MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #7

LAST TIME

- CATEGORIES
- UNIV. CONSTRUCTIONS:
  $I, T, 0, \sqcup, \sqcap$

TOPICS:

I. UNIVERSAL CONSTRUCTIONS (§2.2.1)

II. ABELIAN CATEGORIES (§2.2.2)

I. UNIVERSAL CONSTRUCTIONS : REVIEW

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, "\circlearrowright", shift left=4ex] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowright", shift right=4ex] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

I. UNIVERSAL CONSTRUCTIONS : REVIEW

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
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\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

Univ(X)
DOESN'T HAVE
TO EXIST.

IF EXISTS,
THEN
IT'S UNIQUE
UP TO ISO.

I. UNIVERSAL CONSTRUCTIONS : REVIEW

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, "\circlearrowright", shift left=4] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowright"', shift right=4] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

Univ(X) DOESN'T HAVE TO EXIST.

IF EXISTS, THEN IT'S UNIQUE UP TO ISO.

WE DON'T HAVE "ELEMENTS" TO WORK WITH IN GENERAL

THIS IS THE MAIN WAY WE'LL DO COMPUTATIONS

I. UNIVERSAL CONSTRUCTIONS : REVIEW

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[rd, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[rd, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathscr{C}$:

AN OBJECT $I \in \mathscr{C}$ IS INITIAL IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd}
I \arrow[d, dashed, "\exists! \vec{0}_X"] \\
X
\end{tikzcd}$$

AN OBJECT $T \in \mathscr{C}$ IS TERMINAL IF $\forall X \in \mathscr{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd}
X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\
T
\end{tikzcd}$$

A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$	$\{e\}$	$\mathbb{Z}$	0_{VS}
T	$\{\bullet\}$	$\{e\}$	0_{RING}	0_{VS}
0	N/A	$\{e\}$	N/A	0_{VS}

I. UNIVERSAL CONSTRUCTIONS : REVIEW

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, shift left=4ex, "\circlearrowleft"] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

EXAMPLES	Set	Group	Ring	Vec
$\sqcup$	$\uplus$ DISJOINT UNION	$\circledast$ FREE PRODUCT	$\circledast$ FREE PRODUCT	$\oplus$ DIRECT SUM
$\sqcap$	$\times$ CARTESIAN PRODUCT	$\times$ DIRECT PRODUCT	$\times$ DIRECT PRODUCT	$\times$ DIRECT PRODUCT

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[dr, "\forall \beta'_X"] \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, phantom, shift right=4ex, "\circlearrowleft"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

COPRODUCT OF X AND Y IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, shift left=4ex, "\circlearrowleft"] & X \sqcup Y \arrow[d, dashed, "\exists! \gamma"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \arrow[dl, phantom, shift right=4ex, "\circlearrowleft"] \\
& C &
\end{tikzcd}$$

PRODUCT OF X AND Y IS AN OBJECT $X \sqcap Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha'_X : X \sqcap Y \to X$ AND $\alpha'_Y : X \sqcap Y \to Y$ SATISFYING:

$$\begin{tikzcd}
& P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists! \gamma'"] \arrow[dr, "\forall \beta'_Y"] \arrow[dl, phantom, shift left=4ex, "\circlearrowleft"] \arrow[dr, phantom, shift right=4ex, "\circlearrowleft"] & \\
X & X \sqcap Y \arrow[l, "\alpha'_X"'] \arrow[r, "\alpha'_Y"] & Y
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : REVIEW

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X", "\circlearrowright"'] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"', "\circlearrowright"] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

AN OBJECT $I \in \mathcal{C}$ IS **INITIAL** IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd}
I \arrow[d, dashed, "\exists! \vec{0}_X"] \\
X
\end{tikzcd}$$

AN OBJECT $T \in \mathcal{C}$ IS **TERMINAL** IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd}
X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\
T
\end{tikzcd}$$

A **ZERO** OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

COPRODUCT OF X AND Y IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists! \gamma\ \circlearrowright", "\circlearrowright"'] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\
 & C &
\end{tikzcd}$$

PRODUCT OF X AND Y IS AN OBJECT $X \sqcap Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha'_X : X \sqcap Y \to X$ AND $\alpha'_Y : X \sqcap Y \to Y$ SATISFYING:

$$\begin{tikzcd}
 & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists! \gamma'\ \circlearrowright", "\circlearrowright"'] \arrow[dr, "\forall \beta'_Y"] & \\
X & X \sqcap Y \arrow[l, "\alpha'_X"'] \arrow[r, "\alpha'_Y"] & Y
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : REVIEW

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

“STARTING” & “ENDING” OBJECTS →

AN OBJECT $I \in \mathcal{C}$ IS INITIAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd} I \arrow[d, dashed, "\exists!\, \vec{0}_X"] \\ X \end{tikzcd}$$

AN OBJECT $T \in \mathcal{C}$ IS TERMINAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!\, {}_X\vec{0}"] \\ T \end{tikzcd}$$

A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

WAYS OF COMBINING OBJECTS →

COPRODUCT OF X AND Y IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists!\, \gamma"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C \arrow[ul, phantom, "\circlearrowleft", near end] \arrow[ur, phantom, "\circlearrowleft", near end] & \end{tikzcd}$$

PRODUCT OF X AND Y IS AN OBJECT $X \sqcap Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha'_X : X \sqcap Y \to X$ AND $\alpha'_Y : X \sqcap Y \to Y$ SATISFYING:

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists!\, \gamma'"] \arrow[dr, "\forall \beta'_Y"] & \\ X \arrow[ur, phantom, "\circlearrowleft", near end] & X \sqcap Y \arrow[l, "\alpha'_X"] \arrow[r, "\alpha'_Y"'] & Y \arrow[ul, phantom, "\circlearrowleft", near end] \end{tikzcd}$$

MORE UNIVERSAL CONSTRUCTIONS NEXT—

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & Univ(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & Arb(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
Arb(X) \arrow[dr, "\forall \beta'_X"] \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, phantom, "\circlearrowleft"'] & \\
Univ(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathscr{C}$:

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \\
Y &
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

↑ OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
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\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

IS AN OBJECT

$$X \sqcup_Z Y := X \sqcup_{Z,f,g} Y \in \mathcal{C},$$

EQUIPPED WITH MORPHISMS

$$\alpha_X : X \to X \sqcup_Z Y \ \& \ \alpha_Y : Y \to X \sqcup_Z Y$$

WHERE $\alpha_X f = \alpha_Y g$,

SATISFYING

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\alpha_X"'] \\
Y \arrow[r, "\alpha_Y"] & X \sqcup_Z Y
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowright"'] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

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PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

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$$X \sqcup_Z Y := X \sqcup_{Z,f,g} Y \in \mathcal{C},$$

EQUIPPED WITH MORPHISMS

$\alpha_X : X \to X \sqcup_Z Y$ & $\alpha_Y : Y \to X \sqcup_Z Y$

WHERE $\alpha_X f = \alpha_Y g$,

SATISFYING

$\forall \begin{cases} \beta_X : X \to C \\ \beta_Y : Y \to C \end{cases}$ WHERE $\beta_X f = \beta_Y g$,

WE GET:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[rdd, bend left, "\forall \beta_X"] & \\ Y \arrow[r, "\alpha_Y"] \arrow[rrd, bend right, "\forall \beta_Y"'] & X \sqcup_Z Y \arrow[dr, phantom, "\circlearrowright"] & \\ & & C \end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

↑ OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & Univ(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & Arb(X) \end{tikzcd}$$

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$$\begin{tikzcd} Arb(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\ Univ(X) \arrow[r, "\alpha'_X"] \arrow[ur, phantom, "\circlearrowleft"] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

PUSHOUT OF $f : Z \to X$ & $g : Z \to Y$

IS AN OBJECT

$$X \sqcup_Z Y := X \sqcup_{Z,f,g} Y \in \mathcal{C},$$

EQUIPPED WITH MORPHISMS

$\alpha_X : X \to X \sqcup_Z Y$ & $\alpha_Y : Y \to X \sqcup_Z Y$

WHERE $\alpha_X f = \alpha_Y g$,

SATISFYING

$$\forall \begin{cases} \beta_X : X \to C \\ \beta_Y : Y \to C \end{cases} \text{ WHERE } \beta_X f = \beta_Y g,$$

WE GET:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall \beta_X"] & \\ Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall \beta_Y"'] & X \sqcup_Z Y \arrow[dr, dashed, "\exists ! \gamma"] \arrow[ur, phantom, "\circlearrowleft"] & \\ & \arrow[u, phantom, "\circlearrowleft"] & C \end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

↖ OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \\
\arrow[ur, phantom, "\circlearrowleft"] & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \arrow[ul, phantom, "\circlearrowleft"]
\end{tikzcd}$$

