MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #6

LAST TIME

- SIMPLE ALGS.
- SEMISIMPLE ALGS.
- SEPARABLE ALGS.

TOPICS:

I. CATEGORIES (§2.1)

II. UNIVERSAL CONSTRUCTIONS (§2.2.1)

I. CATEGORIES

"CATEGORY THEORY IS THE MATHEMATICS OF MATHEMATICS."

— PROF. EUGENIA CHENG

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A **CATEGORY** $\mathscr{C}$ CONSISTS OF THE DATA:

(a) A COLLECTION OF **OBJECTS** $\mathrm{Ob}(\mathscr{C})$.

WRITE $X \in \mathscr{C}$ FOR $X \in \mathrm{Ob}(\mathscr{C})$.

(b) FOR EVERY PAIR OF OBJECTS $X, Y \in \mathscr{C}$,

A COLLECTION OF **MORPHISMS** $\mathrm{Hom}_{\mathscr{C}}(X,Y)$.

WRITE $g : X \to Y$ FOR $g \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$.

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(c) FOR EVERY OBJECT $X \in \mathscr{C}$,

AN **IDENTITY MORPHISM** $\mathrm{id}_X : X \to X$.

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(b) FOR EVERY PAIR OF OBJECTS $X, Y \in \mathcal{C}$,

A COLLECTION OF MORPHISMS $Hom_{\mathcal{C}}(X,Y)$.

WRITE $g: X \to Y$ FOR $g \in Hom_{\mathcal{C}}(X,Y)$.

(c) FOR EVERY OBJECT $X \in \mathcal{C}$,

AN IDENTITY MORPHISM $id_X : X \to X$.

(d) FOR EVERY PAIR OF MORPHISMS

$f: W \to X$ AND $g: X \to Y$,

A COMPOSITE MORPHISM $gf := g \circ f : W \to Y$.

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WRITE $g: X \to Y$ FOR $g \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$.

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(d) FOR EVERY PAIR OF MORPHISMS

$f: W \to X$ AND $g: X \to Y$,

A **COMPOSITE MORPHISM** $gf := g \circ f : W \to Y$.

THIS DATA MUST SATISFY THE AXIOMS:

ASSOCIATIVITY

$$(hg)f = h(gf) \text{ IN } \mathrm{Hom}_{\mathscr{C}}(W,Z)$$

UNITALITY

$$\mathrm{id}_X f = f \text{ IN } \mathrm{Hom}_{\mathscr{C}}(W,X) \quad \& \quad g\,\mathrm{id}_X = g \text{ IN } \mathrm{Hom}_{\mathscr{C}}(X,Y)$$

$\forall\ f: W \to X,\ g: X \to Y,\ h: Y \to Z$

I. CATEGORIES

USE "COLLECTION" INSTEAD OF "SET"

(... TO AVOID ISSUES WITH "A SET OF SETS" LATER)

DOESN'T EXIST

A CATEGORY $\mathscr{C}$ CONSISTS OF THE DATA:

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(b) FOR EVERY PAIR OF OBJECTS $X, Y \in \mathscr{C}$, A COLLECTION OF MORPHISMS $\mathrm{Hom}_{\mathscr{C}}(X,Y)$.

(c) FOR EVERY OBJECT $X \in \mathscr{C}$, AN IDENTITY MORPHISM $id_X : X \to X$.

(d) FOR EVERY PAIR OF MORPHISMS $f: W \to X$ AND $g: X \to Y$, A COMPOSITE MORPHISM $gf := g \circ f : W \to Y$.

THIS DATA MUST SATISFY THE AXIOMS:

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$(hg)f = h(gf)$ IN $\mathrm{Hom}_{\mathscr{C}}(W,Z)$

UNITALITY

$id_X f = f$ IN $\mathrm{Hom}_{\mathscr{C}}(W,X)$ & $g\, id_X = g$ IN $\mathrm{Hom}_{\mathscr{C}}(X,Y)$

$\forall f: W \to X,\ g: X \to Y,\ h: Y \to Z$

I. CATEGORIES

USE "COLLECTION" INSTEAD OF "SET"

SPECIAL CASES

A SMALL CATEGORY $\mathscr{C}$ CONSISTS OF THE DATA:

(a) A COLLECTION OF OBJECTS $Ob(\mathscr{C})$.

(b) FOR EVERY PAIR OF OBJECTS $X, Y \in \mathscr{C}$,
A COLLECTION OF MORPHISMS $Hom_{\mathscr{C}}(X,Y)$.
& ALL MORPHISMS $Hom(\mathscr{C})$ FORM A SET

(c) FOR EVERY OBJECT $X \in \mathscr{C}$,
AN IDENTITY MORPHISM $id_X : X \to X$.

(d) FOR EVERY PAIR OF MORPHISMS
$f: W \to X$ AND $g: X \to Y$,
A COMPOSITE MORPHISM $gf := g \circ f : W \to Y$.

THIS DATA MUST SATISFY THE AXIOMS:

ASSOCIATIVITY

$(hg)f = h(gf)$
IN
$Hom_{\mathscr{C}}(W,Z)$

UNITALITY

$id_X f = f$ & $g\, id_X = g$
IN $Hom_{\mathscr{C}}(W,X)$ IN $Hom_{\mathscr{C}}(X,Y)$

$\forall\ f: W \to X,\ g: X \to Y,\ h: Y \to Z$

I. CATEGORIES

USE "COLLECTION" INSTEAD OF "SET"

SPECIAL CASES

A LOCALLY SMALL CATEGORY $\mathscr{C}$ CONSISTS OF THE DATA:

(a) A COLLECTION OF OBJECTS $Ob(\mathscr{C})$.

(b) FOR EVERY PAIR OF OBJECTS $X, Y \in \mathscr{C}$, A COLLECTION OF MORPHISMS $Hom_{\mathscr{C}}(X,Y)$.

& THIS IS A SET $\forall X, Y \in \mathscr{C}$.

(c) FOR EVERY OBJECT $X \in \mathscr{C}$, AN IDENTITY MORPHISM $id_X : X \to X$.

(d) FOR EVERY PAIR OF MORPHISMS $f: W \to X$ AND $g: X \to Y$, A COMPOSITE MORPHISM $gf := g \circ f : W \to Y$.

THIS DATA MUST SATISFY THE AXIOMS:

ASSOCIATIVITY

$(hg)f = h(gf)$ IN $Hom_{\mathscr{C}}(W,Z)$

UNITALITY

$id_X f = f$ IN $Hom_{\mathscr{C}}(W,X)$ & $g\, id_X = g$ IN $Hom_{\mathscr{C}}(X,Y)$

$\forall f: W \to X,\ g: X \to Y,\ h: Y \to Z$

I. CATEGORIES

A CATEGORY $\mathscr{C}$

CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ $\forall X, Y \in \mathscr{C}$.

(c) $\mathrm{id}_X : X \to X$ $\forall X \in \mathscr{C}$.

(d) $gf : W \to Y$
$\forall\, f : W \to X$
$g : X \to Y$.

SATISFYING

ASSOCIATIVITY

$(hg)f = h(gf)$

UNITALITY

$\mathrm{id}_X f = f, \ g\, \mathrm{id}_X = g$

ALGEBRAIC CATEGORIES

I. CATEGORIES

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CONSISTS OF:

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$\forall X, Y \in \mathscr{C}$.

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(d) $gf : W \to Y$
$\forall f : W \to X$
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ASSOCIATIVITY
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$\mathrm{id}_X f = f, \; g\,\mathrm{id}_X = g$

Group
GROUPS & GROUP HOMOMS.

ALGEBRAIC CATEGORIES

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ABELIAN GROUPS & GROUP HOMOMS.
NOT "ABELIAN GROUP HOMOMS"
↖ PROPERTY

ALGEBRAIC CATEGORIES

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Ring

UNITAL RINGS & UNITAL RING HOMOMS.

Rng

RINGS & RING HOMOMS.

ALGEBRAIC CATEGORIES

ComRing

UNITAL COMMUTATIVE RINGS & UNITAL RING HOMOMS

I. CATEGORIES

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SATISFYING

ASSOCIATIVITY $(hg)f = h(gf)$

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$\Bbbk$ FIELD ALG. CLOSED & CHAR. 0 (NOT NEEDED HERE)

Group GROUPS & GROUP HOMOMS.

Ab ABELIAN GROUPS & GROUP HOMOMS. NOT "ABELIAN GROUP HOMOMS" ← PROPERTY

Ring UNITAL RINGS & UNITAL RING HOMOMS.

Rng RINGS & RING HOMOMS.

ALGEBRAIC CATEGORIES

ComRing UNITAL COMMUTATIVE RINGS & UNITAL RING HOMOMS

Vec $\Bbbk$-VECTOR SPACES & $\Bbbk$-LINEAR MAPS

FdVec FINITE DIM'L $\Bbbk$-VECTOR SPACES & $\Bbbk$-LINEAR MAPS

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Ring: UNITAL RINGS & UNITAL RING HOMOMS.

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Alg: $\Bbbk$-ALGEBRAS & $\Bbbk$-ALGEBRA HOMOMS.

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A, B $\Bbbk$-ALGS

Rep(A)
REP'NS OF A &
REP'N MORPHISMS

A-Mod
Mod-A
(A,B)-Bimod
A-Bimod
VARIOUS CATEGORIES OF (BI)MODULES

ALGEBRAIC CATEGORIES

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Bim

OBJECTS $\equiv$ $\Bbbk$-ALGEBRAS

MORPHISMS:

$A \to B \iff [{}_A V_B \in (A,B)\text{-Bimod}]$ ~ ISOCLASS ~

WITH: $id_A \equiv$ **???**

$A \to B \to C \equiv$ **???**

Fd Alg: FINITE DIM'L $\Bbbk$-ALGEBRAS & $\Bbbk$-ALGEBRA HOMOMS.

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Bim

OBJECTS = $\Bbbk$-ALGEBRAS

MORPHISMS: ~ ISOCLASS ~

$A \to B \iff [{}_A V_B \in (A,B)\text{-Bimod}]$

WITH: $\mathrm{id}_A = {}_A(A_{\mathrm{reg}})_A$

$A \to B \to C = {}_A V_B \otimes_B {}_B W_C \in (A,C)\text{-Bimod}$

FdAlg: FINITE DIM'L $\Bbbk$-ALGEBRAS & $\Bbbk$-ALGEBRA HOMOMS.

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MORPHISMS: $A \to B \iff [{}_A V_B \in (A,B)\text{-Bimod}]$ ~ISOCLASS~

WITH: $\mathrm{id}_A \equiv {}_A(A_{reg})_A$

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SATISFYING

ASSOCIATIVITY $(hg)f = h(gf)$

UNITALITY $id_X f = f$, $g\, id_X = g$

... MORE (NON-ALGEBRAIC) EXAMPLES LATER

LET'S STUDY MORPHISMS IN DETAIL ...

$$g : X \longrightarrow Y$$

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LET'S STUDY MORPHISMS IN DETAIL...

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DOMAIN OF g

CODOMAIN OF g

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LET'S STUDY MORPHISMS IN DETAIL...

$g : X \to Y$

DOMAIN OF g

CODOMAIN OF g

g IS **MONIC** (OR IS A **MONO**) IF IT IS LEFT-CANCELLATIVE:

$\forall f, f' : W \to X$ WITH $gf = gf'$ WE GET $f = f'$.

I. CATEGORIES

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HERE: $\mathbb{X} := (X, g)$ IS A SUBOBJECT OF X

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DOMAIN OF g — CODOMAIN OF g

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$\forall f, f' : W \to X$ WITH $gf = gf'$ WE GET $f = f'$.

HERE: $X := (X, g)$ IS A **SUBOBJECT** OF X

g IS **EPIC** (OR IS AN **EPI**) IF IT IS RIGHT-CANCELLATIVE:

$\forall h, h' : Y \to Z$ WITH $hg = h'g$ WE GET $h = h'$.

