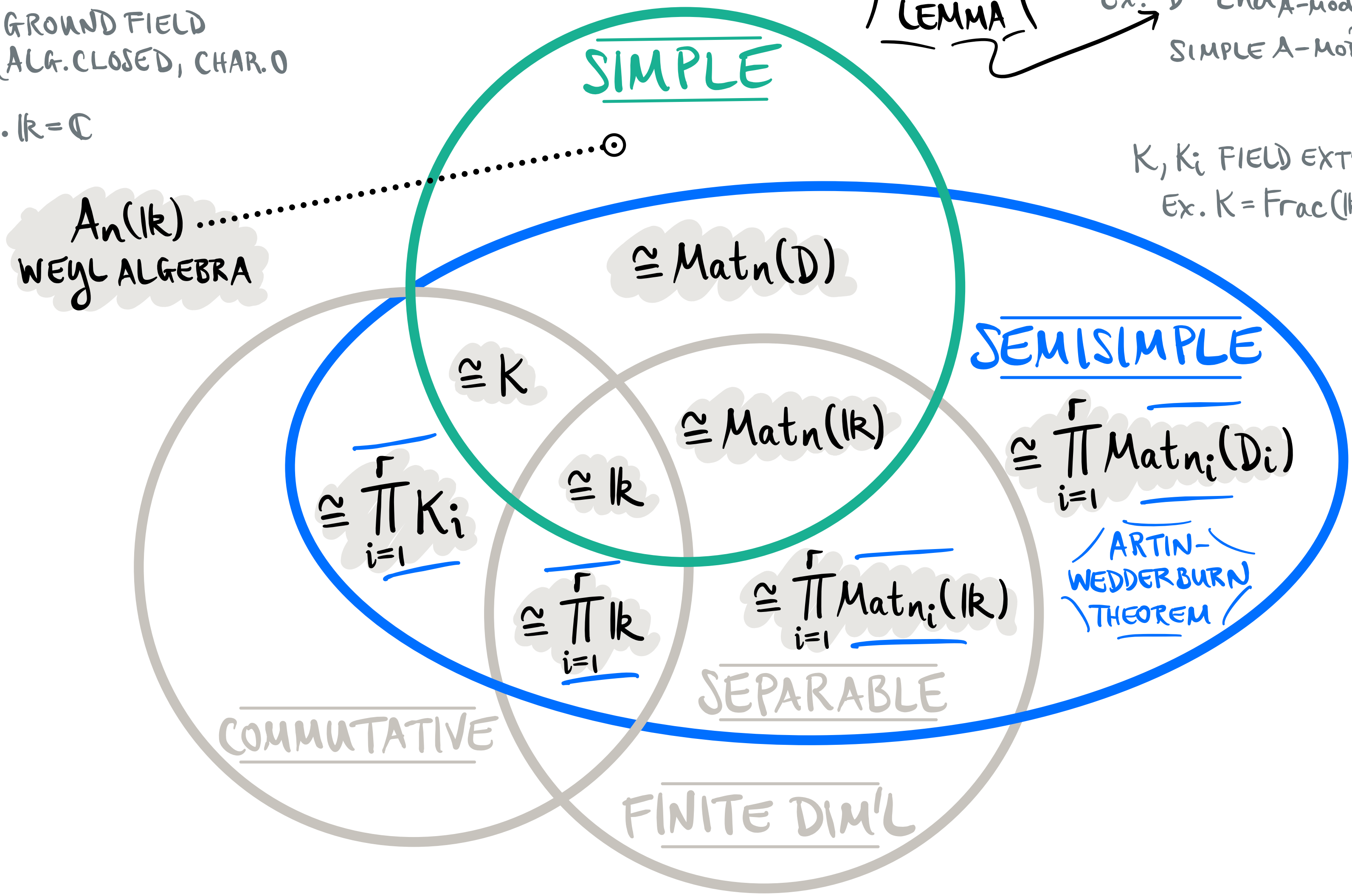
D, D_i DIVISION ALGEBRAS/$\Bbbk$

Ex. $D = \mathrm{End}_{A\text{-mod}}(W)$

SIMPLE A-MODULE

K, K_i FIELD EXT'N OF $\Bbbk$

Ex. $K = \mathrm{Frac}(\Bbbk[x])$



II. SEMISIMPLICITY

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D, D_i DIVISION ALGEBRAS$/\Bbbk$