FORM II

"PULLBACKS" IN $\mathscr{C}$

$\rotatebox{90}{$\cong$}$

PUSHOUTS IN $\mathscr{C}^{op}$

GIVEN A CATEGORY $\mathscr{C}$:

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

IS AN OBJECT

$$X \sqcup_Z Y := X \sqcup_{Z,f,g} Y \in \mathscr{C},$$

EQUIPPED WITH MORPHISMS

$\alpha_X : X \to X \sqcup_Z Y$ & $\alpha_Y : Y \to X \sqcup_Z Y$

WHERE $\alpha_X f = \alpha_Y g$,

SATISFYING

$\forall \begin{cases} \beta_X : X \to C \\ \beta_Y : Y \to C \end{cases}$ WHERE $\beta_X f = \beta_Y g$,

WE GET:

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall \beta_X"] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall \beta_Y"'] & X \sqcup_Z Y \arrow[dr, dashed, "\exists! \gamma"'] & \circlearrowleft \\
& \circlearrowleft & C
\end{tikzcd}$$

PULLBACK OF $f: X \to Z$ & $g: Y \to Z$

IS AN OBJECT

$$X \sqcap_Z Y := X \sqcap_{Z,f,g} Y \in \mathscr{C},$$

EQUIPPED WITH MORPHISMS

$\alpha'_X : X \sqcap_Z Y \to X$ & $\alpha'_Y : X \sqcap_Z Y \to Y$

WHERE $f \alpha'_X = g \alpha'_Y$,

SATISFYING

$\forall \begin{cases} \beta'_X : P \to X \\ \beta'_Y : P \to Y \end{cases}$ WHERE $f \beta'_X = g \beta'_Y$,

WE GET:

$$\begin{tikzcd}
P \arrow[dr, dashed, "\exists! \gamma'"] \arrow[drr, bend left, "\forall \beta'_X"] \arrow[ddr, bend right, "\forall \beta'_Y"'] & \circlearrowleft & \\
& X \sqcap_Z Y \arrow[r, "\alpha'_X"'] \arrow[d, "\alpha'_Y"] & X \arrow[d, "f"] \\
\circlearrowleft & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

↖ OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowright"'] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall \beta_X"] \arrow[dr, phantom, "\circlearrowright"] & \\ Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall \beta_Y"'] \arrow[dr, phantom, "\circlearrowright"] & X \sqcup_Z Y \arrow[dr, dashed, "\exists ! \gamma"] & \\ & & C \end{tikzcd}$$

↑

OF "FORM I"

PULLBACK OF $f: X \to Z$ & $g: Y \to Z$

$$\begin{tikzcd} P \arrow[dr, dashed, "\exists ! \gamma'"] \arrow[drr, bend left, "\forall \beta'_X"] \arrow[ddr, bend right, "\forall \beta'_Y"'] & & \\ & X \sqcap_Z Y \arrow[r, "\alpha'_X"'] \arrow[d, "\alpha'_Y"] \arrow[u, phantom, "\circlearrowright"] \arrow[l, phantom, "\circlearrowright"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

↑

OF "FORM II"

"PULLBACKS" IN $\mathcal{C}$

$\cong$

PUSHOUTS IN $\mathcal{C}^{op}$

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

↑ OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall\beta_X"'] \arrow[dr, phantom, shift left=4, "\circlearrowright"] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\gamma_X"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists!\gamma'_X"'] \arrow[dr, "\forall\beta'_X"] \arrow[dr, phantom, shift right=4, "\circlearrowright"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall\beta_X"] \arrow[ddr, phantom, bend left=15, "\circlearrowright"] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall\beta_Y"'] \arrow[drr, phantom, bend right=15, "\circlearrowright"] & X \sqcup_Z Y \arrow[dr, dashed, "\exists!\gamma"'] & \\
 & & C
\end{tikzcd}$$

PULLBACK OF $f: X \to Z$ & $g: Y \to Z$

$$\begin{tikzcd}
P \arrow[dr, dashed, "\exists!\gamma'"] \arrow[drr, bend left, "\forall\beta'_X"] \arrow[drr, phantom, bend left=15, "\circlearrowright"] \arrow[ddr, bend right, "\forall\beta'_Y"'] \arrow[ddr, phantom, bend right=15, "\circlearrowright"] & & \\
 & X \sqcap_Z Y \arrow[r, "\alpha'_X"'] \arrow[d, "\alpha'_Y"] & X \arrow[d, "f"] \\
 & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

EXERCISE 2.9 FOR $X, Y, Z \in \mathsf{Set}$, WE GET

WITH FUNCTIONS $f: Z \to X$, $g: Z \to Y$:

$$X \sqcup_Z Y \cong (X \uplus Y)/\sim$$

QUOTIENT SET OF $X \uplus Y$

HERE, $f(z) \sim g(z)$ IN $X \uplus Y$ $\forall z \in Z$

WITH FUNCTIONS $f: X \to Z$, $g: Y \to Z$:

$$X \sqcap_Z Y = \{(x,y) \in X \times Y \mid f(x) = g(y) \text{ IN } Z\}$$

SUBSET OF $X \times Y$

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

↖ OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
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FORM I

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\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall \beta_X"] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall \beta_Y"'] & X \sqcup_Z Y \arrow[dr, dashed, "\exists ! \gamma"] & \\
 & & C
\end{tikzcd}$$

PULLBACK OF $f: X \to Z$ & $g: Y \to Z$

$$\begin{tikzcd}
P \arrow[dr, dashed, "\exists ! \gamma'"] \arrow[drr, bend left, "\forall \beta'_X"] \arrow[ddr, bend right, "\forall \beta'_Y"'] & & \\
 & X \sqcap_Z Y \arrow[r, "\alpha'_X"'] \arrow[d, "\alpha'_Y"] & X \arrow[d, "f"] \\
 & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

IN Set

I	$\emptyset$
T	$\{\bullet\}$
$\sqcup$	$\uplus$ DISJOINT UNION
$\sqcap$	$\times$ CARTESIAN PRODUCT

EXERCISE 2.9

FOR $X, Y, Z \in \mathrm{Set}$, WE GET

WITH FUNCTIONS $f: Z \to X$, $g: Z \to Y$:

$$X \sqcup_Z Y \cong (X \uplus Y) / \sim$$

QUOTIENT SET OF $X \uplus Y$

HERE, $f(z) \sim g(z)$ IN $X \uplus Y$ $\forall z \in Z$

WITH FUNCTIONS $f: X \to Z$, $g: Y \to Z$:

$$X \sqcap_Z Y = \{ (x,y) \in X \times Y \mid f(x) = g(y) \text{ IN } Z \}$$

SUBSET OF $X \times Y$

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall\beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[dr, "\forall\beta'_X"] \arrow[d, dashed, "\exists!\gamma'_X"'] \arrow[dr, phantom, "\circlearrowleft"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall\beta_X"] \arrow[dr, phantom, "\circlearrowleft"] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall\beta_Y"'] \arrow[dr, phantom, "\circlearrowleft"] & X \sqcup_Z Y \arrow[dr, dashed, "\exists!\gamma"'] & \\
& & C
\end{tikzcd}$$

PULLBACK OF $f: X \to Z$ & $g: Y \to Z$

$$\begin{tikzcd}
P \arrow[drr, bend left, "\forall\beta'_X"] \arrow[ddr, bend right, "\forall\beta'_Y"'] \arrow[dr, dashed, "\exists!\gamma'"] & & \\
& X \sqcap_Z Y \arrow[r, "\alpha'_X"'] \arrow[d, "\alpha'_Y"] \arrow[ur, phantom, "\circlearrowleft"] \arrow[dl, phantom, "\circlearrowleft"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

IN Set

I	$\emptyset$
T	$\{\bullet\}$
$\sqcup$	$\uplus$ DISJOINT UNION
$\sqcap$	$\times$ CARTESIAN PRODUCT

EXERCISE 2.9 FOR $X, Y, Z \in$ Set, WE GET

WITH FUNCTIONS

$f: \emptyset \to X$, $g: \emptyset \to Y$:

$$X \sqcup_Z Y \cong (X \uplus Y)/\sim$$

QUOTIENT SET OF $X \uplus Y$

HERE, $f(z) \sim g(z)$ IN $X \uplus Y$ $\forall z \in Z$

WITH FUNCTIONS

$f: X \to Z$, $g: Y \to Z$:

$$X \sqcap_Z Y = \{(x,y) \in X \times Y \mid f(x) = g(y) \text{ IN } Z\}$$

SUBSET OF $X \times Y$

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

↖ OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft", near start] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall \beta_X"] \arrow[ddr, phantom, "\circlearrowleft", near start] & \\ Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall \beta_Y"'] \arrow[drr, phantom, "\circlearrowleft", near start] & X \sqcup_Z Y \arrow[dr, dashed, "\exists ! \gamma"'] & \\ & & C \end{tikzcd}$$