HERE: $Y := (Y, g)$ IS A **QUOTIENT OBJECT** OF Y

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LET'S STUDY MORPHISMS IN DETAIL...

$$g : X \longrightarrow Y$$

DOMAIN OF g — CODOMAIN OF g

g IS **MONIC** (OR IS A **MONO**) IF IT IS LEFT-CANCELLATIVE:

$\forall f, f' : W \to X$ WITH $gf = gf'$ WE GET $f = f'$.

HERE: $\mathbb{X} := (X, g)$ IS A **SUBOBJECT** OF X

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EXERCISE 2.2

MONO IN Ab $\equiv$ INJECTIVE GROUP HOMOM.

EPI IN Ab $\equiv$ SURJECTIVE GROUP HOMOM.

I. CATEGORIES

A CATEGORY $\mathscr{C}$ CONSISTS OF:

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HERE: WRITE $g' =: g^{-1}$ AND $X \cong Y$.

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FROM EXERCISE 2.2

ISO IN Ab $\equiv$ BIJECTIVE GROUP HOMOM. $\equiv$ GROUP ISOM.

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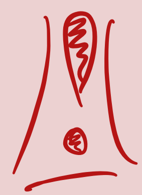

ISO $\neq$ MONO + EPI

IN GENERAL

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<u>EXERCISE 2.2</u>
$\mathbb{Z} \hookrightarrow \mathbb{Q}$ IN Ring IS MONIC & EPIC YET IS NOT AN ISO.

I. CATEGORIES

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ISO $\Leftarrow$ MONO + EPI

FOR "ABELIAN" CATS (STUDIED LATER)

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Ab ~~Ring~~

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EXERCISE 2.1 (1) SHOW: g ISO $\Rightarrow$ g MONIC & EPIC. *youdo!*

(2) $g : X \to Y$ IS $\begin{cases} \text{SPLIT-MONIC} & \text{IF } \exists h : Y \to X \text{ s.t. } hg = id_X. \\ \text{SPLIT-EPIC} & \text{IF } \exists f : Y \to X \text{ s.t. } gf = id_Y. \end{cases}$

SHOW: g SPLIT-MONIC EPI (OR SPLIT-EPIC MONO) $\Rightarrow$ g ISO.

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✓ ISO $=$ MONO $+$ EPI

(Ab) ~~(Ring)~~

FOR "ABELIAN" CATS (STUDIED LATER)

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SUCH THAT

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- $f \in \mathrm{Hom}(\mathcal{D}) \Rightarrow \mathrm{domain}(f), \mathrm{codomain}(f) \in \mathrm{Ob}(\mathcal{D})$.
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A SUBCATEGORY $\mathscr{D}$ OF $\mathscr{C}$ IS FULL IF

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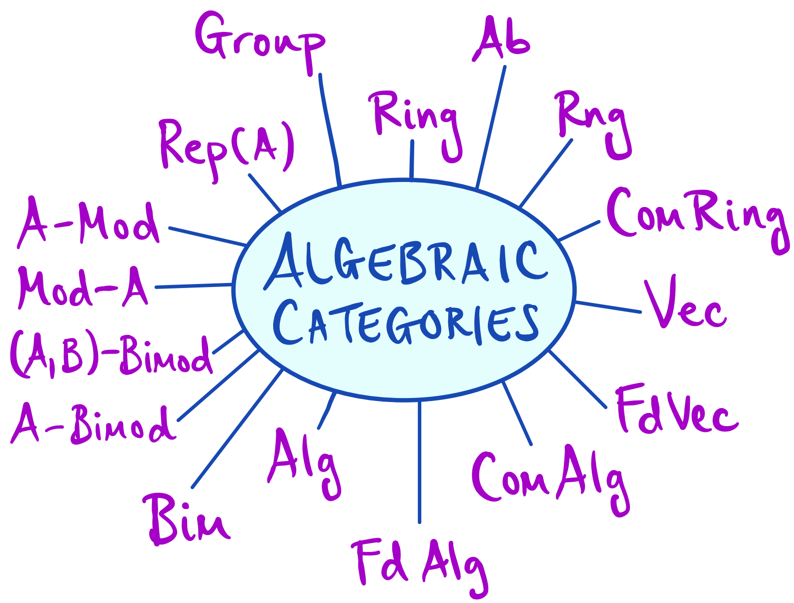

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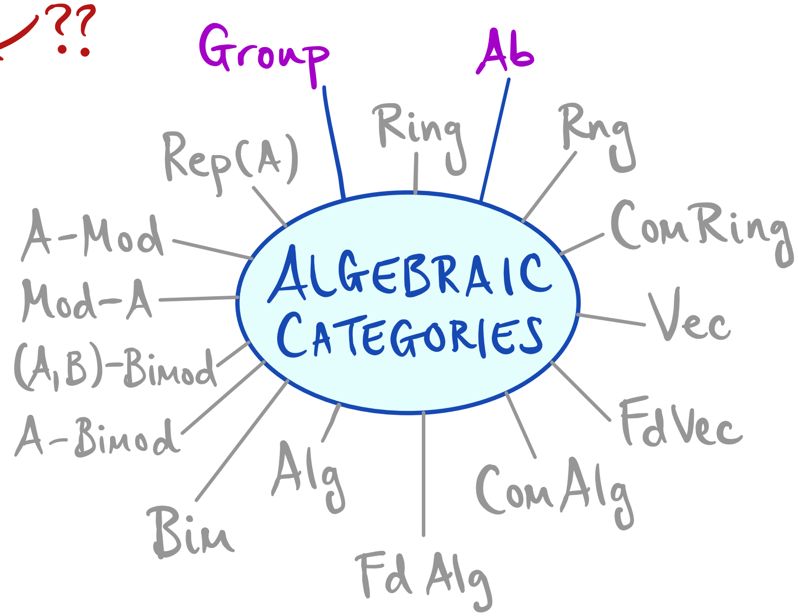

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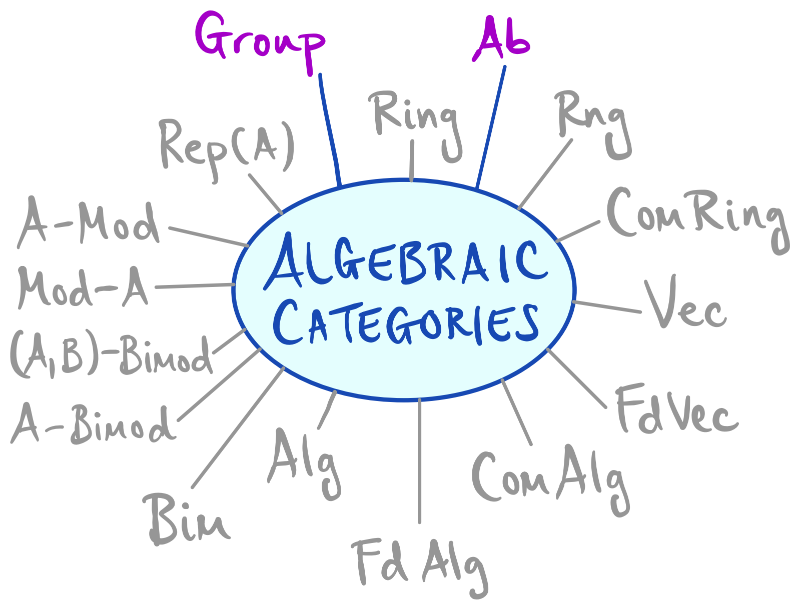

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Ab $\equiv$ SUBCATEGORY OF **Group**

I. CATEGORIES

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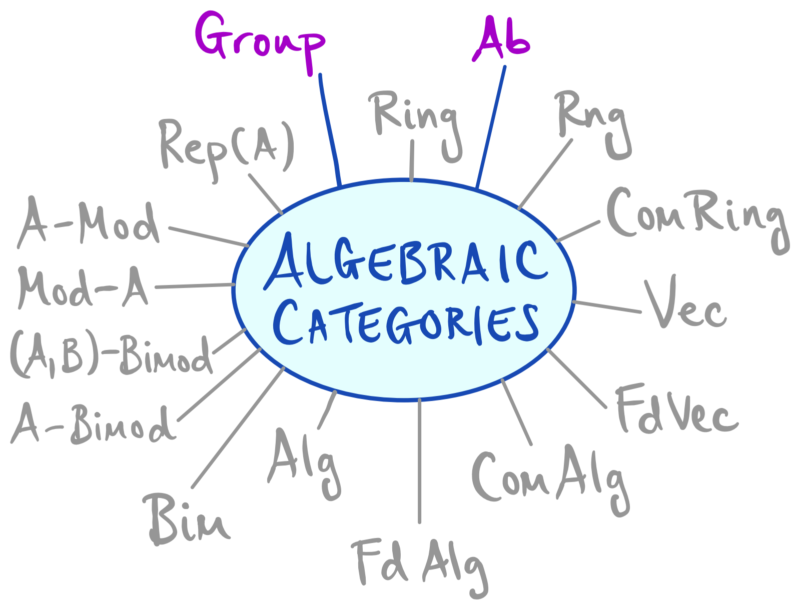


Ab ≡ SUBCATEGORY OF Group

FULL BECAUSE $\forall G, G' \in \mathrm{Ab}$:

$f \in \mathrm{Hom}_{\mathrm{Ab}}(G,G')$ IS A GROUP HOMOM.

SO $f \in \mathrm{Hom}_{\mathrm{Group}}(G,G')$

& VICE VERSA.

I. CATEGORIES

A **CATEGORY** $\mathscr{C}$ CONSISTS OF:

(a) **OBJECTS**.

(b) **MORPHISMS** $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ $\forall X, Y \in \mathscr{C}$.

(c) $id_X : X \to X$ $\forall X \in \mathscr{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY $(hg)f = h(gf)$

UNITALITY $id_X f = f$, $g\, id_X = g$

.... LET'S CHECK OUT SUBSTRUCTURES & EXAMPLES

A **SUBCATEGORY** $\mathscr{D}$ OF $\mathscr{C}$ CONSISTS OF:

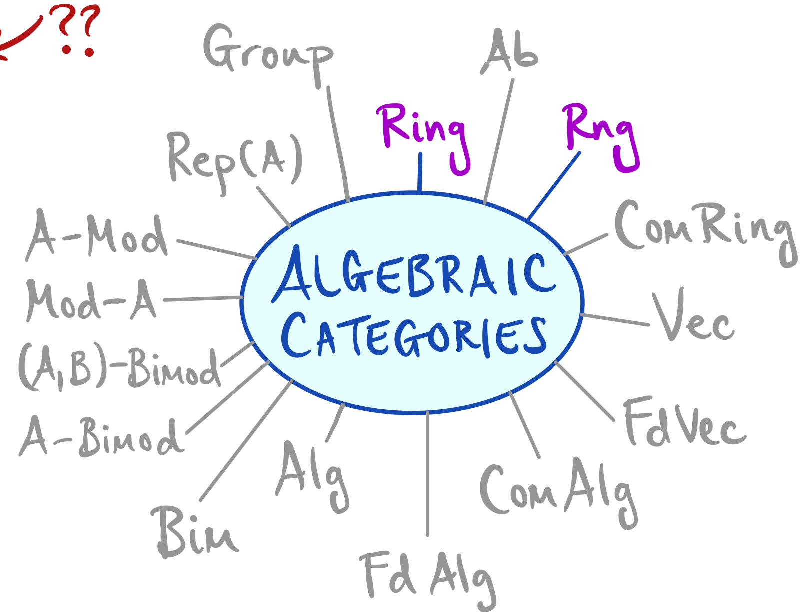
(a) SUBCOLLECTION $Ob(\mathscr{D})$ OF $OB(\mathscr{C})$.