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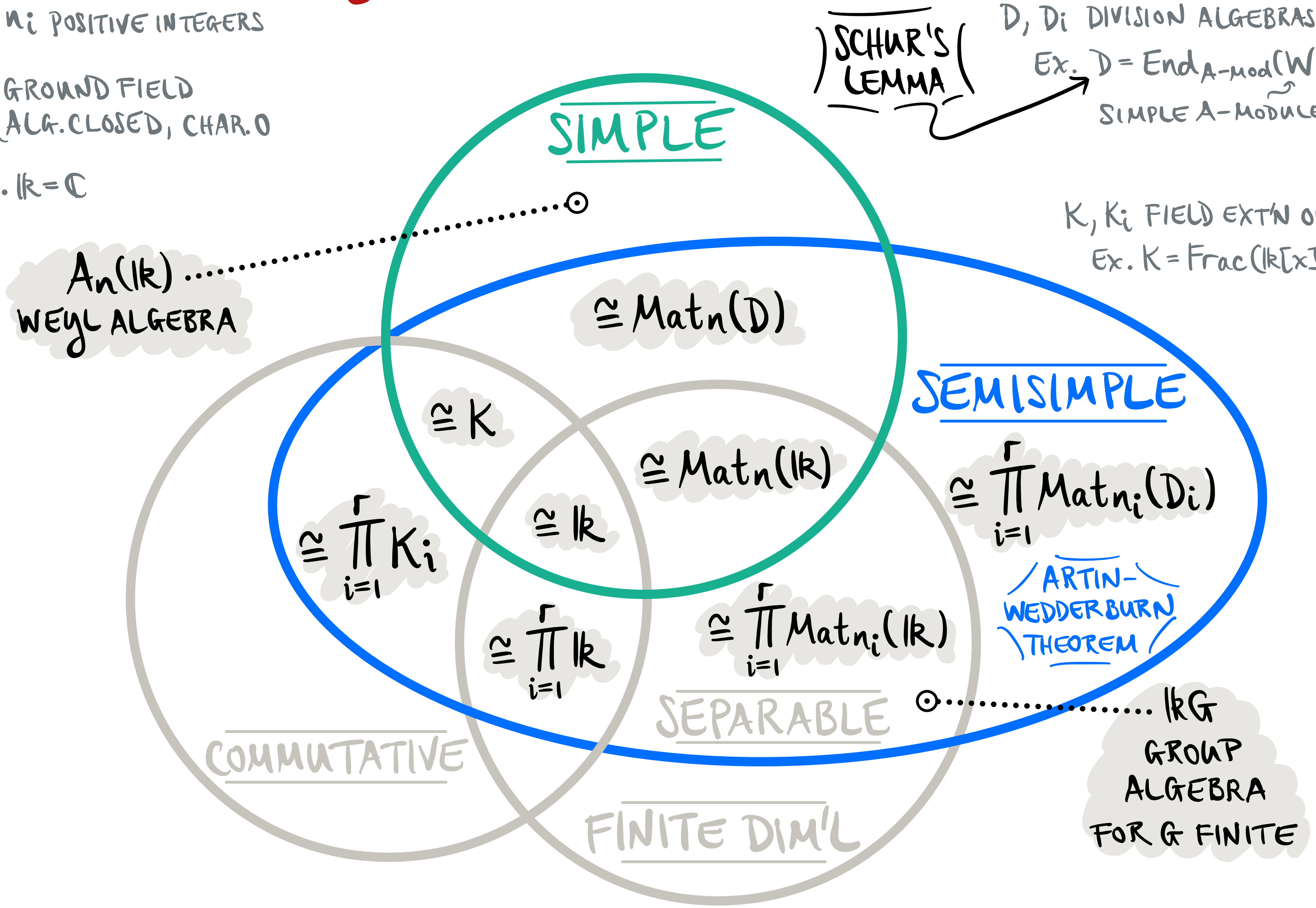