PULLBACK OF $f: X \to Z$ & $g: Y \to Z$

$$\begin{tikzcd} P \arrow[dr, dashed, "\exists ! \gamma'"] \arrow[drr, bend left, "\forall \beta'_X"] \arrow[ddr, bend right, "\forall \beta'_Y"'] \arrow[drr, phantom, "\circlearrowleft"] \arrow[ddr, phantom, "\circlearrowleft"] & & \\ & X \sqcap_Z Y \arrow[r, "\alpha'_X"'] \arrow[d, "\alpha'_Y"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

IN Set

I	$\emptyset$
T	$\{\bullet\}$
$\sqcup$	$\uplus$ DISJOINT UNION
$\sqcap$	$\times$ CARTESIAN PRODUCT

EXERCISE 2.9 FOR $X, Y, Z \in \mathsf{Set}$, WE GET

WITH FUNCTIONS $f: \emptyset \to X$, $g: \emptyset \to Y$:

$$X \sqcup_{\emptyset} Y \cong (X \uplus Y) / \sim$$

$\parallel$

~~QUOTIENT SET OF~~ $X \uplus Y$

HERE, $f(z) \sim g(z)$ ← VACUOUS IN $X \uplus Y$ $\forall z \in Z$

WITH FUNCTIONS $f: X \to Z$, $g: Y \to Z$:

$$X \sqcap_Z Y = \{ (x,y) \in X \times Y \mid f(x) = g(y) \text{ IN } Z \}$$

SUBSET OF $X \times Y$

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

↖ OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowright"', near start] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

PUSHOUT OF $f : Z \to X$ & $g : Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall \beta_X"] \arrow[dr, phantom, "\circlearrowright"] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall \beta_Y"'] \arrow[dr, phantom, "\circlearrowright"] & X \sqcup_Z Y \arrow[dr, dashed, "\exists ! \gamma"'] & \\
 & & C
\end{tikzcd}$$

PULLBACK OF $f : X \to Z$ & $g : Y \to Z$

$$\begin{tikzcd}
P \arrow[dr, dashed, "\exists ! \gamma'"] \arrow[drr, bend left, "\forall \beta'_X"] \arrow[ddr, bend right, "\forall \beta'_Y"'] & & \\
 & X \sqcap_Z Y \arrow[r, "\alpha'_X"'] \arrow[d, "\alpha'_Y"] \arrow[u, phantom, "\circlearrowleft"] \arrow[l, phantom, "\circlearrowleft"] & X \arrow[d, "f"] \\
 & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

IN Set

I	$\emptyset$
T	$\{\bullet\}$
$\sqcup$	$\uplus$ DISJOINT UNION
$\sqcap$	$\times$ CARTESIAN PRODUCT

EXERCISE 2.9

FOR $X, Y, Z \in \mathsf{Set}$, WE GET

WITH FUNCTIONS

$f : \emptyset \to X$, $g : \emptyset \to Y$:

$$X \sqcup_{\emptyset} Y \cong (X \uplus Y) / \sim$$

$\|$

~~QUOTIENT SET OF~~ $X \uplus Y$

VACUOUS

HERE, $f(z) \sim g(z)$ IN $X \uplus Y$ $\forall z \in Z$

WITH FUNCTIONS

$f : X \to \{\bullet\}$, $g : Y \to \{\bullet\}$:

$$X \sqcap_{\{\bullet\}} Y = \{ (x,y) \in X \times Y \mid \underbrace{f(x) = g(y) \text{ IN } Z}_{} \}$$

VACUOUS →

$\|$

~~SUBSET OF~~ $X \times Y$

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

↑ OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, shift left=4, "\circlearrowleft"] & Univ(X) \arrow[d, dashed, "\exists! \gamma_X"] \\
 & Arb(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
Arb(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, shift right=4, "\circlearrowleft"] & \\
Univ(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall \beta_X"] \arrow[ddr, phantom, shift left=6, "\circlearrowleft"] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall \beta_Y"'] \arrow[drr, phantom, shift right=4, "\circlearrowleft"] & X \sqcup_Z Y \arrow[dr, dashed, "\exists! \gamma"'] & \\
 & & C
\end{tikzcd}$$

PULLBACK OF $f: X \to Z$ & $g: Y \to Z$

$$\begin{tikzcd}
P \arrow[dr, dashed, "\exists! \gamma'"] \arrow[drr, bend left, "\forall \beta'_X"] \arrow[drr, phantom, shift left=4, "\circlearrowleft"] \arrow[ddr, bend right, "\forall \beta'_Y"'] \arrow[ddr, phantom, shift right=4, "\circlearrowleft"] & & \\
 & X \sqcap_Z Y \arrow[r, "\alpha'_X"'] \arrow[d, "\alpha'_Y"] & X \arrow[d, "f"] \\
 & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

EXER 2.10

$$X \sqcup_{I, \vec{0}_X, \vec{0}_Y} Y$$
$$\cong$$
$$X \sqcup Y$$

$$X \sqcap_{T, {}_X\vec{0}, {}_Y\vec{0}} Y$$
$$\cong$$
$$X \sqcap Y$$

EXERCISE 2.9 FOR $X, Y, Z \in Set$, WE GET

WITH FUNCTIONS

$f: \emptyset \to X$, $g: \emptyset \to Y$:

$$X \sqcup_\emptyset Y \cong (X \uplus Y)/\sim$$

$\|$

~~QUOTIENT SET OF~~ $X \uplus Y$

← VACUOUS

HERE, $f(z) \sim g(z)$ IN $X \uplus Y$ $\forall z \in Z$

WITH FUNCTIONS

$f: X \to \{\bullet\}$, $g: Y \to \{\bullet\}$:

$$X \sqcap_{\{\bullet\}} Y = \{ (x,y) \in X \times Y \mid \underbrace{f(x) = g(y) \text{ IN } Z}_{\text{VACUOUS}} \}$$

$\|$

~~SUBSET OF~~ $X \times Y$

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

↑ OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"', shift right=3] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

PUSHOUT OF $f : Z \to X$ & $g : Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall \beta_X"] \arrow[ddr, phantom, "\circlearrowleft", shift left=2] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall \beta_Y"'] \arrow[drr, phantom, "\circlearrowleft"', shift right=2] & X \sqcup_Z Y \arrow[dr, dashed, "\exists ! \gamma"'] & \\
& & C
\end{tikzcd}$$

PULLBACK OF $f : X \to Z$ & $g : Y \to Z$

$$\begin{tikzcd}
P \arrow[dr, dashed, "\exists ! \gamma'"] \arrow[drr, bend left, "\forall \beta'_X"] \arrow[ddr, bend right, "\forall \beta'_Y"'] \arrow[drr, phantom, "\circlearrowleft", shift left=2] \arrow[ddr, phantom, "\circlearrowleft"', shift right=2] & & \\
& X \sqcap_Z Y \arrow[r, "\alpha'_X"'] \arrow[d, "\alpha'_Y"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

HINT FOR $X \sqcup_I Y \cong X \sqcup Y$

EXER 2.10

$$\begin{gathered} X \sqcup_{I, \vec{0}_X, \vec{0}_Y} Y \\ \cong \\ X \sqcup Y \end{gathered}$$

$$\begin{gathered} X \sqcap_{T, {}_X\vec{0}, {}_Y\vec{0}} Y \\ \cong \\ X \sqcap Y \end{gathered}$$

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall\beta_X"'] \arrow[dr, phantom, shift left=4, "\circlearrowright"] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\gamma_X"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists!\gamma'_X"'] \arrow[dr, "\forall\beta'_X"] \arrow[dr, phantom, shift right=4, "\circlearrowright"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall\beta_X"] \arrow[ddr, phantom, "\circlearrowright"] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall\beta_Y"'] \arrow[drr, phantom, "\circlearrowright"] & X \sqcup_Z Y \arrow[dr, dashed, "\exists!\gamma"] & \\
 & & C
\end{tikzcd}$$

PULLBACK OF $f: X \to Z$ & $g: Y \to Z$

$$\begin{tikzcd}
P \arrow[dr, dashed, "\exists!\gamma'"] \arrow[drr, bend left, "\forall\beta'_X"] \arrow[ddr, bend right, "\forall\beta'_Y"'] \arrow[drr, phantom, "\circlearrowright"] \arrow[ddr, phantom, "\circlearrowright"] & & \\
 & X \sqcap_Z Y \arrow[r, "\alpha'_X"'] \arrow[d, "\alpha'_Y"] & X \arrow[d, "f"] \\
 & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