(b) SUBCOLLECTION $\mathrm{Hom}(\mathscr{D})$ OF $\mathrm{Hom}(\mathscr{C})$.

∋.

- $X \in \mathscr{D} \Rightarrow id_X \in \mathrm{Hom}(\mathscr{D})$.
- $f \in \mathrm{Hom}(\mathscr{D}) \Rightarrow$ $dom(f), codom(f) \in Ob(\mathscr{D})$.
- $f, g \in \mathrm{Hom}(\mathscr{D})$ WITH $codom(f) = dom(g)$ $\Rightarrow gf \in \mathrm{Hom}(\mathscr{D})$.

??



SUBCAT $\mathscr{D}$ OF $\mathscr{C}$ IS **FULL** IF

$$\mathrm{Hom}_{\mathscr{D}}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(X,Y) \quad \forall X, Y \in \mathscr{D}.$$

??

I. CATEGORIES

.... LET'S CHECK OUT SUBSTRUCTURES & EXAMPLES

A **CATEGORY** $\mathcal{C}$ CONSISTS OF:

(a) **OBJECTS**.

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ASSOCIATIVITY $(hg)f = h(gf)$

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A **SUBCATEGORY** $\mathcal{D}$ OF $\mathcal{C}$ CONSISTS OF:

(a) SUBCOLLECTION $\mathrm{Ob}(\mathcal{D})$ OF $\mathrm{Ob}(\mathcal{C})$.

(b) SUBCOLLECTION $\mathrm{Hom}(\mathcal{D})$ OF $\mathrm{Hom}(\mathcal{C})$.

$\ni$.

- $X \in \mathcal{D} \Rightarrow \mathrm{id}_X \in \mathrm{Hom}(\mathcal{D})$.
- $f \in \mathrm{Hom}(\mathcal{D}) \Rightarrow \mathrm{dom}(f), \mathrm{codom}(f) \in \mathrm{Ob}(\mathcal{D})$.
- $f, g \in \mathrm{Hom}(\mathcal{D})$ WITH $\mathrm{codom}(f) = \mathrm{dom}(g)$ $\Rightarrow gf \in \mathrm{Hom}(\mathcal{D})$.

SUBCAT $\mathcal{D}$ OF $\mathcal{C}$ IS **FULL** IF

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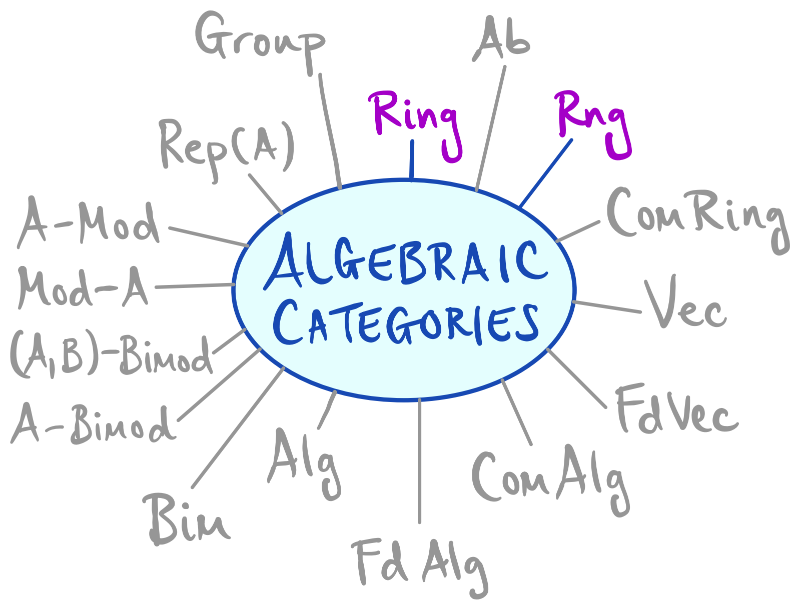

Ring ≡ SUBCATEGORY OF **Rng**

I. CATEGORIES

A CATEGORY $\mathscr{C}$ CONSISTS OF:

(a) OBJECTS.

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SATISFYING

ASSOCIATIVITY $(hg)f = h(gf)$

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.... LET'S CHECK OUT SUBSTRUCTURES & EXAMPLES

A SUBCATEGORY $\mathscr{D}$ OF $\mathscr{C}$ CONSISTS OF:

(a) SUBCOLLECTION $\mathrm{Ob}(\mathscr{D})$ OF $\mathrm{OB}(\mathscr{C})$.

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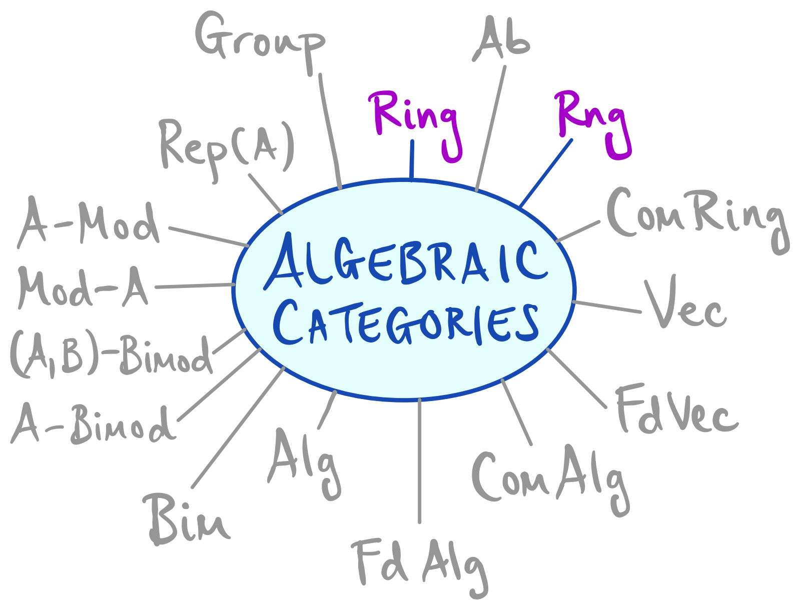
∋.

- $X \in \mathscr{D} \Rightarrow id_X \in \mathrm{Hom}(\mathscr{D})$.
- $f \in \mathrm{Hom}(\mathscr{D}) \Rightarrow$ $\mathrm{dom}(f), \mathrm{codom}(f) \in \mathrm{Ob}(\mathscr{D})$.
- $f, g \in \mathrm{Hom}(\mathscr{D})$ WITH $\mathrm{codom}(f) = \mathrm{dom}(g)$ $\Rightarrow gf \in \mathrm{Hom}(\mathscr{D})$.

SUBCAT $\mathscr{D}$ OF $\mathscr{C}$ IS FULL IF

$$\mathrm{Hom}_{\mathscr{D}}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(X,Y) \quad \forall X, Y \in \mathscr{D}.$$



Ring $\equiv_{\vee}$ SUBCATEGORY OF Rng

NOT FULL BECAUSE $\forall R, R' \in$ Ring:

$f \in \mathrm{Hom}_{Rng}(R,R')$ IS A RING HOMOM.

BUT IT DOESN'T NEED TO BE UNITAL

$\therefore\ \mathrm{Hom}_{Rng}(R,R') \neq \mathrm{Hom}_{Ring}(R,R')$.

I. CATEGORIES

A CATEGORY $\mathscr{C}$

CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS

$\mathrm{Hom}_{\mathscr{C}}(X,Y)$

$\forall X, Y \in \mathscr{C}$.

(c) $id_X : X \to X$

$\forall X \in \mathscr{C}$.

(d) $gf : W \to Y$

$\forall f : W \to X$

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SATISFYING

ASSOCIATIVITY

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UNITALITY

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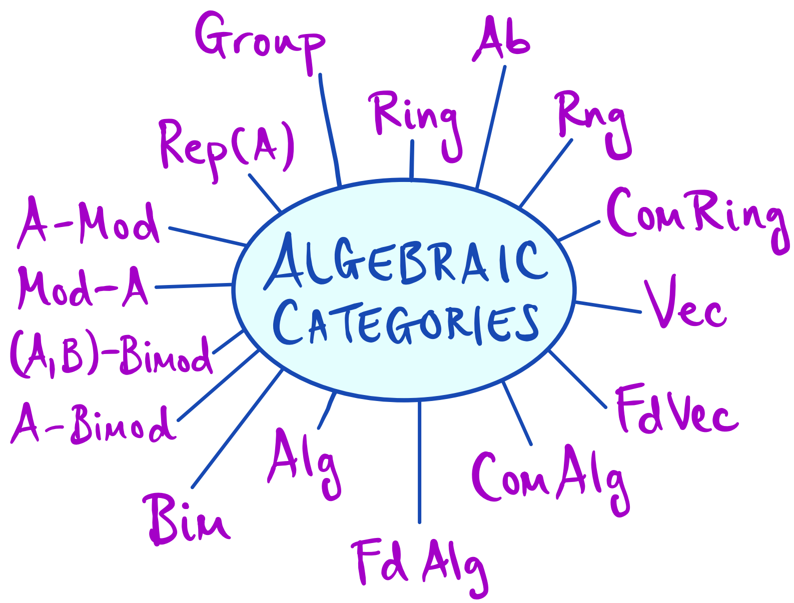
.....MORE EXAMPLES



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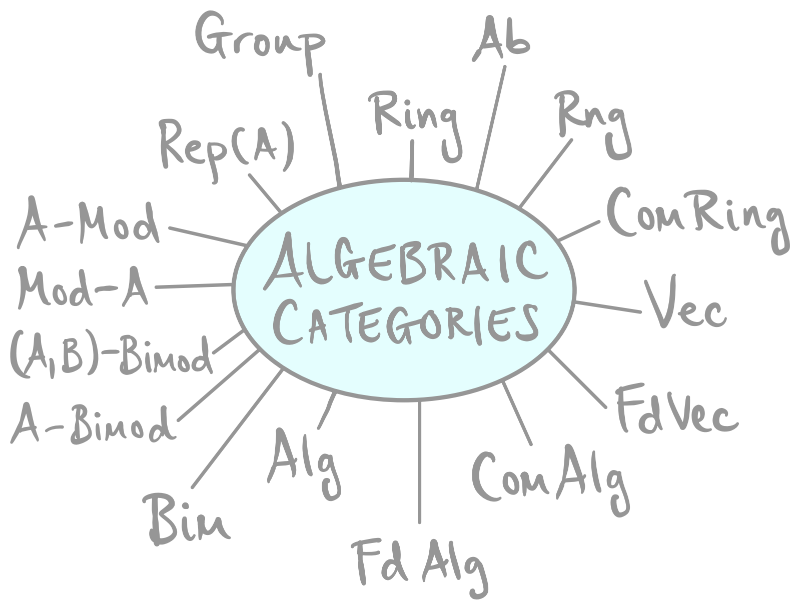

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SATISFYING

ASSOCIATIVITY
$(hg)f = h(gf)$

UNITALITY
$\mathrm{id}_X f = f, \; g\, \mathrm{id}_X = g$

.....MORE EXAMPLES

$\varnothing$
NO OBJECTS
NO MORPHISMS

LOGICAL/CATEGORICAL CATEGORIES

ALGEBRAIC CATEGORIES

Group, Ab, Ring, Rng, Rep(A), ComRing, A-Mod, Vec, Mod-A, (A,B)-Bimod, FdVec, A-Bimod, Alg, ComAlg, Bim, Fd Alg