II. SEMISIMPLICITY

Ex. $\Bbbk G$ FOR G FINITE GROUP

IS

SEMISIMPLE

& FINITE DIM

$\therefore$

$$\boxed{\Bbbk G \cong \prod_{i=1}^{r} \mathrm{Mat}_{n_i}(\Bbbk)}$$

FOR SOME UNIQUE CHOICE OF POSITIVE INTEGERS n_i, r

MASCHKE'S THEOREM

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FOR SOME UNIQUE CHOICE OF POSITIVE INTEGERS n_i, r

MASCHKE'S THEOREM

Ex. $G = C_2$

ISOM CLASSES OF SIMPLE $\Bbbk C_2$-MODULES REPRESENTED BY

TRIVIAL MODULE	"SIGN" MODULE
$\triangleright : \Bbbk C_2 \times \Bbbk v \longrightarrow \Bbbk v$	$\triangleright : \Bbbk C_2 \times \Bbbk v \longrightarrow \Bbbk v$
$e \triangleright v = v$	$e \triangleright v = v$
$g \triangleright v = v$	$g \triangleright v = -v$

II. SEMISIMPLICITY

Ex. $\Bbbk G$ FOR G FINITE GROUP

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1-DIM'L $(n_1 = n_2 = 1)$

$$\therefore \Bbbk C_2 \cong \mathrm{End}_{\Bbbk C_2\text{-mod}}(V_{triv}) \times \mathrm{End}_{\Bbbk C_2\text{-mod}}(V_{sign})$$

$\cong \Bbbk \times \Bbbk$ AS ALGS

II. SEMISIMPLICITY

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Ex. $G = S_3$ $\exists \begin{cases} \text{TWO 1-DIM} \\ \text{ONE 2-DIM} \end{cases}$ SIMPLE $\Bbbk S_3$-MODS UP TO $\cong$

$$\therefore\ \Bbbk S_3 \cong \Bbbk^{\times 2} \times \mathrm{Mat}_2(\Bbbk)$$ AS ALGS

II. SEMISIMPLICITY

Ex. $\Bbbk G$ FOR G FINITE GROUP

IS

SEMISIMPLE

& FINITE DIM

$\therefore$

$$\boxed{\Bbbk G \cong \prod_{i=1}^{r} \mathrm{Mat}_{n_i}(\Bbbk)}$$

FOR SOME UNIQUE CHOICE OF POSITIVE INTEGERS n_i, r

MASCHKE'S THEOREM

THE ISOCLASSES OF SIMPLE $\Bbbk G$-MODS HAVE DIM : $\{n_1, n_2, \ldots, n_r\}$

Ex. $G = C_2$

ISOM CLASSES OF SIMPLE $\Bbbk C_2$-MODULES REPRESENTED BY

TRIVIAL MODULE

$\triangleright : \Bbbk C_2 \times \Bbbk v \longrightarrow \Bbbk v$

$e \triangleright v = v$

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"SIGN" MODULE

$\triangleright : \Bbbk C_2 \times \Bbbk v \longrightarrow \Bbbk v$

$e \triangleright v = v$

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Ex. $G = S_3$ $\exists \left\{ \begin{array}{l} \text{TWO 1-DIM} \\ \text{ONE 2-DIM} \end{array} \right.$ SIMPLE $\Bbbk S_3$-MODS UP TO $\cong$

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III. SEPARABILITY

TAKE A $\Bbbk$-ALGEBRA (A, m, u)

RECALL

SEMISIMPLICITY IS MODULE-THEORETIC

III. SEPARABILITY

TAKE A $\Bbbk$-ALGEBRA (A, m, u)

A IS SEPARABLE IF

THE A-BIMODULE MAP

$$\mu : A\otimes A^{op} \longrightarrow A$$
$$a\otimes b \longmapsto m(a\otimes b) = ab$$

HAS A RIGHT INVERSE

HERE: $\triangleright_{A\otimes A^{op}} = m\otimes \mathrm{id}_{A^{op}}$

$\triangleleft_{A\otimes A^{op}} = \mathrm{id}_{A}\otimes m$

$\triangleright_{A} = m = \triangleleft_{A}$

RECALL

SEMISIMPLICITY IS MODULE-THEORETIC

SEPARABILITY IS INTRINSIC TO THE ALGEBRA STRUCTURE (DEPENDS ON $m = m_A$)