EXER 2.10

$$X \sqcup_{I, \vec{0}_X, \vec{0}_Y} Y$$
$$\cong$$
$$X \sqcup Y$$

$$X \sqcap_{T, {}_X\vec{0}, {}_Y\vec{0}} Y$$
$$\cong$$
$$X \sqcap Y$$

HINT FOR $X \sqcup_I Y \cong X \sqcup Y$

UNIV PROP OF $X \sqcup Y$ YIELDS

A MORPHISM $\bar{\gamma} : X \sqcup Y \to X \sqcup_I Y$

VIA:

$$\begin{tikzcd}
X \arrow[r, "\bar\alpha_X"] \arrow[dr, "\alpha_X"'] \arrow[dr, phantom, shift left=4, "\circlearrowright"] & X \sqcup Y \arrow[d, dashed, "\exists!\bar\gamma"] & Y \arrow[l, "\bar\alpha_Y"'] \arrow[dl, "\alpha_Y"] \arrow[dl, phantom, shift right=4, "\circlearrowright"] \\
 & X \sqcup_I Y &
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "\bar\alpha_X"] \arrow[dr, "\forall\bar\beta_X"'] \arrow[dr, phantom, shift left=4, "\circlearrowright"] & X \sqcup Y \arrow[d, dashed, "\exists!\bar\gamma"] & Y \arrow[l, "\bar\alpha_Y"'] \arrow[dl, "\forall\bar\beta_Y"] \arrow[dl, phantom, shift right=4, "\circlearrowright"] \\
 & C &
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, shift left=4ex, "\circlearrowright"] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, shift right=4ex, "\circlearrowright"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

PUSHOUT OF $f : Z \to X$ & $g : Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall \beta_X"] \arrow[ddr, phantom, "\circlearrowright"] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall \beta_Y"'] \arrow[drr, phantom, "\circlearrowright"] & X \sqcup_Z Y \arrow[dr, dashed, "\exists! \gamma"'] & \\
 & & C
\end{tikzcd}$$

PULLBACK OF $f : X \to Z$ & $g : Y \to Z$

$$\begin{tikzcd}
P \arrow[dr, dashed, "\exists! \gamma'"] \arrow[drr, bend left, "\forall \beta'_X"] \arrow[drr, phantom, "\circlearrowright"] \arrow[ddr, bend right, "\forall \beta'_Y"'] \arrow[ddr, phantom, "\circlearrowright"] & & \\
 & X \sqcap_Z Y \arrow[r, "\alpha'_X"'] \arrow[d, "\alpha'_Y"] & X \arrow[d, "f"] \\
 & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

EXER 2.10

$X \sqcup_{I, \vec{0}_X, \vec{0}_Y} Y$

$\cong$

$X \sqcup Y$

$X \sqcap_{T, {}_X\vec{0}, {}_Y\vec{0}} Y$

$\cong$

$X \sqcap Y$

HINT FOR $X \sqcup_I Y \cong X \sqcup Y$

UNIV PROP OF $X \sqcup Y$ YIELDS

A MORPHISM $\overline{\gamma} : X \sqcup Y \to X \sqcup_I Y$

VIA:

$$\begin{tikzcd}
X \arrow[r, "\overline{\alpha}_X"] \arrow[dr, "\alpha_X"'] \arrow[dr, phantom, shift left=4ex, "\circlearrowright"] & X \sqcup Y \arrow[d, dashed, "\exists! \overline{\gamma}"'] & Y \arrow[l, "\overline{\alpha}_Y"'] \arrow[dl, "\alpha_Y"] \arrow[dl, phantom, shift right=4ex, "\circlearrowright"] \\
 & X \sqcup_I Y &
\end{tikzcd}$$

UNIV PROP OF $X \sqcup_I Y$ YIELDS

A MORPHISM $\gamma : X \sqcup_I Y \to X \sqcup Y$

VIA:

$$\begin{tikzcd}
I \arrow[r, "\vec{0}_X"] \arrow[d, "\vec{0}_Y"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\overline{\alpha}_X"] \arrow[ddr, phantom, "\circlearrowright"] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\overline{\alpha}_Y"'] \arrow[drr, phantom, "\circlearrowright"] & X \sqcup_I Y \arrow[dr, dashed, "\exists! \gamma"] & \\
 & & X \sqcup Y
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

↖ OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowright"', near start] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall \beta_X"] \arrow[ddr, phantom, "\circlearrowright"] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall \beta_Y"'] \arrow[drr, phantom, "\circlearrowright"'] & X \sqcup_Z Y \arrow[dr, dashed, "\exists ! \gamma"'] & \\
 & & C
\end{tikzcd}$$

PULLBACK OF $f: X \to Z$ & $g: Y \to Z$

$$\begin{tikzcd}
P \arrow[dr, dashed, "\exists ! \gamma'"'] \arrow[drr, bend left, "\forall \beta'_X"] \arrow[ddr, bend right, "\forall \beta'_Y"'] & & \\
 & X \sqcap_Z Y \arrow[r, "\alpha'_X"'] \arrow[d, "\alpha'_Y"] \arrow[ur, phantom, "\circlearrowright"] \arrow[dl, phantom, "\circlearrowright"] & X \arrow[d, "f"] \\
 & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

HINT FOR $X \sqcup_I Y \cong X \sqcup Y$

EXER 2.10

$$X \sqcup_{I, \vec{0}_X, \vec{0}_Y} Y$$
$$\cong$$
$$X \sqcup Y$$

$$X \sqcap_{T, {}_X\vec{0}, {}_Y\vec{0}} Y$$
$$\cong$$
$$X \sqcap Y$$

UNIV PROP OF $X \sqcup Y$ YIELDS

A MORPHISM $\bar{\gamma}: X \sqcup Y \to X \sqcup_I Y$

VIA:

$$\begin{tikzcd}
X \arrow[r, "\bar{\alpha}_X"] \arrow[dr, "\alpha_X"'] & X \sqcup Y \arrow[d, dashed, "\exists ! \bar{\gamma}"'] \arrow[dl, phantom, "\circlearrowright"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[l, "\bar{\alpha}_Y"'] \arrow[dl, "\alpha_Y"] \\
 & X \sqcup_I Y &
\end{tikzcd}$$

UNIV PROP OF $X \sqcup_I Y$ YIELDS

A MORPHISM $\gamma: X \sqcup_I Y \to X \sqcup Y$

VIA:

$$\begin{tikzcd}
I \arrow[r, "\vec{0}_X"] \arrow[d, "\vec{0}_Y"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\bar{\alpha}_X"] \arrow[ddr, phantom, "\circlearrowright"] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\bar{\alpha}_Y"'] \arrow[drr, phantom, "\circlearrowright"'] \arrow[ur, phantom, "\circlearrowright"] & X \sqcup_I Y \arrow[dr, dashed, "\exists ! \gamma"] & \\
 & & X \sqcup Y
\end{tikzcd}$$

INDEED, $\bar{\alpha}_X \vec{0}_X = \bar{\alpha}_Y \vec{0}_Y = \vec{0}_{X \sqcup Y}$

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"', "\circlearrowright"] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X", "\circlearrowright"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall \beta_X", "\circlearrowright"'] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall \beta_Y"', "\circlearrowright"] & X \sqcup_Z Y \arrow[dr, dashed, "\exists! \gamma"'] & \\
& & C
\end{tikzcd}$$

PULLBACK OF $f: X \to Z$ & $g: Y \to Z$

$$\begin{tikzcd}
P \arrow[dr, dashed, "\exists! \gamma'"] \arrow[drr, bend left, "\forall \beta'_X", "\circlearrowright"'] \arrow[ddr, bend right, "\forall \beta'_Y"', "\circlearrowright"] & & \\
& X \sqcap_Z Y \arrow[r, "\alpha'_X"'] \arrow[d, "\alpha'_Y"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

EXER 2.10

$$X \sqcup_{I, \vec{0}_X, \vec{0}_Y} Y \cong X \sqcup Y$$

$$X \sqcap_{T, {}_X\vec{0}, {}_Y\vec{0}} Y \cong X \sqcap Y$$

HINT FOR $X \sqcup_I Y \cong X \sqcup Y$

UNIV PROP OF $X \sqcup Y$ YIELDS A MORPHISM $\bar{\gamma}: X \sqcup Y \to X \sqcup_I Y$ VIA:

$$\begin{tikzcd}
X \arrow[r, "\bar{\alpha}_X"] \arrow[dr, "\alpha_X"', "\circlearrowright"] & X \sqcup Y \arrow[d, dashed, "\exists! \bar{\gamma}"] & Y \arrow[l, "\bar{\alpha}_Y"'] \arrow[dl, "\alpha_Y", "\circlearrowright"'] \\
& X \sqcup_I Y &
\end{tikzcd}$$

UNIV PROP OF $X \sqcup_I Y$ YIELDS A MORPHISM $\gamma: X \sqcup_I Y \to X \sqcup Y$ VIA:

$$\begin{tikzcd}
I \arrow[r, "\vec{0}_X"] \arrow[d, "\vec{0}_Y"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\bar{\alpha}_X", "\circlearrowright"'] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\bar{\alpha}_Y"', "\circlearrowright"] & X \sqcup_I Y \arrow[dr, dashed, "\exists! \gamma"] & \\
& & X \sqcup Y
\end{tikzcd}$$

SHOW γ & $\bar{\gamma}$ ARE MUTUALLY INV.