I. CATEGORIES

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UNITALITY

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.....MORE EXAMPLES

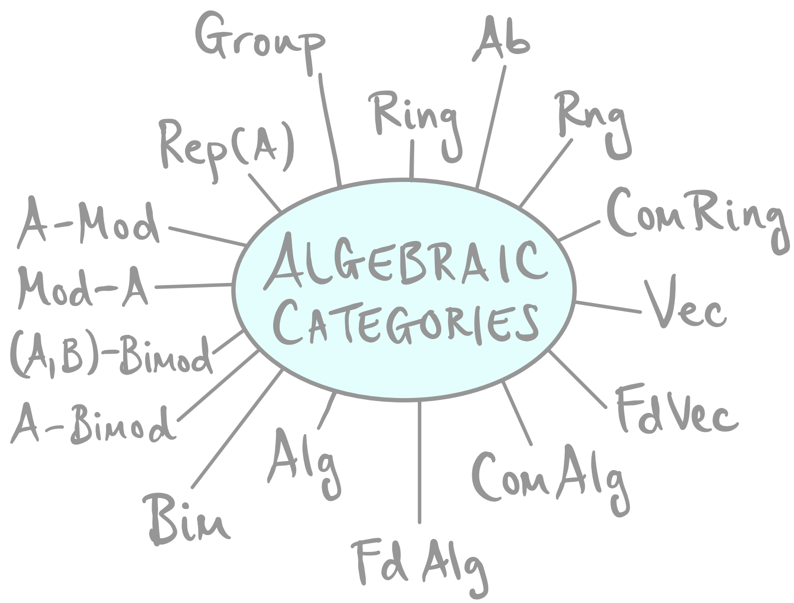
$\varnothing$

- NO OBJECTS
- NO MORPHISMS

Set

- SETS
- FUNCTIONS

LOGICAL/ CATEGORICAL CATEGORIES



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SATISFYING

ASSOCIATIVITY
$(hg)f = h(gf)$

UNITALITY
$id_X f = f, \; g\, id_X = g$

....MORE EXAMPLES

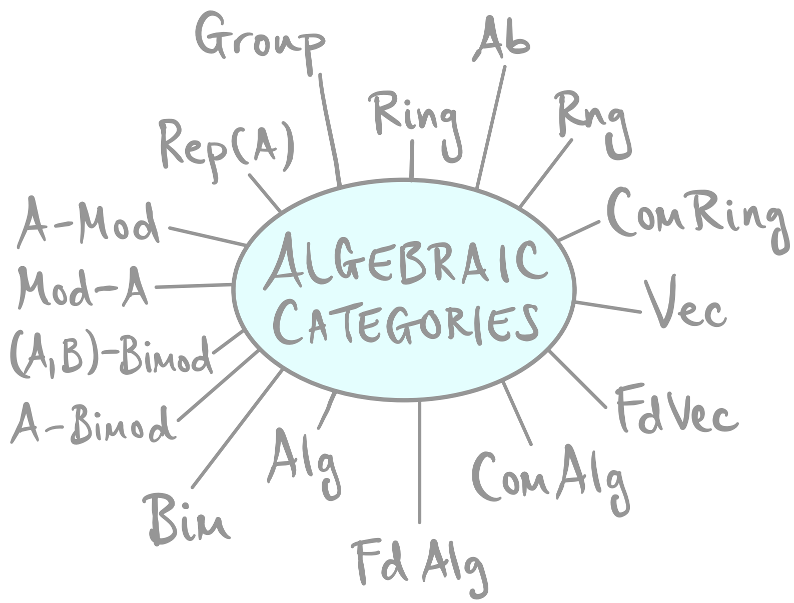
$\emptyset$
- NO OBJECTS
- NO MORPHISMS

Set
- SETS
- FUNCTIONS

LOGICAL/ CATEGORICAL CATEGORIES

Cat
- SMALL CATEGORIES
- "FUNCTORS"

LECTURE 8



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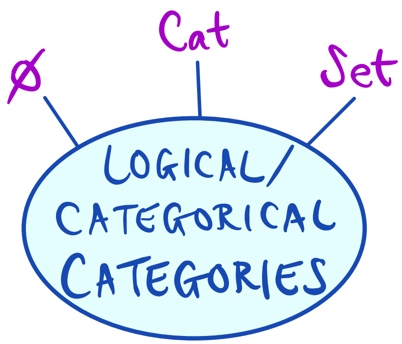

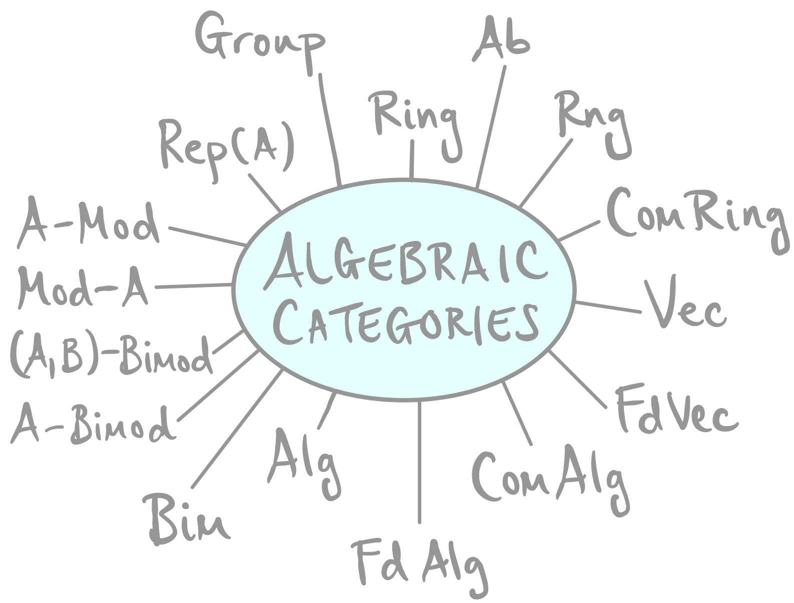

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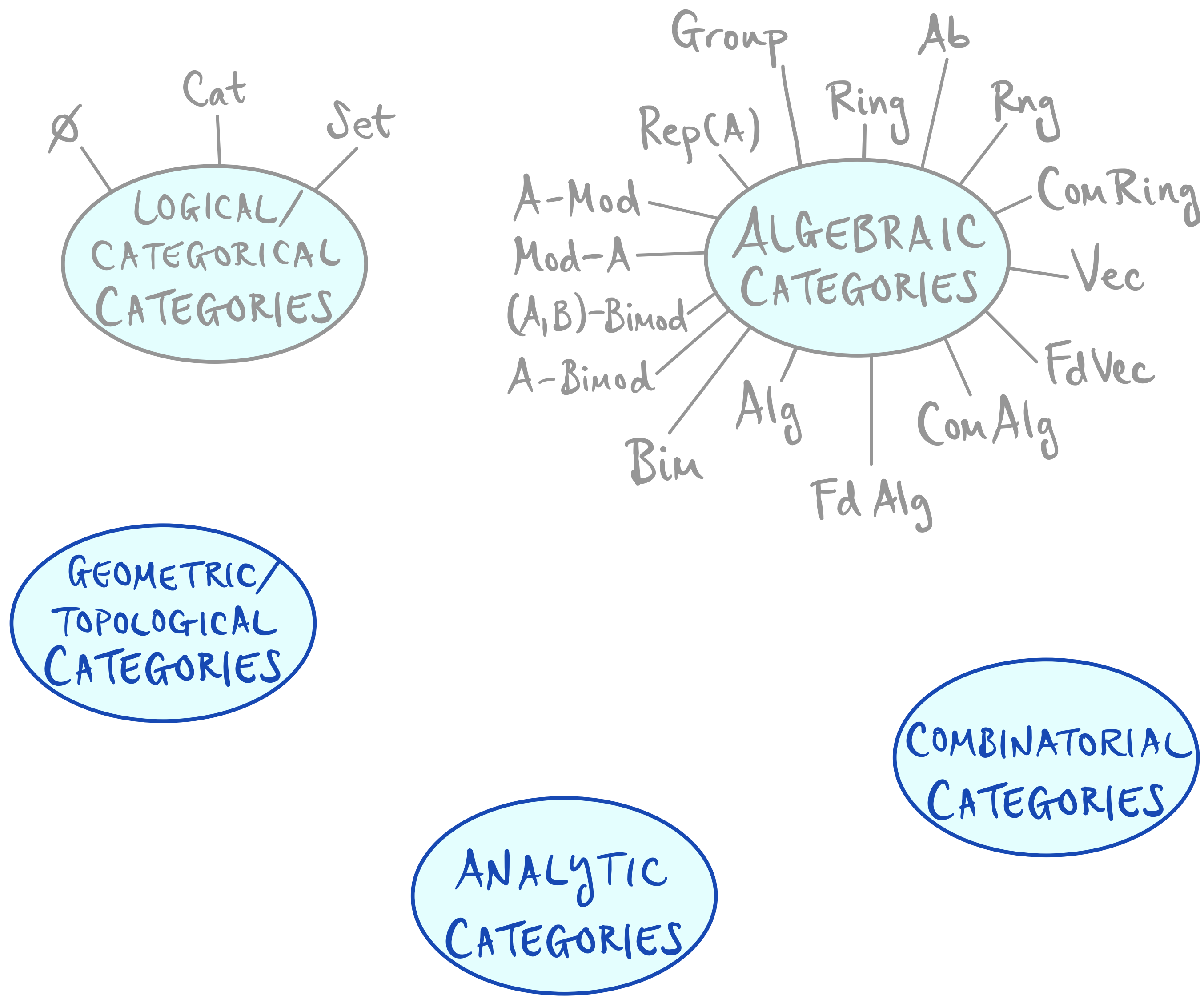

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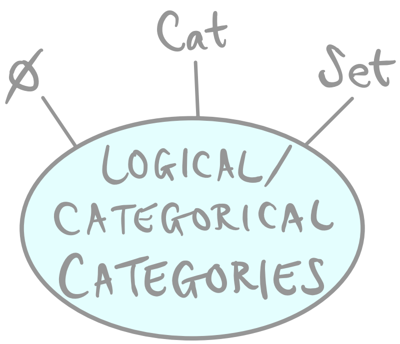

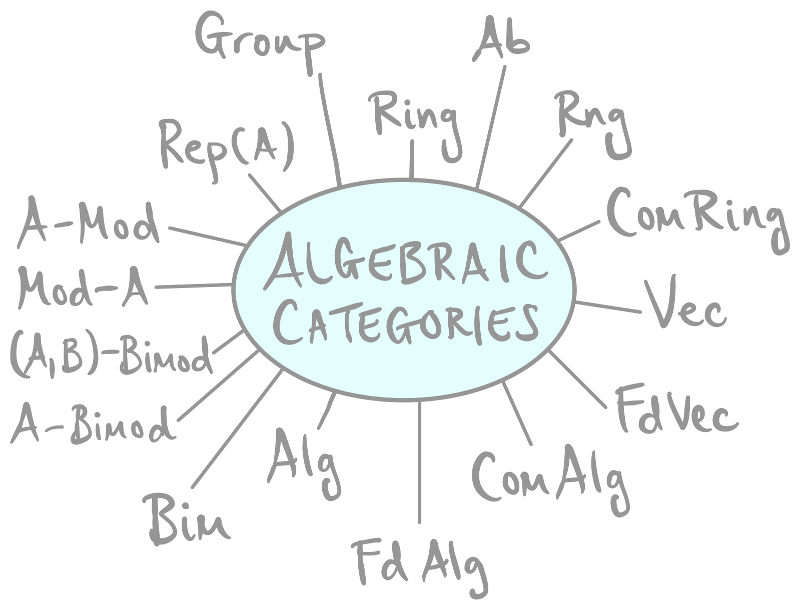

Aff $\begin{cases} \text{AFFINE VARIETIES} \\ \text{REGULAR MAPS} \end{cases}$

GEOMETRIC/
TOPOLOGICAL
CATEGORIES

Top $\begin{cases} \text{TOPOLOGICAL SPACES} \\ \text{CONTINUOUS MAPS} \end{cases}$