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TAKE A $\Bbbk$-ALGEBRA (A, m, u)

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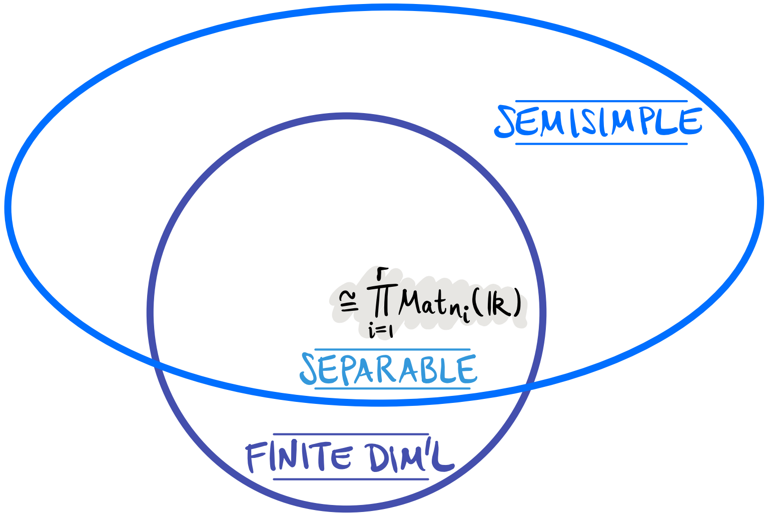

RECALL

SEMISIMPLICITY IS MODULE-THEORETIC

SEPARABILITY IS INTRINSIC TO THE ALGEBRA STRUCTURE (DEPENDS ON $m = m_A$)

A IS SEPARABLE

$\Updownarrow$

A FINITEDIM & SEMISIMPLE

$\Updownarrow$ ARTIN-WED. THM

$$A \cong \prod_{i=1}^{r} Mat_{n_i}(\Bbbk)$$ AS ALGS

ON THE CLASSIFICATION OF NICE ALGEBRAS —

n, n_i POSITIVE INTEGERS

$\Bbbk$ $\left\{ \begin{array}{l} \text{GROUND FIELD} \\ \text{ALG. CLOSED, CHAR. 0} \end{array} \right.$