I. UNIVERSAL CONSTRUCTIONS : PUSHOUTS AND PULLBACKS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_x"] \arrow[dr, "\forall \beta_x"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_x"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_x"'] \arrow[dr, "\forall \beta'_x"] \arrow[dr, phantom, shift right=3, "\circlearrowleft"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_x"] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

PUSHOUT OF $f : Z \to X$ & $g : Z \to Y$

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_x"'] \arrow[ddr, bend left, "\forall \beta_X"] \arrow[ddr, phantom, bend left=15, "\circlearrowleft"] & \\ Y \arrow[r, "\alpha_y"] \arrow[drr, bend right, "\forall \beta_Y"'] \arrow[drr, phantom, bend right=15, "\circlearrowleft"] & X \sqcup_Z Y \arrow[dr, dashed, "\exists ! \gamma"] & \\ & & C \end{tikzcd}$$

PULLBACK OF $f : X \to Z$ & $g : Y \to Z$

$$\begin{tikzcd} P \arrow[dr, dashed, "\exists ! \gamma'"] \arrow[drr, bend left, "\forall \beta'_X"] \arrow[ddr, bend right, "\forall \beta'_Y"'] \arrow[drr, phantom, bend left=15, "\circlearrowleft"] \arrow[ddr, phantom, bend right=15, "\circlearrowleft"] & & \\ & X \sqcap_Z Y \arrow[r, "\alpha'_x"'] \arrow[d, "\alpha'_y"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

EXER 2.10

$$X \sqcup_I Y \cong X \sqcup Y$$

$$X \sqcap_T Y \cong X \sqcap Y$$

IF $\left\{ \begin{array}{l} \text{PUSHOUTS} \\ \text{PULLBACKS} \end{array} \right.$ & $\left\{ \begin{array}{l} I \\ T \end{array} \right.$ EXIST IN $\mathcal{C}$,

SO DO $\left\{ \begin{array}{l} \text{COPRODUCTS} \\ \text{PRODUCTS.} \end{array} \right.$

I. UNIVERSAL CONSTRUCTIONS : COEQUALIZERS & EQUALIZERS

OPERATION ON PARALLEL MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"', pos=0.3] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

I. UNIVERSAL CONSTRUCTIONS : COEQUALIZERS & EQUALIZERS

OPERATION ON PARALLEL MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathscr{C}$:

COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

I. UNIVERSAL CONSTRUCTIONS : COEQUALIZERS & EQUALIZERS

OPERATION ON PARALLEL MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, shift left=4, "\circlearrowright"] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, shift right=4, "\circlearrowright"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

IS AN OBJECT $\mathrm{coeq}(f,g) \in \mathcal{C}$

EQUIPPED WITH A MORPHISM

$\alpha : Y \longrightarrow \mathrm{coeq}(f,g)$

$\ni.$ $\alpha f = \alpha g$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] & \mathrm{coeq}(f,g)
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : COEQUALIZERS & EQUALIZERS

OPERATION ON PARALLEL MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[r, phantom, shift right=6, "\circlearrowleft"] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[d, phantom, shift left=6, "\circlearrowleft"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

IS AN OBJECT $\mathrm{coeq}(f,g) \in \mathcal{C}$

EQUIPPED WITH A MORPHISM

$\alpha : Y \longrightarrow \mathrm{coeq}(f,g)$

$\ni.\ \alpha f = \alpha g$

WHERE $\forall \beta : Y \longrightarrow C \ \ni.\ \beta f = \beta g$

WE GET:

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] \arrow[r, phantom, shift right=6, "\circlearrowleft"] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists! \gamma"] \\
 & & C
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : COEQUALIZERS & EQUALIZERS

OPERATION ON PARALLEL MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright" description, shift left=2ex] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowright" description, shift left=2ex] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

COEQUALIZER OF $X \underset{g}{\overset{f}{\rightrightarrows}} Y$

IS AN OBJECT $\mathrm{coeq}(f,g) \in \mathcal{C}$

EQUIPPED WITH A MORPHISM

$\alpha : Y \longrightarrow \mathrm{coeq}(f,g)$

$\ni.\ \alpha f = \alpha g$

WHERE $\forall \beta : Y \longrightarrow C \ni.\ \beta f = \beta g$

WE GET:

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowright" description, shift left=2ex] \\
 & & C
\end{tikzcd}$$

EQUALIZER OF $X \underset{g}{\overset{f}{\rightrightarrows}} Y$

IS AN OBJECT $\mathrm{eq}(f,g) \in \mathcal{C}$

EQUIPPED WITH A MORPHISM

$\alpha' : \mathrm{eq}(f,g) \longrightarrow X$

$\ni.\ f\alpha' = g\alpha'$

WHERE $\forall \beta' : K \longrightarrow X \ni.\ f\beta' = g\beta'$

WE GET:

$$\begin{tikzcd}
K \arrow[d, dashed, "\exists! \gamma'"'] \arrow[dr, "\forall \beta'"] & & \\
\mathrm{eq}(f,g) \arrow[r, "\alpha'"'] \arrow[ur, phantom, "\circlearrowright" description, shift left=2ex] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : COEQUALIZERS & EQUALIZERS

OPERATION ON PARALLEL MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, shift left=4, "\circlearrowright"] & Univ(X) \arrow[d, dashed, "\exists ! \gamma_X"] \\
& Arb(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
Arb(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, shift right=4, "\circlearrowright"] & \\
Univ(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] \arrow[dr, phantom, shift left=4, "\circlearrowright"] & coeq(f,g) \arrow[d, dashed, "\exists ! \gamma"] \\
& & C
\end{tikzcd}$$

EQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
K \arrow[d, dashed, "\exists ! \gamma'"'] \arrow[dr, "\forall \beta'"] \arrow[dr, phantom, shift right=4, "\circlearrowright"] & & \\
eq(f,g) \arrow[r, "\alpha'"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

SOME

SPECIAL

CASES

I. UNIVERSAL CONSTRUCTIONS : COEQUALIZERS & EQUALIZERS

OPERATION ON PARALLEL MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & Univ(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\
 & Arb(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
Arb(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
Univ(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowright"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & coeq(f,g) \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowright"] \\
 & & C
\end{tikzcd}$$

EQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
K \arrow[d, dashed, "\exists! \gamma'"'] \arrow[dr, "\forall \beta'"] & & \\
eq(f,g) \arrow[r, "\alpha'"'] \arrow[ur, phantom, "\circlearrowright"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

SOME
SPECIAL
CASES

EXERCISE 2.11

LET $f, g : X \to Y$
BE FUNCTIONS IN **Set**.

THEN:

$eq(f,g) = \{x \in X \mid f(x) = g(x)\}$
SUBSET OF X

I. UNIVERSAL CONSTRUCTIONS : COEQUALIZERS & EQUALIZERS

OPERATION ON PARALLEL MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"', near start] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists ! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & & C
\end{tikzcd}$$

EQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
K \arrow[d, dashed, "\exists ! \gamma'"'] \arrow[dr, "\forall \beta'"] \arrow[dr, phantom, "\circlearrowleft"', near start] & & \\
\mathrm{eq}(f,g) \arrow[r, "\alpha'"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

RECALL FOR $\mathbb{k}$-ALG. A
W/ MODULES V_A & ${}_AW$
GET:

$$V \otimes_A W \cong (V \otimes_{\mathbb{k}} W) / R$$

$$\mathrm{SPAN}_{\mathbb{k}} \langle (v \triangleleft a) \otimes_{\mathbb{k}} w - v \otimes_{\mathbb{k}} (a \triangleright w) \rangle$$
$v \in V,\ w \in W,\ a \in A$

EXERCISE 2.11

LET $f, g : X \to Y$
BE FUNCTIONS IN **Set**.