ANALYTIC
CATEGORIES

COMBINATORIAL
CATEGORIES

I. CATEGORIES

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.....MORE EXAMPLES

LOGICAL/CATEGORICAL CATEGORIES: $\varnothing$, Cat, Set

ALGEBRAIC CATEGORIES: Group, Ab, Ring, Rng, Rep(A), ComRing, A-Mod, Mod-A, Vec, (A,B)-Bimod, A-Bimod, FdVec, Bim, Alg, ComAlg, Fd Alg

GEOMETRIC/TOPOLOGICAL CATEGORIES: Aff, Top

Hilb $\begin{cases} \text{HILBERT SPACES} \\ \text{BOUNDED LINEAR MAPS} \end{cases}$

ANALYTIC CATEGORIES

COMBINATORIAL CATEGORIES

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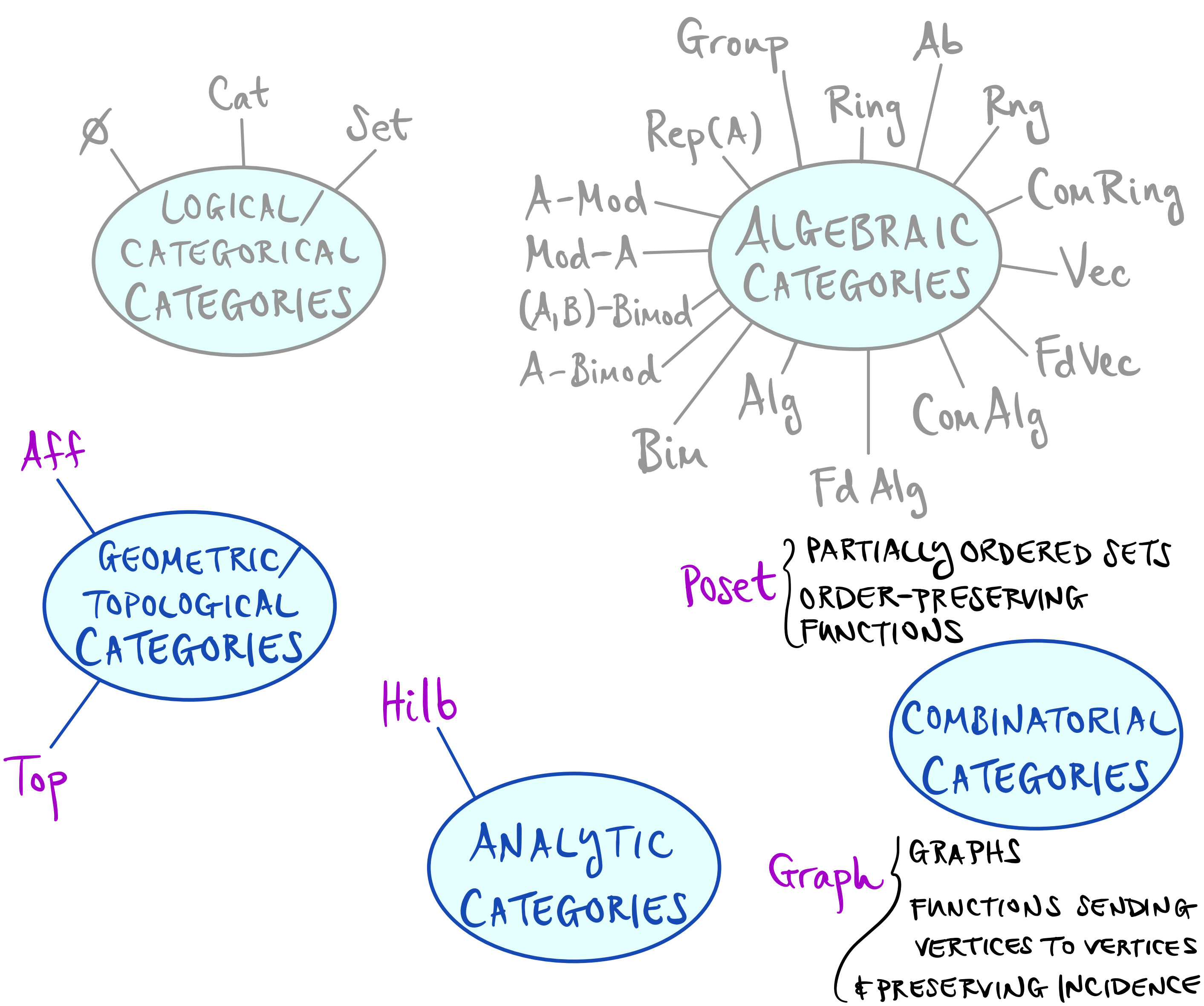

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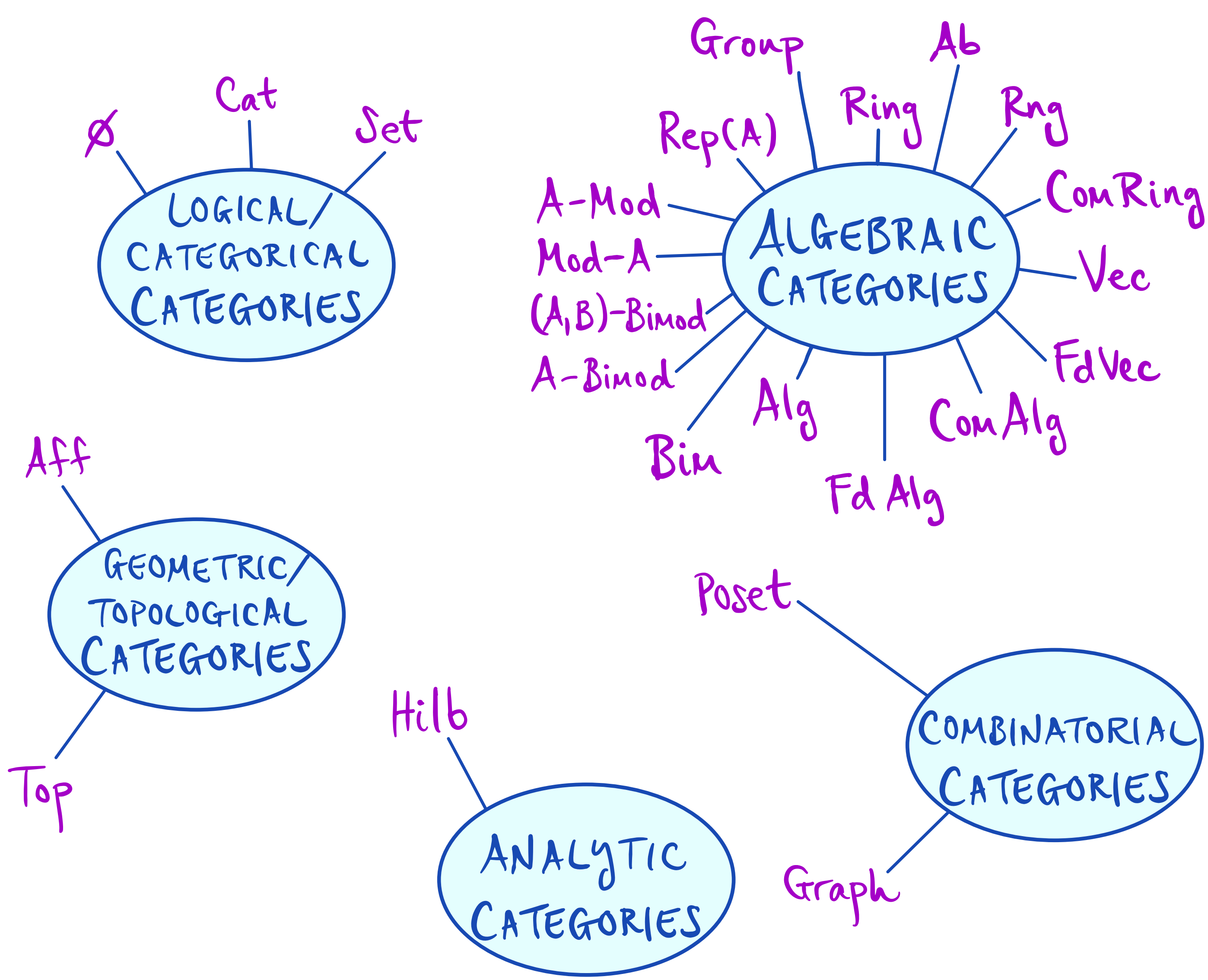

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.....MORE EXAMPLES??

EXERCISE 2.6 IS THE FOLLOWING A CATEGORY?

80s Music :

- OBJECTS = PERSONS
- $\exists f \in \mathrm{Hom}_{80s\,Music}(\text{Person A}, \text{Person B})$

$\Longleftrightarrow$ { Person A & Person B BOTH LIKE A CERTAIN TRACK FROM THE 1980s.

I. CATEGORIES

A CATEGORY $\mathscr{C}$ CONSISTS OF:

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.....MORE EXAMPLES ??

EXERCISE 2.6 IS THE FOLLOWING A CATEGORY?

80s Music :

- OBJECTS $\equiv$ PERSONS
- $\exists f \in \mathrm{Hom}_{\text{80s Music}}(\text{Person A}, \text{Person B})$

$\iff$ { Person A & Person B BOTH LIKE A CERTAIN TRACK FROM THE 1980s.

MAKE UP A WEIRD EXAMPLE

I. CATEGORIES

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SOME OPERATIONS ON CATEGORIES –

I. CATEGORIES

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SOME OPERATIONS ON CATEGORIES —

GIVEN A CATEGORY $\mathscr{C}$,

ITS **OPPOSITE CATEGORY** $\mathscr{C}^{op}$ IS A CATEGORY DEFINED BY

- $Ob(\mathscr{C}^{op}) = Ob(\mathscr{C})$
- $\exists f \in \mathrm{Hom}_{\mathscr{C}^{op}}(X,Y) \iff \exists f \in \mathrm{Hom}_{\mathscr{C}}(Y,X)$

≡ REVERSE DIRECTION OF MORPHISMS ≡

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- $\mathrm{Ob}(\mathcal{C}^{op}) = \mathrm{Ob}(\mathcal{C})$
- $\exists f \in \mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) \iff \exists f \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$

≡ REVERSE DIRECTION OF MORPHISMS ≡

GIVEN CATEGORIES $\mathcal{C}$ AND $\mathcal{C}'$,

ITS **PRODUCT CATEGORY** $\mathcal{C} \times \mathcal{C}'$ IS A CATEGORY DEFINED BY

- $\mathrm{Ob}(\mathcal{C} \times \mathcal{C}') = \{(X,X') \mid X \in \mathcal{C},\ X' \in \mathcal{C}'\}$
- $\mathrm{Hom}_{\mathcal{C}\times\mathcal{C}'}((X,X'),(Y,Y'))$
  $= \{(g,g') \mid g \in \mathrm{Hom}_{\mathcal{C}}(X,Y),\ g' \in \mathrm{Hom}_{\mathcal{C}'}(X',Y')\}$

≡ THINK ABOUT COMPOSITION OF MORPHISMS ≡

II. UNIVERSAL CONSTRUCTIONS

RECALL UNIVERSAL PROPERTY ...

GIVEN A GADGET X,

A UNIVERSAL STRUCTURE ATTACHED TO X VIA α (OR α')

IS A STRUCTURE $\mathrm{Univ}(X)$

$\ni$. $\forall$ ARBITRARY STRUCTURES $\mathrm{Arb}(X)$ ATTACHED TO X VIA β (OR β')

$\exists!$ STRUCTURE MAP γ (OR γ') MAKING THE DIAGRAM COMMUTE:

$$
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[dr, "\forall\beta"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}
$$

FORM I

$$
\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists!\gamma'"'] \arrow[dr, "\forall\beta'"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'"] \arrow[ur, phantom, "\circlearrowleft"] & X
\end{tikzcd}
$$

FORM II

II. UNIVERSAL CONSTRUCTIONS

RECALL UNIVERSAL PROPERTY ...