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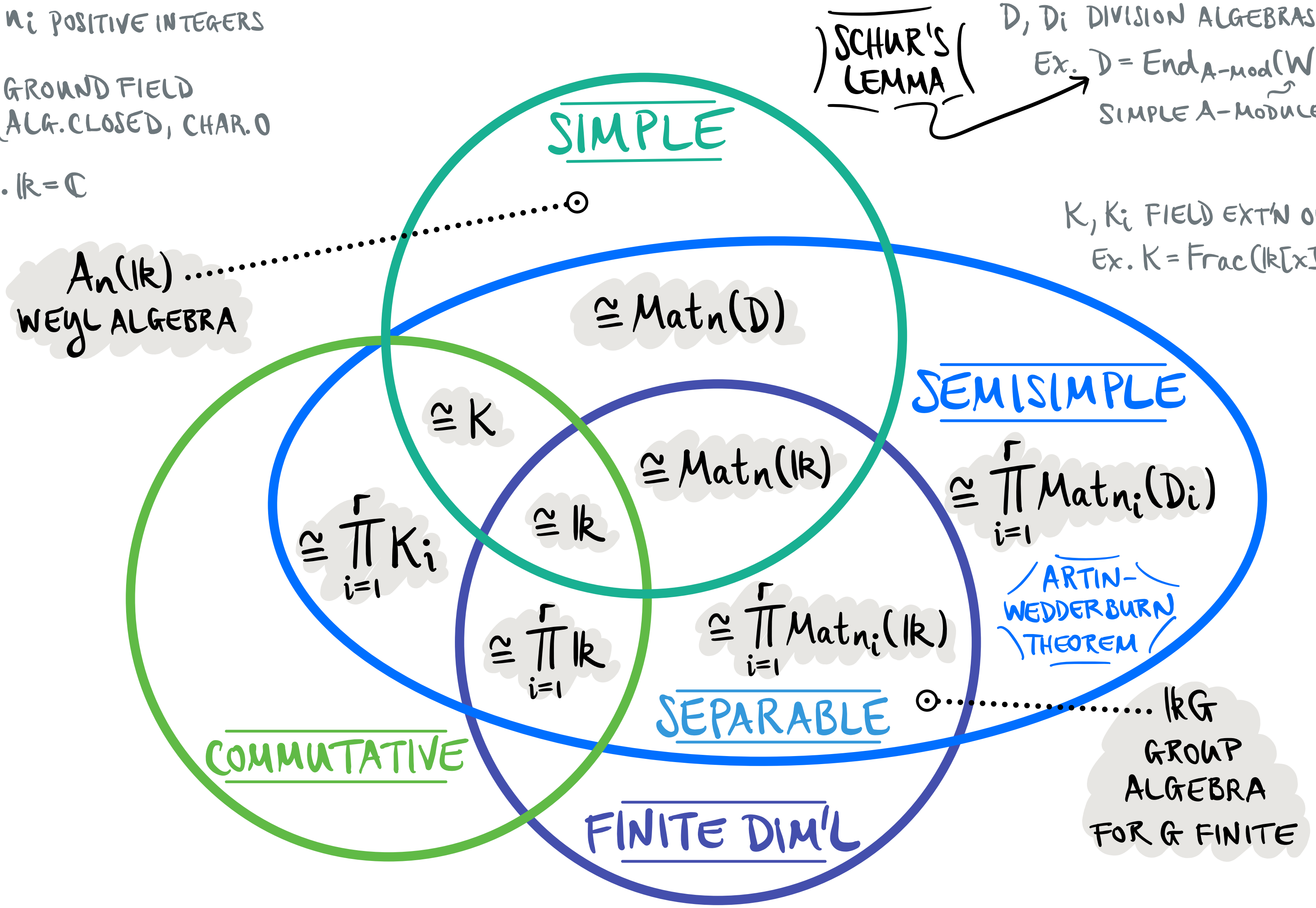
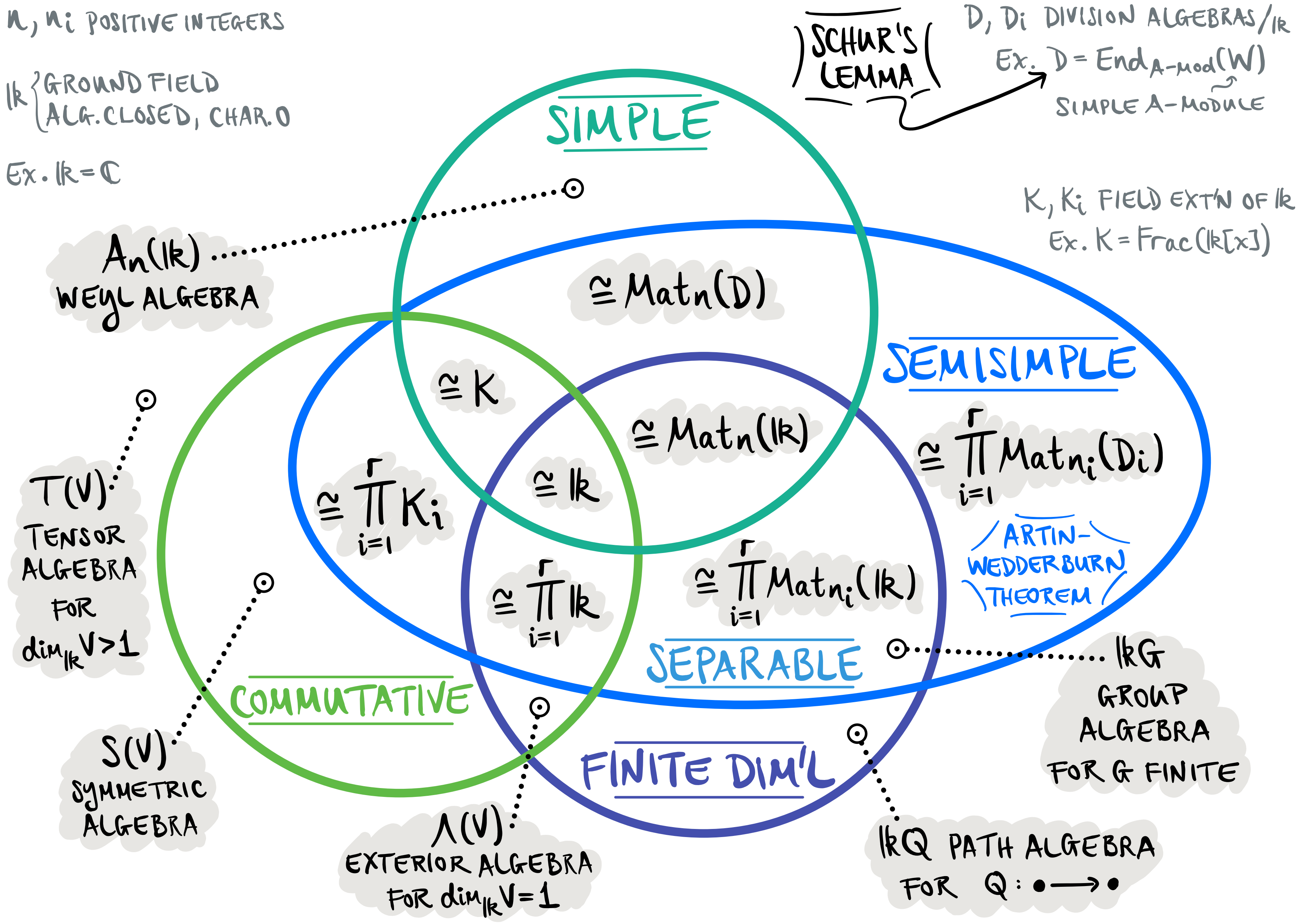
Ex. $K = \mathrm{Frac}(\Bbbk[x])$