THEN:

$$\mathrm{eq}(f,g) = \{ x \in X \mid f(x) = g(x) \}$$
SUBSET OF X

I. UNIVERSAL CONSTRUCTIONS : COEQUALIZERS & EQUALIZERS

OPERATION ON PARALLEL MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists ! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & & C
\end{tikzcd}$$

EQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
K \arrow[d, dashed, "\exists ! \gamma'"'] \arrow[dr, "\forall \beta'"] \arrow[dr, phantom, "\circlearrowleft"'] & & \\
\mathrm{eq}(f,g) \arrow[r, "\alpha'"'] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

RECALL FOR $\Bbbk$-ALG. A W/ MODULES V_A & ${}_AW$ GET:

$$V \otimes_A W \cong (V \otimes_{\Bbbk} W) / R$$

$$\mathrm{SPAN}_{\Bbbk} \langle (v \triangleleft a) \otimes_{\Bbbk} w - v \otimes_{\Bbbk} (a \triangleright w) \rangle$$

$v \in V, w \in W, a \in A$

EXERCISE 2.11 WE GET

$$V \otimes_A W \cong \mathrm{Coeq}(f,g)$$

FOR

$$f : V \otimes_{\Bbbk} A \otimes_{\Bbbk} W \xrightarrow{\triangleleft \otimes_{\Bbbk} \mathrm{id}_W} V \otimes_{\Bbbk} W$$

$$g : V \otimes_{\Bbbk} A \otimes_{\Bbbk} W \xrightarrow{\mathrm{id}_V \otimes_{\Bbbk} \triangleright} V \otimes_{\Bbbk} W$$

EXERCISE 2.11

LET $f, g : X \to Y$ BE FUNCTIONS IN Set.

THEN:

$$\mathrm{eq}(f,g) = \{x \in X \mid f(x) = g(x)\}$$

SUBSET OF X

I. UNIVERSAL CONSTRUCTIONS : COEQUALIZERS & EQUALIZERS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, shift right=3, "\circlearrowleft"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

COEQUALIZER OF $X \underset{g}{\overset{f}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists ! \gamma"] \\
 & & C
\end{tikzcd}$$

ARISES FROM OTHER UNIV. CONSTRUCTIONS

I. UNIVERSAL CONSTRUCTIONS : COEQUALIZERS & EQUALIZERS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, shift left=4ex, "\circlearrowleft"] & Univ(X) \arrow[d, dashed, "\exists ! \gamma_X"] \\
& Arb(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
Arb(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, shift right=4ex, "\circlearrowleft"] & \\
Univ(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] \arrow[dr, phantom, shift left=4ex, "\circlearrowleft"] & coeq(f,g) \arrow[d, dashed, "\exists ! \gamma"] \\
& & C
\end{tikzcd}$$

ARISES FROM OTHER UNIV. CONSTRUCTIONS

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\alpha_X"'] \arrow[ddr, bend left, "\forall \beta_X"] \arrow[ddr, phantom, shift left=3ex, "\circlearrowleft"] & \\
Y \arrow[r, "\alpha_Y"] \arrow[drr, bend right, "\forall \beta_Y"'] \arrow[drr, phantom, shift right=2ex, "\circlearrowleft"] & X \sqcup_Z Y \arrow[dr, dashed, "\exists ! \gamma"'] & \\
& & C
\end{tikzcd}$$

COPRODUCT OF X AND Y

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, shift left=4ex, "\circlearrowleft"] & X \sqcup Y \arrow[d, dashed, "\exists ! \gamma"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \arrow[dl, phantom, shift right=4ex, "\circlearrowleft"] \\
& C &
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : COEQUALIZERS & EQUALIZERS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowright"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists ! \gamma"] \arrow[dl, phantom, "\circlearrowright"] \\
& & C
\end{tikzcd}$$

ARISES FROM OTHER
UNIV. CONSTRUCTIONS

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\bar{\alpha}_X"'] \arrow[ddr, bend left, "\forall \bar{\beta}_X"] \arrow[ddr, phantom, "\circlearrowright"] & \\
Y \arrow[r, "\bar{\alpha}_Y"] \arrow[drr, bend right, "\forall \bar{\beta}_Y"'] \arrow[drr, phantom, "\circlearrowright"] & X \sqcup_Z Y \arrow[dr, dashed, "\exists ! \bar{\gamma}"] & \\
& & C
\end{tikzcd}$$

COPRODUCT OF X AND Y

$$\begin{tikzcd}
X \arrow[r, "\bar{\bar{\alpha}}_X"] \arrow[dr, "\forall \bar{\bar{\beta}}_X"'] & X \sqcup Y \arrow[d, dashed, "\exists ! \bar{\bar{\gamma}}"] \arrow[dl, phantom, "\circlearrowright"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[l, "\bar{\bar{\alpha}}_Y"'] \arrow[dl, "\forall \bar{\bar{\beta}}_Y"] \\
& C &
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : COEQUALIZERS & EQUALIZERS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists ! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
& & C
\end{tikzcd}$$

ARISES FROM OTHER UNIV. CONSTRUCTIONS

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\bar{\alpha}_X"'] \arrow[ddr, bend left, "\forall \bar{\beta}_X"] \arrow[dr, phantom, "\circlearrowleft"] & \\
Y \arrow[r, "\bar{\alpha}_Y"] \arrow[drr, bend right, "\forall \bar{\beta}_Y"'] \arrow[dr, phantom, "\circlearrowleft"] & X \sqcup_Z Y \arrow[dr, dashed, "\exists ! \bar{\gamma}"'] & \\
& & C
\end{tikzcd}$$

EXERCISE 2.12 FOR $f, g: X \to Y \in \mathcal{C}$

HAVE:

$$\begin{tikzcd}
X \arrow[r, "\bar{\bar{\alpha}}_X"] \arrow[dr, "f"'] & X \sqcup Y \arrow[d, "\bar{\bar{\gamma}}_f"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[l, "\bar{\bar{\alpha}}_Y"'] \arrow[dl, "\mathrm{id}_Y"] \\
& Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\bar{\bar{\alpha}}_X"] \arrow[dr, "g"'] & X \sqcup Y \arrow[d, "\bar{\bar{\gamma}}_g"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[l, "\bar{\bar{\alpha}}_Y"'] \arrow[dl, "\mathrm{id}_Y"] \\
& Y &
\end{tikzcd}$$

COPRODUCT OF X AND Y

$$\begin{tikzcd}
X \arrow[r, "\bar{\bar{\alpha}}_X"] \arrow[dr, "\forall \bar{\bar{\beta}}_X"'] & X \sqcup Y \arrow[d, dashed, "\exists ! \bar{\bar{\gamma}}"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[l, "\bar{\bar{\alpha}}_Y"'] \arrow[dl, "\forall \bar{\bar{\beta}}_Y"] \\
& C &
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : COEQUALIZERS & EQUALIZERS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowright"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

COEQUALIZER OF $X \underset{g}{\overset{f}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowright"] \\
& & C
\end{tikzcd}$$

EXERCISE 2.12 FOR $f, g : X \to Y \in \mathcal{C}$

HAVE:

$$\begin{tikzcd}
X \arrow[r, "\bar{\bar{\alpha}}_X"] \arrow[dr, "f"'] & X \sqcup Y \arrow[d, "\bar{\bar{\gamma}}_f"] & Y \arrow[l, "\bar{\bar{\alpha}}_Y"'] \arrow[dl, "\mathrm{id}_Y"] & X \arrow[r, "\bar{\bar{\alpha}}_X"] \arrow[dr, "g"'] & X \sqcup Y \arrow[d, "\bar{\bar{\gamma}}_g"] & Y \arrow[l, "\bar{\bar{\alpha}}_Y"'] \arrow[dl, "\mathrm{id}_Y"] \\
& Y & & & Y &
\end{tikzcd}$$

SHOW:

$$\mathrm{Coeq}(f,g) \cong Y \sqcup_{X \sqcup Y, \bar{\bar{\gamma}}_f, \bar{\bar{\gamma}}_g} Y$$

$$\cong Y \sqcup_{X \sqcup Y} Y$$

ARISES FROM OTHER UNIV. CONSTRUCTIONS

PUSHOUT OF $f : Z \to X$ & $g : Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\bar{\alpha}_X"'] \arrow[ddr, bend left, "\forall \bar{\beta}_X"] & \\
Y \arrow[r, "\bar{\alpha}_Y"] \arrow[drr, bend right, "\forall \bar{\beta}_Y"'] & X \sqcup_Z Y \arrow[dr, dashed, "\exists! \bar{\gamma}"'] & \\
& & C
\end{tikzcd}$$

COPRODUCT OF X AND Y

$$\begin{tikzcd}
X \arrow[r, "\bar{\bar{\alpha}}_X"] \arrow[dr, "\forall \bar{\bar{\beta}}_X"'] & X \sqcup Y \arrow[d, dashed, "\exists! \bar{\bar{\gamma}}"] & Y \arrow[l, "\bar{\bar{\alpha}}_Y"'] \arrow[dl, "\forall \bar{\bar{\beta}}_Y"] \\
& C &
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : COEQUALIZERS & EQUALIZERS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowright"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists ! \gamma"] \arrow[dl, phantom, "\circlearrowright"] \\
 & & C
\end{tikzcd}$$