GIVEN A GADGET X,

A UNIVERSAL STRUCTURE ATTACHED TO X VIA α (OR α')

IS A STRUCTURE $\mathrm{Univ}(X)$

∴ $\forall$ ARBITRARY STRUCTURES $\mathrm{Arb}(X)$ ATTACHED TO X VIA β (OR β')

$\exists!$ STRUCTURE MAP γ (OR γ') MAKING THE DIAGRAM COMMUTE:

$\mathrm{Univ}(X)$ DOESN'T HAVE TO EXIST.

IF EXISTS, THEN IT'S UNIQUE UP TO ISO.

$$
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[dr, "\forall\beta"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}
$$

FORM I

$$
\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists!\gamma'"'] \arrow[dr, "\forall\beta'"] \arrow[dr, phantom, shift right=3, "\circlearrowleft"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'"] & X
\end{tikzcd}
$$

FORM II

II. UNIVERSAL CONSTRUCTIONS

WE DON'T HAVE "ELEMENTS" TO WORK WITH IN GENERAL. THIS IS THE MAIN WAY WE'LL DO COMPUTATIONS.

RECALL UNIVERSAL PROPERTY ...

GIVEN A GADGET X,

A UNIVERSAL STRUCTURE ATTACHED TO X VIA α_X (OR α'_X)

IS A STRUCTURE $\mathrm{Univ}(X)$

s.t. $\forall$ ARBITRARY STRUCTURES $\mathrm{Arb}(X)$ ATTACHED TO X VIA β_X (OR β'_X)

$\exists!$ STRUCTURE MAP γ_X (OR γ'_X) MAKING THE DIAGRAM COMMUTE:

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IF EXISTS, THEN IT'S UNIQUE UP TO ISO.

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X \arrow[ul, phantom, "\circlearrowright"]
\end{tikzcd}$$

FORM II

II. UNIVERSAL CONSTRUCTIONS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowright"'] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

FORM II

Univ(X) DOESN'T HAVE TO EXIST.

IF EXISTS, THEN IT'S UNIQUE UP TO ISO.

WE DON'T HAVE "ELEMENTS" TO WORK WITH IN GENERAL. THIS IS THE MAIN WAY WE'LL DO COMPUTATIONS

NAMING

Univ(X) IS ATTACHED TO X VIA α_X (OR α'_X)

II. UNIVERSAL CONSTRUCTIONS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

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Univ(X) IS ATTACHED TO X VIA α_X (OR α'_X)

SOMETIMES WE ONLY NAME THIS WHEN BUILDING UNIVERSAL CONSTRUCTIONS

II. UNIVERSAL CONSTRUCTIONS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

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$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists!\gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X \end{tikzcd}$$

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SOMETIMES WE ONLY NAME THIS

WHEN BUILDING UNIVERSAL CONSTRUCTIONS

II. UNIVERSAL CONSTRUCTIONS

UNIVERSAL PROPERTY

$$
\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}
$$

FORM I

$$
\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X
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II. UNIVERSAL CONSTRUCTIONS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X \end{tikzcd}$$

FORM II

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SOMETIMES WE NAME BOTH

WHEN BUILDING UNIVERSAL CONSTRUCTIONS

Ex.

WILL BUILD "KERNEL" OF A MORPHISM $f: X \to Y$ $\equiv$ $\left\{ \begin{array}{l} \text{OBJECT "ker(f)"} \\ \text{EQUIPPED WITH MORPHISM} \\ \ker(f) \xrightarrow{\alpha'_f} X \end{array} \right.$

II. UNIVERSAL CONSTRUCTIONS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

Univ(X) DOESN'T HAVE TO EXIST.

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SOMETIMES WE NAME BOTH

WHEN BUILDING UNIVERSAL CONSTRUCTIONS

Ex.

WILL BUILD "KERNEL" OF A MORPHISM $f: X \to Y$ $\equiv$ $\left\{ \begin{array}{l} \text{OBJECT "}\ker(f)\text{"} \\ \text{EQUIPPED WITH MORPHISM} \\ \ker(f) \xrightarrow{\alpha'_f} X \end{array} \right.$

II. UNIVERSAL CONSTRUCTIONS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

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WE DON'T HAVE "ELEMENTS" TO WORK WITH IN GENERAL. THIS IS THE MAIN WAY WE'LL DO COMPUTATIONS

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X \end{tikzcd}$$

FORM II

NAMING

Univ(X) IS ATTACHED TO X VIA α_X (OR α'_X)

SOMETIMES WE NAME BOTH

WHEN BUILDING UNIVERSAL CONSTRUCTIONS

Ex.

WILL BUILD "KERNEL" OF A MORPHISM $f: X \to Y$ $\equiv$ $\left\{ \begin{array}{l} \text{OBJECT "ker(f)"} \\ \text{EQUIPPED WITH} \\ \text{MORPHISM} \\ \ker(f) \xrightarrow{\alpha'_f} X \end{array} \right.$

II. UNIVERSAL CONSTRUCTIONS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[d, phantom, shift left=6, "\circlearrowleft"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

Univ(X) DOESN'T HAVE TO EXIST.

IF EXISTS, THEN IT'S UNIQUE UP TO ISO.

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Univ(X) IS ATTACHED TO X VIA α_X (OR α'_X)

SOMETIMES WE NAME BOTH

WHEN BUILDING UNIVERSAL CONSTRUCTIONS

Ex.

WILL BUILD "KERNEL" OF A MORPHISM $f : X \to Y$ $\equiv$ $\left\{ \begin{array}{l} \text{OBJECT "ker(f)"} \\ \text{EQUIPPED WITH MORPHISM} \\ \ker(f) \xrightarrow{\alpha'_f} X \end{array} \right.$

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X
\end{tikzcd}$$

FORM II

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathscr{C}$:

AN OBJECT $I \in \mathscr{C}$ IS INITIAL IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd} I \arrow[d, dashed, "\exists! \vec{0}_X"] \\ X \end{tikzcd}$$

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

≡ THINK ABOUT THE LINK ≡

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowright"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

AN OBJECT $I \in \mathcal{C}$ IS INITIAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd}
I \arrow[d, dashed, "\exists! \vec{0}_X"] \\
X
\end{tikzcd}$$

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathscr{C}$:

$$\begin{tikzcd}
X \arrow[rr, "\alpha_X"] \arrow[drr, "\forall \beta_X"'] & & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \\
& \arrow[ur, phantom, "\circlearrowleft"] & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

AN OBJECT $I \in \mathscr{C}$ IS INITIAL IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd}
I \arrow[d, dashed, "\exists! \vec{0}_X"] \\
X
\end{tikzcd}$$

AN OBJECT $T \in \mathscr{C}$ IS TERMINAL IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd}
X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\
T
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[drr, "\forall \beta'_X"] & \arrow[d, phantom, "\circlearrowleft"] & \\
\mathrm{Univ}(X) \arrow[rr, "\alpha'_X"'] & & X
\end{tikzcd}$$

FORM II

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowright"] & X
\end{tikzcd}$$

FORM II

≡ THINK ABOUT THE LINK ≡

GIVEN A CATEGORY $\mathscr{C}$:

AN OBJECT $I \in \mathscr{C}$ IS **INITIAL** IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd}
I \arrow[d, dashed, "\exists! \vec{0}_X"] \\
X
\end{tikzcd}$$

AN OBJECT $T \in \mathscr{C}$ IS **TERMINAL** IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd}
X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\
T
\end{tikzcd}$$

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathscr{C}$:

AN OBJECT $I \in \mathscr{C}$ IS INITIAL IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM $\vec{o}_X : I \to X$.

$$\begin{tikzcd} I \arrow[d, dashed, "\exists! \vec{o}_X"] \\ X \end{tikzcd}$$

AN OBJECT $T \in \mathscr{C}$ IS TERMINAL IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM ${}_X\vec{o} : X \to T$.

$$\begin{tikzcd} X \arrow[d, dashed, "\exists! {}_X\vec{o}"] \\ T \end{tikzcd}$$

A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

AN OBJECT $I \in \mathcal{C}$ IS INITIAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd} I \arrow[d, dashed, "\exists! \vec{0}_X"] \\ X \end{tikzcd}$$

AN OBJECT $T \in \mathcal{C}$ IS TERMINAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd} X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\ T \end{tikzcd}$$

A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I				
T				
0				

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowright"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

AN OBJECT $I \in \mathcal{C}$ IS **INITIAL** IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd}
I \arrow[d, dashed, "\exists! \vec{0}_X"] \\
X
\end{tikzcd}$$

AN OBJECT $T \in \mathcal{C}$ IS **TERMINAL** IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd}
X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\
T
\end{tikzcd}$$

A **ZERO** OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$			
T	$\{\bullet\}$			
0	N/A			

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathscr{C}$:

AN OBJECT $I \in \mathscr{C}$ IS INITIAL IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd}
I \arrow[d, dashed, "\exists! \vec{0}_X"] \\
X
\end{tikzcd}$$

AN OBJECT $T \in \mathscr{C}$ IS TERMINAL IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd}
X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\
T
\end{tikzcd}$$

A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$ ⇜ IS A SUBSET OF ANY SET INCLUDING $\emptyset$			
T	$\{\bullet\}$			
0	N/A			

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\,\gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists!\,\gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

AN OBJECT $I \in \mathcal{C}$ IS **INITIAL** IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd}
I \arrow[d, dashed, "\exists!\,\vec{0}_X"] \\
X
\end{tikzcd}$$

AN OBJECT $T \in \mathcal{C}$ IS **TERMINAL** IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd}
X \arrow[d, dashed, "\exists!\,{}_X\vec{0}"] \\
T
\end{tikzcd}$$

A **ZERO** OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$			
T	$\{\bullet\}$			
0	N/A			

$\emptyset$ ⇜ IS A SUBSET OF ANY SET INCLUDING $\emptyset$

$\{\bullet\}$ ⇜ ANY FUNCTION MUST SEND ELEMENTS TO ELEMENTS

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowright"] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathscr{C}$:

AN OBJECT $I \in \mathscr{C}$ IS INITIAL IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd} I \arrow[d, dashed, "\exists! \vec{0}_X"] \\ X \end{tikzcd}$$

AN OBJECT $T \in \mathscr{C}$ IS TERMINAL IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd} X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\ T \end{tikzcd}$$

A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I	$\varnothing$	$\{e\}$		
T	$\{\bullet\}$	$\{e\}$		
0	N/A	$\{e\}$		

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathscr{C}$:

AN OBJECT $I \in \mathscr{C}$ IS INITIAL IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd} I \arrow[d, dashed, "\exists! \vec{0}_X"] \\ X \end{tikzcd}$$

AN OBJECT $T \in \mathscr{C}$ IS TERMINAL IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd} X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\ T \end{tikzcd}$$

A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I	$\varnothing$	$\{e\}$ ⇜ $\exists!\ \{e\} \to G$ INCLUSION		
T	$\{\bullet\}$	$\{e\}$ ⇜ $\exists!\ G \twoheadrightarrow G/G = \{e\}$		
0	N/A	$\{e\}$		

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathscr{C}$:

AN OBJECT $I \in \mathscr{C}$ IS INITIAL IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd}
I \arrow[d, dashed, "\exists! \vec{0}_X"] \\
X
\end{tikzcd}$$

AN OBJECT $T \in \mathscr{C}$ IS TERMINAL IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd}
X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\
T
\end{tikzcd}$$

A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I	$\varnothing$	$\{e\}$	$\mathbb{Z}$	
T	$\{\bullet\}$	$\{e\}$	0_{RING}	
0	N/A	$\{e\}$	N/A	

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathscr{C}$:

AN OBJECT $I \in \mathscr{C}$ IS **INITIAL** IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd}
I \arrow[d, dashed, "\exists! \vec{0}_X"] \\
X
\end{tikzcd}$$

AN OBJECT $T \in \mathscr{C}$ IS **TERMINAL** IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd}
X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\
T
\end{tikzcd}$$