ON THE CLASSIFICATION OF NICE ALGEBRAS —



MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

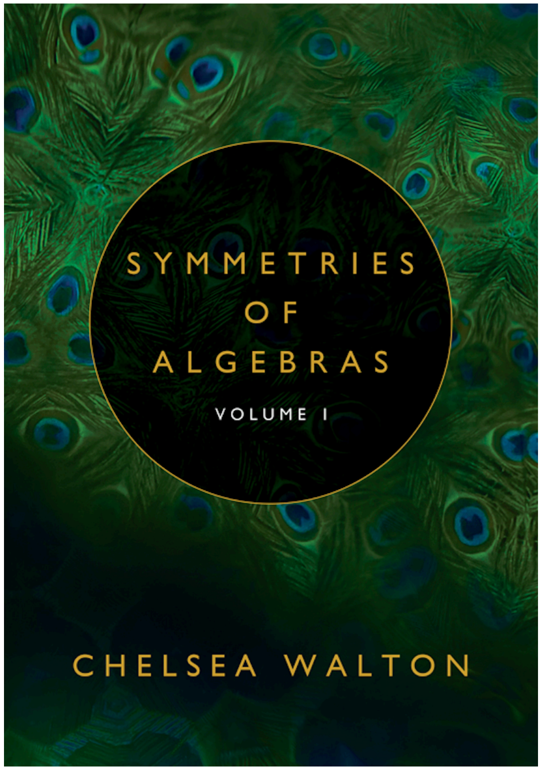
LECTURE #5

THIS ENDS
CHAPTER 1
ON ALGEBRAS/$\Bbbk$

TOPICS:

- ✓ I. SIMPLICITY (§1.5)
- ✓ II. SEMISIMPLICITY (§1.6)
- ✓ III. SEPARABILITY (§1.7)

NEXT TIME
CATEGORY
THEORY!



Lecture #5 keywords: Artin-Wedderburn Theorem, Cartan-Wedderburn Theorem, Maschke's Theorem, indecomposable module, Schur's Lemma, semisimple algebra, separable algebra, simple algebra/module