ARISES FROM OTHER UNIV. CONSTRUCTIONS

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\bar{\alpha}_X"'] \arrow[ddr, bend left, "\forall \bar{\beta}_X"] \arrow[ddr, phantom, "\circlearrowright"] & \\
Y \arrow[r, "\bar{\alpha}_Y"] \arrow[drr, bend right, "\forall \bar{\beta}_Y"'] \arrow[drr, phantom, "\circlearrowright"'] & X \sqcup_Z Y \arrow[dr, dashed, "\exists ! \bar{\gamma}"'] & \\
 & & C
\end{tikzcd}$$

COPRODUCT OF X AND Y

$$\begin{tikzcd}
X \arrow[r, "\bar{\bar{\alpha}}_X"] \arrow[dr, "\forall \bar{\bar{\beta}}_X"'] & X \sqcup Y \arrow[d, dashed, "\exists ! \bar{\bar{\gamma}}"] \arrow[dl, phantom, "\circlearrowright"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[l, "\bar{\bar{\alpha}}_Y"'] \arrow[dl, "\forall \bar{\bar{\beta}}_Y"] \\
 & C &
\end{tikzcd}$$

PUSHOUTS
&
COPRODUCTS
EXIST
⇓
COEQUALIZERS
EXIST

EXERCISE 2.12 FOR $f, g: X \to Y \in \mathcal{C}$

HAVE:

$$\begin{tikzcd}
X \arrow[r, "\bar{\bar{\alpha}}_X"] \arrow[dr, "f"'] & X \sqcup Y \arrow[d, "\bar{\bar{\gamma}}_f"] \arrow[dl, phantom, "\circlearrowright"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[l, "\bar{\bar{\alpha}}_Y"'] \arrow[dl, "\mathrm{id}_Y"] \\
 & Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\bar{\bar{\alpha}}_X"] \arrow[dr, "g"'] & X \sqcup Y \arrow[d, "\bar{\bar{\gamma}}_g"] \arrow[dl, phantom, "\circlearrowright"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[l, "\bar{\bar{\alpha}}_Y"'] \arrow[dl, "\mathrm{id}_Y"] \\
 & Y &
\end{tikzcd}$$

SHOW:

$$\mathrm{Coeq}(f,g) \cong Y \sqcup_{X \sqcup Y,\, \bar{\bar{\gamma}}_f,\, \bar{\bar{\gamma}}_g} Y$$
$$\cong Y \sqcup_{X \sqcup Y} Y$$

I. UNIVERSAL CONSTRUCTIONS : RECAP

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[rd, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[ld, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[rd, "\forall \beta'_X"] \arrow[rd, phantom, "\circlearrowleft"', near start] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[rd, "\forall \beta"'] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists ! \gamma"] \arrow[ld, phantom, "\circlearrowleft"] \\
 & & C
\end{tikzcd}$$

PUSHOUT OF $f : Z \to X$ & $g : Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\bar{\alpha}_X"'] \arrow[rdd, bend left, "\forall \bar{\beta}_X"] \arrow[rdd, phantom, "\circlearrowleft", near start] & \\
Y \arrow[r, "\bar{\alpha}_Y"] \arrow[rrd, bend right, "\forall \bar{\beta}_Y"'] \arrow[rrd, phantom, "\circlearrowleft", near start] & X \sqcup_Z Y \arrow[rd, dashed, "\exists ! \bar{\gamma}"'] & \\
 & & C
\end{tikzcd}$$

$I \in \mathcal{C}$ INITIAL

$$\begin{tikzcd}
I \arrow[d, dashed, "\exists ! \vec{0}_X"] \\
X
\end{tikzcd}$$

COPRODUCT OF X AND Y

$$\begin{tikzcd}
X \arrow[r, "\bar{\bar{\alpha}}_X"] \arrow[rd, "\forall \bar{\bar{\beta}}_X"'] & X \sqcup Y \arrow[d, dashed, "\exists ! \bar{\bar{\gamma}}"] & Y \arrow[l, "\bar{\bar{\alpha}}_Y"'] \arrow[ld, "\forall \bar{\bar{\beta}}_Y"] \\
 & C &
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : RECAP

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

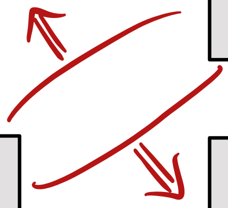
COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists ! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & & C
\end{tikzcd}$$

HAVE THE FOLLOWING EXISTENCE IMPLICATIONS

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\bar{\alpha}_X"'] \arrow[ddr, bend left, "\forall \bar{\beta}_X"] & \\
Y \arrow[r, "\bar{\alpha}_Y"] \arrow[drr, bend right, "\forall \bar{\beta}_Y"'] & X \sqcup_Z Y \arrow[dr, dashed, "\exists ! \bar{\gamma}"'] \arrow[r, phantom, "\circlearrowleft"] \arrow[d, phantom, "\circlearrowleft"] & {} \\
 & {} & C
\end{tikzcd}$$



$I \in \mathcal{C}$ INITIAL

$$\begin{tikzcd}
I \arrow[d, dashed, "\exists ! \vec{o}_X"] \\
X
\end{tikzcd}$$

COPRODUCT OF X AND Y

$$\begin{tikzcd}
X \arrow[r, "\bar{\bar{\alpha}}_X"] \arrow[dr, "\forall \bar{\bar{\beta}}_X"'] & X \sqcup Y \arrow[d, dashed, "\exists ! \bar{\bar{\gamma}}"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[l, "\bar{\bar{\alpha}}_Y"'] \arrow[dl, "\forall \bar{\bar{\beta}}_Y"] \\
 & C &
\end{tikzcd}$$

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I. UNIVERSAL CONSTRUCTIONS : RECAP

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathscr{C}$:

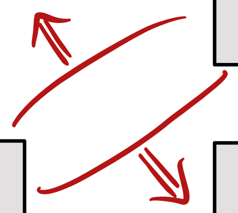
COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists ! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & & C
\end{tikzcd}$$

HAVE THE FOLLOWING EXISTENCE IMPLICATIONS

PUSHOUT OF $f : Z \to X$ & $g : Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\bar{\alpha}_X"] \arrow[ddr, bend left, "\forall \bar{\beta}_X"] & \\
Y \arrow[r, "\bar{\alpha}_Y"] \arrow[drr, bend right, "\forall \bar{\beta}_Y"'] & X \sqcup_Z Y \arrow[dr, dashed, "\exists ! \bar{\gamma}"] & \\
 & & C
\end{tikzcd}$$



$I \in \mathscr{C}$ INITIAL

$$\begin{tikzcd}
I \arrow[d, dashed, "\exists ! \vec{o}_X"] \\
X
\end{tikzcd}$$

COPRODUCT OF X AND Y

$$\begin{tikzcd}
X \arrow[r, "\bar{\bar{\alpha}}_X"] \arrow[dr, "\forall \bar{\bar{\beta}}_X"'] & X \sqcup Y \arrow[d, dashed, "\exists ! \bar{\bar{\gamma}}"] & Y \arrow[l, "\bar{\bar{\alpha}}_Y"'] \arrow[dl, "\forall \bar{\bar{\beta}}_Y"] \\
 & C &
\end{tikzcd}$$

& LIKEWISE FOR

EQUALIZERS

PULLBACKS

TERMINAL OBJECT

PRODUCTS

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, "\circlearrowleft" above right, shift left=3] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft" below left, shift right=3] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_x"] \arrow[dr, "\forall \beta_x"'] \arrow[dr, phantom, shift left=3ex, "\circlearrowleft"] & Univ(X) \arrow[d, dashed, "\exists! \gamma_x"] \\ & Arb(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} Arb(X) \arrow[d, dashed, "\exists! \gamma'_x"'] \arrow[dr, "\forall \beta'_x"] \arrow[dr, phantom, shift right=3ex, "\circlearrowleft"] & \\ Univ(X) \arrow[r, "\alpha'_x"] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

A MORPHISM $g: X \to Y$ IS

A ZERO MORPHISM IF

$gf = gf'$ & $hg = h'g$

$\forall f, f' : W \to X \in \mathcal{C}$ $\forall h, h' : Y \to Z \in \mathcal{C}$

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

A MORPHISM $g : X \to Y$ IS

A ZERO MORPHISM IF

$$gf = gf' \quad \forall f, f' : W \to X \in \mathcal{C} \qquad \& \qquad hg = h'g \quad \forall h, h' : Y \to Z \in \mathcal{C}$$

Ex.

IF $\exists$ ZERO OBJECT IN $\mathcal{C}$, THEN:

$$\vec{0}_{X,Y} : X \xrightarrow{{}_X\vec{0}} 0 \xrightarrow{\vec{0}_Y} Y$$

IS A ZERO MORPHISM.