A **ZERO** OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$	$\{e\}$	$\mathbb{Z} \rightsquigarrow \exists!\, \mathbb{Z} \to R$, $1_{\mathbb{Z}} \mapsto 1_R$	
T	$\{\bullet\}$	$\{e\}$	$0_{\mathrm{RING}} \rightsquigarrow \exists!\, R \twoheadrightarrow R/R = 0_{\mathrm{RING}}$	
0	N/A	$\{e\}$	N/A	

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathscr{C}$:

AN OBJECT $I \in \mathscr{C}$ IS **INITIAL** IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd} I \arrow[d, dashed, "\exists! \vec{0}_X"] \\ X \end{tikzcd}$$

AN OBJECT $T \in \mathscr{C}$ IS **TERMINAL** IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd} X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\ T \end{tikzcd}$$

A **ZERO** OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$	$\{e\}$	$\mathbb{Z}$	0_{VS}
T	$\{\bullet\}$	$\{e\}$	0_{RING}	0_{VS}
0	N/A	$\{e\}$	N/A	0_{VS}

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

AN OBJECT $I \in \mathcal{C}$ IS INITIAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd} I \arrow[d, dashed, "\exists! \vec{0}_X"] \\ X \end{tikzcd}$$

AN OBJECT $T \in \mathcal{C}$ IS TERMINAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd} X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\ T \end{tikzcd}$$

A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$	$\{e\}$	$\mathbb{Z}$	0_{VS}
T	$\{\bullet\}$	$\{e\}$	0_{RING}	0_{VS}
0	N/A	$\{e\}$	N/A	0_{VS}

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

$$
\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\
& \mathrm{Arb}(X)
\end{tikzcd}
$$

FORM I

$$
\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowright"] & X
\end{tikzcd}
$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

COPRODUCT OF X AND Y	PRODUCT OF X AND Y

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[r, phantom, bend right=40, "\circlearrowleft"] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

COPRODUCT OF X AND Y	PRODUCT OF X AND Y
IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING: $\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists ! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$	

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

COPRODUCT OF X AND Y	PRODUCT OF X AND Y
IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:	

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$$

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowright"] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

COPRODUCT OF X AND Y IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowright"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$$

PRODUCT OF X AND Y IS AN OBJECT $X \sqcap Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha'_X : X \sqcap Y \to X$ AND $\alpha'_Y : X \sqcap Y \to Y$ SATISFYING:

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists! \gamma'"] \arrow[dr, "\forall \beta'_Y"] & \\ X \arrow[ur, phantom, "\circlearrowright"] & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] \arrow[ur, phantom, "\circlearrowright"] & Y \end{tikzcd}$$

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

COPRODUCT OF X AND Y IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:	PRODUCT OF X AND Y IS AN OBJECT $X \sqcap Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha'_X : X \sqcap Y \to X$ AND $\alpha'_Y : X \sqcap Y \to Y$ SATISFYING:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists! \gamma"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$$

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists! \gamma'"] \arrow[dr, "\forall \beta'_Y"] & \\ X & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] & Y \end{tikzcd}$$

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

COPRODUCT OF X AND Y IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists ! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$$

PRODUCT OF X AND Y IS AN OBJECT $X \sqcap Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha'_X : X \sqcap Y \to X$ AND $\alpha'_Y : X \sqcap Y \to Y$ SATISFYING:

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists ! \gamma'"] \arrow[dr, "\forall \beta'_Y"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & \\ X & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] & Y \end{tikzcd}$$

EXAMPLES

	Set	Group	Ring	Vec
$\sqcup$				
$\sqcap$				

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

COPRODUCT OF X AND Y IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\
 & C &
\end{tikzcd}$$

PRODUCT OF X AND Y IS AN OBJECT $X \sqcap Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha'_X : X \sqcap Y \to X$ AND $\alpha'_Y : X \sqcap Y \to Y$ SATISFYING:

$$\begin{tikzcd}
 & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists! \gamma'"] \arrow[dr, "\forall \beta'_Y"] \arrow[dl, phantom, "\circlearrowleft"'] \arrow[dr, phantom, "\circlearrowleft"] & \\
X & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] & Y
\end{tikzcd}$$

EXAMPLES	Set	Group	Ring	Vec
$\sqcup$	$\uplus$ DISJOINT UNION			
$\sqcap$	$\times$ CARTESIAN PRODUCT			

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists!\gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

COPRODUCT	PRODUCT
COPRODUCT OF X AND Y IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:	PRODUCT OF X AND Y IS AN OBJECT $X \sqcap Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha'_X : X \sqcap Y \to X$ AND $\alpha'_Y : X \sqcap Y \to Y$ SATISFYING:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists!\gamma"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$$

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists!\gamma'"] \arrow[dr, "\forall \beta'_Y"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"'] & \\ X & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] & Y \end{tikzcd}$$

EXAMPLES	Set	Group	Ring	Vec
$\sqcup$	$\uplus$ DISJOINT UNION	$*$ FREE PRODUCT		
$\sqcap$	$\times$ CARTESIAN PRODUCT	$\times$ DIRECT PRODUCT		

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathscr{C}$ & OBJECTS $X, Y \in \mathscr{C}$:

COPRODUCT OF X AND Y IS AN OBJECT $X \sqcup Y \in \mathscr{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists! \gamma"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\
 & C &
\end{tikzcd}$$

PRODUCT OF X AND Y IS AN OBJECT $X \sqcap Y \in \mathscr{C}$ EQUIPPED WITH MORPHISMS $\alpha'_X : X \sqcap Y \to X$ AND $\alpha'_Y : X \sqcap Y \to Y$ SATISFYING:

$$\begin{tikzcd}
 & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists! \gamma'"] \arrow[dr, "\forall \beta'_Y"] & \\
X & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] & Y
\end{tikzcd}$$

EXAMPLES

	Set	Group	Ring	Vec
$\sqcup$	$\uplus$ DISJOINT UNION	$\circledast$ FREE PRODUCT	$\circledast$ FREE PRODUCT	
$\sqcap$	$\times$ CARTESIAN PRODUCT	$\times$ DIRECT PRODUCT	$\times$ DIRECT PRODUCT	

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

COPRODUCT	PRODUCT
COPRODUCT OF X AND Y IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:	PRODUCT OF X AND Y IS AN OBJECT $X \sqcap Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha'_X : X \sqcap Y \to X$ AND $\alpha'_Y : X \sqcap Y \to Y$ SATISFYING:

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\
& C &
\end{tikzcd}$$

$$\begin{tikzcd}
& P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists! \gamma'"] \arrow[dr, "\forall \beta'_Y"] & \\
X \arrow[ur, phantom, "\circlearrowleft"] & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] \arrow[ur, phantom, "\circlearrowleft"] & Y
\end{tikzcd}$$

EXAMPLES

	Set	Group	Ring	Vec
$\sqcup$	$\uplus$ DISJOINT UNION	$\circledast$ FREE PRODUCT	$\circledast$ FREE PRODUCT	$\oplus$ DIRECT SUM
$\sqcap$	$\times$ CARTESIAN PRODUCT	$\times$ DIRECT PRODUCT	$\times$ DIRECT PRODUCT	$\times$ DIRECT PRODUCT

II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

$$
\begin{tikzcd}
I \arrow[d, dashed, "\exists!\, \vec{0}_X"] \\
X
\end{tikzcd}
$$

INITIAL OBJECT

$$
\begin{tikzcd}
X \arrow[d, dashed, "\exists!\, {}_X\vec{0}"] \\
T
\end{tikzcd}
$$

TERMINAL OBJECT

$$
\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists!\, \gamma"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\
& C &
\end{tikzcd}
$$

COPRODUCT OF OBJECTS

$$
\begin{tikzcd}
& P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists!\, \gamma'"] \arrow[dr, "\forall \beta'_Y"] & \\
X & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] & Y
\end{tikzcd}
$$

PRODUCT OF OBJECTS

II. Universal Constructions: A Computation

$$
\begin{tikzcd}
I \arrow[d, dashed, "\exists!\, \vec{0}_X"] \\
X
\end{tikzcd}
$$

Initial object

$$
\begin{tikzcd}
X \arrow[d, dashed, "\exists!\, {}_X\vec{0}"] \\
T
\end{tikzcd}
$$

Terminal object

$$
\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists!\, \gamma"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\
& C &
\end{tikzcd}
$$

Coproduct of objects

$$
\begin{tikzcd}
& P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists!\, \gamma'"] \arrow[dr, "\forall \beta'_Y"] & \\
X & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] & Y
\end{tikzcd}
$$

Product of objects

Example: $X \sqcup I \cong X$ for any $X \in \mathcal{C}$.

II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

$$\begin{tikzcd}
I \arrow[d, dashed, "\exists!\,\vec{o}_X"] \\
X
\end{tikzcd}$$

INITIAL OBJECT

$$\begin{tikzcd}
X \arrow[d, dashed, "\exists!\,{}_X\vec{o}"] \\
T
\end{tikzcd}$$

TERMINAL OBJECT

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall\beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists!\,\gamma"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall\beta_Y"] \\
& C &
\end{tikzcd}$$

COPRODUCT OF OBJECTS

$$\begin{tikzcd}
& P \arrow[dl, "\forall\beta'_X"'] \arrow[d, dashed, "\exists!\,\gamma'"] \arrow[dr, "\forall\beta'_Y"] & \\
X & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] & Y
\end{tikzcd}$$

PRODUCT OF OBJECTS

EXAMPLE: $X \sqcup I \cong X$ for any $X \in \mathscr{C}$.

PF/ HAVE

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\beta_X = \mathrm{id}_X"'] & X \sqcup I \arrow[d, dashed, "\gamma"] & I \arrow[l, "\alpha_I"'] \arrow[dl, "\vec{o}_X = \beta_I"] \\
& X = C &
\end{tikzcd}$$

$\therefore\ \gamma\,\alpha_X = \mathrm{id}_X$ STS: $\alpha_X\,\gamma = \mathrm{id}_{X \sqcup I}$

II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

$$\begin{tikzcd} I \arrow[d, dashed, "\exists!\, \vec{o}_X"] \\ X \end{tikzcd}$$

INITIAL OBJECT

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!\, {}_X\vec{o}"] \\ T \end{tikzcd}$$

TERMINAL OBJECT

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists!\, \gamma"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$$

COPRODUCT OF OBJECTS

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists!\, \gamma'"] \arrow[dr, "\forall \beta'_Y"] & \\ X & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] & Y \end{tikzcd}$$

PRODUCT OF OBJECTS

<u>EXAMPLE</u>: $X \sqcup I \cong X$ for any $X \in \mathcal{C}$.

PF/ HAVE

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\beta_X = \mathrm{id}_X"'] & X \sqcup I \arrow[d, dashed, "\gamma"] & I \arrow[l, "\alpha_I"'] \arrow[dl, "\vec{o}_X = \beta_I"] \\ & X = C & \end{tikzcd}$$

$\therefore\ \gamma\, \alpha_X = \mathrm{id}_X$ <u>STS</u>: $\alpha_X\, \gamma = \mathrm{id}_{X \sqcup I}$

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\mathrm{id}_X"'] \arrow[ddr, bend right, "\alpha_X"'] & X \sqcup I \arrow[d, dashed, "\gamma"] & I \arrow[l, "\alpha_I"'] \arrow[dl, "\vec{o}_X"] \arrow[ddl, bend left, "\alpha_I"] \\ & X \arrow[d, "\alpha_X"] & \\ & X \sqcup I & \end{tikzcd}$$

II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

$$\begin{tikzcd} I \arrow[d, dashed, "\exists!\, \vec{0}_X"] \\ X \end{tikzcd}$$

INITIAL OBJECT

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!\, {}_X\vec{0}"] \\ T \end{tikzcd}$$

TERMINAL OBJECT

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, bend left=20, "\circlearrowleft"] & X \sqcup Y \arrow[d, dashed, "\exists!\,\gamma"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \arrow[dl, phantom, bend right=20, "\circlearrowleft"] \\
& C &
\end{tikzcd}$$

COPRODUCT OF OBJECTS

$$\begin{tikzcd}
& P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists!\,\gamma'"] \arrow[dr, "\forall \beta'_Y"] \arrow[dl, phantom, bend left=20, "\circlearrowleft"] \arrow[dr, phantom, bend right=20, "\circlearrowleft"] & \\
X & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] & Y
\end{tikzcd}$$

PRODUCT OF OBJECTS

EXAMPLE: $X \sqcup I \cong X$ for any $X \in \mathscr{C}$.

PF/ HAVE

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\beta_X = \mathrm{id}_X"'] \arrow[dr, phantom, bend left=20, "\circlearrowleft"] & X \sqcup I \arrow[d, dashed, "\gamma"] & I \arrow[l, "\alpha_I"'] \arrow[dl, "\vec{0}_X = \beta_I"] \arrow[dl, phantom, bend right=20, "\circlearrowleft"] \\
& X = C &
\end{tikzcd}$$

$\therefore\ \gamma\, \alpha_X = \mathrm{id}_X$ STS: $\alpha_X\, \gamma = \mathrm{id}_{X \sqcup I}$

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\mathrm{id}_X"'] \arrow[ddr, bend right, "\alpha_X"'] \arrow[dr, phantom, bend left=20, "\circlearrowleft"] & X \sqcup I \arrow[d, dashed, "\gamma"] & I \arrow[l, "\alpha_I = \vec{0}_{X \sqcup I}"'] \arrow[dl, "\vec{0}_X"] \arrow[ddl, bend left, "\alpha_I = \vec{0}_{X \sqcup I}"] \arrow[dl, phantom, bend right=20, "\circlearrowleft"] \\
& X \arrow[d, "\alpha_X"] & \\
& X \sqcup I &
\end{tikzcd}$$

BECAUSE I INITIAL

$\circlearrowleft$ BECAUSE UNIQUENESS OF $\vec{0}_*$

II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

$$\begin{tikzcd} I \arrow[d, dashed, "\exists!\, \vec{0}_X"] \\ X \end{tikzcd}$$

INITIAL OBJECT

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!\, {}_X\vec{0}"] \\ T \end{tikzcd}$$

TERMINAL OBJECT

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists!\, \gamma"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$$

COPRODUCT OF OBJECTS

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists!\, \gamma'"] \arrow[dr, "\forall \beta'_Y"] & \\ X & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] & Y \end{tikzcd}$$

PRODUCT OF OBJECTS

EXAMPLE: $X \sqcup I \cong X$ for any $X \in \mathscr{C}$.

PF/ HAVE

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\beta_X = \mathrm{id}_X"'] & X \sqcup I \arrow[d, dashed, "\gamma"] & I \arrow[l, "\alpha_I"'] \arrow[dl, "\vec{0}_X = \beta_I"] \\ & X \arrow[d, equal] & \\ & C & \end{tikzcd}$$

$\therefore\ \gamma\, \alpha_X = \mathrm{id}_X$

STS: $\alpha_X\, \gamma = \mathrm{id}_{X \sqcup I}$

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[ddr, bend right, "\alpha_X"'] & X \sqcup I \arrow[d, dashed, "\gamma"] & I \arrow[l, "\vec{0}_{X \sqcup I}"'] \arrow[ddl, bend left, "\vec{0}_{X \sqcup I}"] \\ & X \arrow[d, "\alpha_X"] & \\ & X \sqcup I & \end{tikzcd}$$

II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

$$\begin{tikzcd} I \arrow[d, dashed, "\exists!\, \vec{0}_X"] \\ X \end{tikzcd}$$

INITIAL OBJECT

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!\, {}_X\vec{0}"] \\ T \end{tikzcd}$$

TERMINAL OBJECT

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists!\, \gamma"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$$

COPRODUCT OF OBJECTS

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists!\, \gamma'"] \arrow[dr, "\forall \beta'_Y"] & \\ X & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] & Y \end{tikzcd}$$

PRODUCT OF OBJECTS

EXAMPLE: $X \sqcup I \cong X$ for any $X \in \mathscr{C}$.

PF/ HAVE

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\beta_X = \mathrm{id}_X"'] & X \sqcup I \arrow[d, dashed, "\gamma"] & I \arrow[l, "\alpha_I"'] \arrow[dl, "\vec{0}_X = \beta_I"] \\ & X = C & \end{tikzcd}$$

$\therefore\ \gamma\, \alpha_X = \mathrm{id}_X$ STS: $\alpha_X\, \gamma = \mathrm{id}_{X \sqcup I}$

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[ddr, bend right, "\alpha_X"'] & X \sqcup I \arrow[d, dashed, "\gamma"] & I \arrow[l, "\vec{0}_{X \sqcup I}"'] \arrow[ddl, bend left, "\vec{0}_{X \sqcup I}"] \\ & X \arrow[d, "\alpha_X"] & \\ & X \sqcup I & \end{tikzcd}$$

ALSO HAVE

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\alpha_X"'] & X \sqcup I \arrow[d, "\mathrm{id}"] & I \arrow[l, "\alpha_I = \vec{0}_{X \sqcup I}"'] \arrow[dl, "\vec{0}_{X \sqcup I}"] \\ & X \sqcup I & \end{tikzcd}$$

II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

$$
\begin{tikzcd}
I \arrow[d, dashed, "\exists!\, \vec{0}_X"] \\
X
\end{tikzcd}
$$

INITIAL OBJECT

$$
\begin{tikzcd}
X \arrow[d, dashed, "\exists!\, {}_X\vec{0}"] \\
T
\end{tikzcd}
$$

TERMINAL OBJECT

$$
\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists!\, \gamma"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\
& C &
\end{tikzcd}
$$

COPRODUCT OF OBJECTS

$$
\begin{tikzcd}
& P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists!\, \gamma'"] \arrow[dr, "\forall \beta'_Y"] & \\
X & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] & Y
\end{tikzcd}
$$

PRODUCT OF OBJECTS

EXAMPLE: $X \sqcup I \cong X$ FOR ANY $X \in \mathscr{C}$.

PF/ HAVE

$$
\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\beta_X = \mathrm{id}_X"'] & X \sqcup I \arrow[d, dashed, "\gamma"] & I \arrow[l, "\alpha_I"'] \arrow[dl, "\vec{0}_X = \beta_I"] \\
& X = C &
\end{tikzcd}
$$

$\therefore\ \gamma\, \alpha_X = \mathrm{id}_X$

~~STS~~: $\alpha_X\, \gamma = \mathrm{id}_{X \sqcup I}$

$$
\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[ddr, bend right, "\alpha_X"'] & X \sqcup I \arrow[d, dashed, "\gamma"] & I \arrow[l, "\vec{0}_{X \sqcup I}"'] \arrow[ddl, bend left, "\vec{0}_{X \sqcup I}"] \\
& X \arrow[d, "\alpha_X"] & \\
& X \sqcup I &
\end{tikzcd}
$$

UNIQUENESS OF γ IN COPRODUCT CONSTRUCTION

$$
\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\alpha_X"'] & X \sqcup I \arrow[d, "\mathrm{id}"] & I \arrow[l, "\alpha_I = \vec{0}_{X \sqcup I}"'] \arrow[dl, "\vec{0}_{X \sqcup I}"] \\
& X \sqcup I &
\end{tikzcd}
$$

II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

$$\begin{tikzcd} I \arrow[d, dashed, "\exists!\, \vec{0}_X"] \\ X \end{tikzcd}$$

INITIAL OBJECT

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!\, {}_X\vec{0}"] \\ T \end{tikzcd}$$

TERMINAL OBJECT

$$\begin{tikzcd}[column sep=large] X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists!\, \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$$

COPRODUCT OF OBJECTS

$$\begin{tikzcd}[column sep=large] & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists!\, \gamma'"] \arrow[dr, "\forall \beta'_Y"] & \\ X \arrow[ur, phantom, "\circlearrowleft"] & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] & Y \arrow[ul, phantom, "\circlearrowleft"] \end{tikzcd}$$

PRODUCT OF OBJECTS

<u>LIKEWISE</u> $X \sqcup I \cong X \cong I \sqcup X$ AND $X \sqcap T \cong X \cong T \sqcap X$

for any $X \in \mathcal{C}$.

II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

$$\begin{tikzcd} I \arrow[d, dashed, "\exists!\, \vec{o}_X"] \\ X \end{tikzcd}$$

INITIAL OBJECT

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!\, {}_X\vec{o}"] \\ T \end{tikzcd}$$

TERMINAL OBJECT

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists!\, \gamma"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$$

COPRODUCT OF OBJECTS

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists!\, \gamma'"] \arrow[dr, "\forall \beta'_Y"] & \\ X & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] & Y \end{tikzcd}$$

PRODUCT OF OBJECTS

LIKEWISE $X \sqcup I \cong X \cong I \sqcup X$ AND $X \sqcap T \cong X \cong T \sqcap X$ FOR ANY $X \in \mathscr{C}$.

THINK ABOUT THIS IN THE CONTEXT OF:

EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$	$\{e\}$	$\mathbb{Z}$	0_{VS}
T	$\{\bullet\}$	$\{e\}$	0_{Ring}	0_{VS}
0	N/A	$\{e\}$	N/A	0_{VS}

EXAMPLES	Set	Group	Ring	Vec
$\sqcup$	$\biguplus$ DISJOINT UNION	$\circledast$ FREE PRODUCT	$\circledast$ FREE PRODUCT	$\oplus$ DIRECT SUM
$\sqcap$	$\times$ CARTESIAN PRODUCT	$\times$ DIRECT PRODUCT	$\times$ DIRECT PRODUCT	$\times$ DIRECT PRODUCT

MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #6

TOPICS:

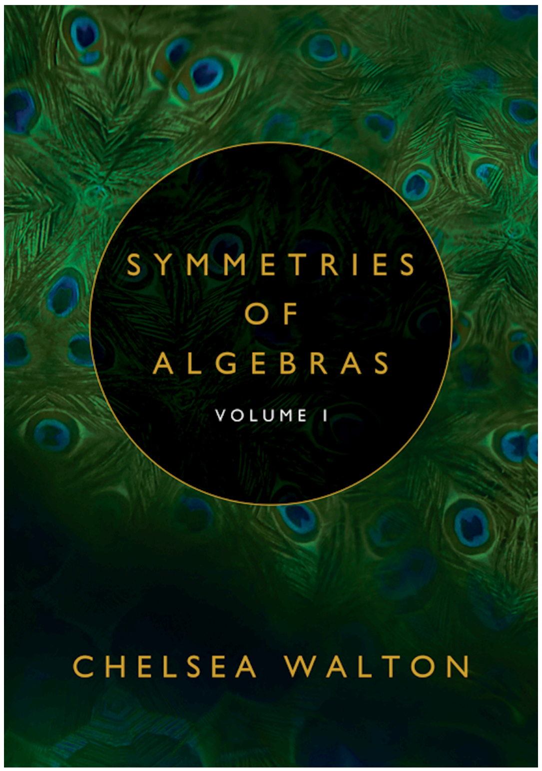
~~I~~. CATEGORIES (§2.1)

II. UNIVERSAL CONSTRUCTIONS (§2.2.1) → I, T, 0 ✓ ; ⊔, ⊓ ✓

NEXT TIME: MORE & ABELIAN CATEGORIES

Enjoy this lecture?
You'll enjoy the textbook!

C. Walton's "Symmetries of Algebras, Volume 1" (2024)



Lecture #6 keywords: category, coproduct of objects, initial object, morphism, product of objects, object, terminal object, universal construction, zero object