$\rightsquigarrow$ $\forall$ MORPHISMS $f \in \mathcal{C}$:

$$f \circ \vec{0} = \vec{0} \quad \& \quad \vec{0} \circ f = \vec{0}$$

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

OPERATION ON MORPHISMS

GIVEN A CATEGORY $\mathscr{C}$ W/ZERO OBJ.:

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

$\circlearrowleft$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

$\circlearrowleft$

FORM II

g IS A ZERO MORPH.

IF $gf = gf'$ $\forall f, f'$

& $hg = h'g$ $\forall h, h'$

Ex.

$$\vec{0}_{X,Y} : X \xrightarrow{{}_X\vec{0}} 0 \xrightarrow{\vec{0}_Y} Y$$

$\circlearrowleft$

$$f \circ \vec{0} = \vec{0} \quad \& \quad \vec{0} \circ f = \vec{0} \quad \forall f \in \mathscr{C}$$

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

OPERATION ON MORPHISMS

GIVEN A CATEGORY $\mathcal{C}$ W/ ZERO OBJ.:

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

FORM II

g IS A ZERO MORPH.

IF $gf = gf' \ \forall f, f'$

& $hg = h'g \ \forall h, h'$

Ex.

$$\vec{0}_{X,Y} : X \xrightarrow{{}_X\vec{0}} 0 \xrightarrow{\vec{0}_Y} Y$$

$$\Downarrow$$

$$f \circ \vec{0} = \vec{0} \ \& \ \vec{0} \circ f = \vec{0}$$
$$\forall f \in \mathcal{C}$$

COKERNEL OF $f: X \to Y$

IS AN OBJECT $\mathrm{coker}(f) \in \mathcal{C}$

EQUIPPED WITH A MORPHISM

$$\alpha : Y \longrightarrow \mathrm{coker}(f)$$

$$\ni. \ \alpha f = \vec{0}_{X, \mathrm{coker}(f)}$$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] & Y \arrow[r, "\alpha"] & \mathrm{coker}(f) \end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[rd, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[ld, phantom, "\circlearrowright"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[rd, "\forall \beta'_X"] \arrow[rd, phantom, "\circlearrowright"'] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X \end{tikzcd}$$

FORM II

g IS A ZERO MORPH. IF $gf = gf' \ \forall f, f'$

& $hg = h'g \ \forall h, h'$

Ex.

$\vec{0}_{X,Y} : X \xrightarrow{{}_X\vec{0}} 0 \xrightarrow{\vec{0}_Y} Y$

$\rightsquigarrow$

$f \circ \vec{0} = \vec{0}$ & $\vec{0} \circ f = \vec{0}$ $\forall f \in \mathcal{C}$

GIVEN A CATEGORY $\mathcal{C}$ W/ ZERO OBJ.:

COKERNEL OF $f : X \to Y$

IS AN OBJECT $\mathrm{coker}(f) \in \mathcal{C}$

EQUIPPED WITH A MORPHISM

$\alpha : Y \longrightarrow \mathrm{coker}(f)$

$\ni.$ $\alpha f = \vec{0}_{X, \mathrm{coker}(f)}$

WHERE $\forall \beta : Y \longrightarrow C \ \ni. \ \beta f = \vec{0}_{X,C}$

WE GET:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[rrd, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[rd, "\forall \beta"'] & \mathrm{coker}(f) \arrow[d, dashed, "\exists! \gamma"] \arrow[ld, phantom, "\circlearrowright"] \\ & & C \end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft", near start] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

g IS A ZERO MORPH.

IF $gf = gf' \ \forall f, f'$

& $hg = h'g \ \forall h, h'$

Ex.

$\vec{0}_{X,Y} : X \xrightarrow{{}_X\vec{0}} 0 \xrightarrow{\vec{0}_Y} Y$

$\Downarrow$

$f \circ \vec{0} = \vec{0}$ & $\vec{0} \circ f = \vec{0}$
$\forall f \in \mathcal{C}$

GIVEN A CATEGORY $\mathcal{C}$ W/ZERO OBJ.:

COKERNEL OF $f : X \to Y$

IS AN OBJECT $\mathrm{coker}(f) \in \mathcal{C}$

EQUIPPED WITH A MORPHISM

$\alpha : Y \longrightarrow \mathrm{coker}(f)$

$\ni.\ \alpha f = \vec{0}_{X, \mathrm{coker}(f)}$

WHERE $\forall \beta : Y \longrightarrow C \ \ni.\ \beta f = \vec{0}_{X,C}$

WE GET:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[drr, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coker}(f) \arrow[d, dashed, "\exists ! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & & C
\end{tikzcd}$$

KERNEL OF $f : X \to Y$

IS AN OBJECT $\ker(f) \in \mathcal{C}$

EQUIPPED WITH A MORPHISM

$\alpha' : \ker(f) \longrightarrow X$

$\ni.\ f \alpha' = \vec{0}_{\ker(f), Y}$

WHERE $\forall \beta' : K \longrightarrow X \ \ni.\ f\beta' = \vec{0}_{K,Y}$

WE GET:

$$\begin{tikzcd}
K \arrow[d, dashed, "\exists ! \gamma'"'] \arrow[dr, "\forall \beta'"] \arrow[drr, bend left, "\vec{0}"] \arrow[dr, phantom, "\circlearrowleft", near end, shift right=3] & & \\
\ker(f) \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X
\end{tikzcd}$$

FORM II

g IS A ZERO MORPH.

IF $gf = gf'$ $\forall f, f'$

& $hg = h'g$ $\forall h, h'$

Ex.

$\vec{0}_{X,Y} : X \xrightarrow{{}_X\vec{0}} 0 \xrightarrow{\vec{0}_Y} Y$

$\downarrow$

$f \circ \vec{0} = \vec{0}$ & $\vec{0} \circ f = \vec{0}$ $\forall f \in \mathcal{C}$

GIVEN A CATEGORY $\mathcal{C}$ W/ ZERO OBJ.:

COKERNEL OF $f : X \to Y$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[drr, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coker}(f) \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & & C
\end{tikzcd}$$

KERNEL OF $f : X \to Y$

$$\begin{tikzcd}
K \arrow[d, dashed, "\exists! \gamma'"'] \arrow[dr, "\forall \beta'"] \arrow[drr, bend left, "\vec{0}"] & & \\
\ker(f) \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] \arrow[ur, phantom, "\circlearrowleft"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

↖ OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$ W/ZERO OBJ.:

COKERNEL OF $f: X \to Y$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[drr, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coker}(f) \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & & C
\end{tikzcd}$$

KERNEL OF $f: X \to Y$

$$\begin{tikzcd}
K \arrow[d, dashed, "\exists! \gamma'"'] \arrow[dr, "\forall \beta'"] \arrow[drr, bend left, "\vec{0}"] \arrow[dr, phantom, shift right=3, "\circlearrowleft"] & & \\
\ker(f) \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

↑ RECOVERS THE USUAL NOTION OF (CO)KERNELS IN MANY CATEGORIES

EXERCISE 2.15

g IS A ZERO MORPH. IF $gf = gf'$ $\forall f, f'$ & $hg = h'g$ $\forall h, h'$

Ex. $\vec{0}_{X,Y} : X \xrightarrow{{}_X\vec{0}} 0 \xrightarrow{\vec{0}_Y} Y$

$\downarrow$

$f \circ \vec{0} = \vec{0}$ & $\vec{0} \circ f = \vec{0}$ $\forall f \in \mathcal{C}$

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

↖ OPERATION ON MORPHISMS

GIVEN A CATEGORY $\mathcal{C}$ W/ZERO OBJ.:

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

COKERNEL OF $f: X \to Y$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[drr, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coker}(f) \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & & C
\end{tikzcd}$$

KERNEL OF $f: X \to Y$

$$\begin{tikzcd}
K \arrow[d, dashed, "\exists! \gamma'"'] \arrow[dr, "\forall \beta'"] \arrow[drr, bend left, "\vec{0}"] \arrow[d, phantom, "\circlearrowleft"', shift left=6] & & \\
\mathrm{ker}(f) \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

g IS A ZERO MORPH. IF $gf = gf'$ $\forall f, f'$

& $hg = h'g$ $\forall h, h'$

Ex.

$$\vec{0}_{X,Y} : X \xrightarrow{{}_X\vec{0}} 0 \xrightarrow{\vec{0}_Y} Y$$

$\rightsquigarrow$

$f \circ \vec{0} = \vec{0}$ & $\vec{0} \circ f = \vec{0}$ $\forall f \in \mathcal{C}$

↑ RECOVERS THE USUAL NOTION OF (CO)KERNELS IN MANY CATEGORIES

EXERCISE 2.15

↑ IS A SPECIAL CASE OF A (CO)EQUALIZER

EXERCISE 2.13

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"', shift right=3] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$ w/ZERO OBJ.:

COKERNEL OF $f: X \to Y$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[drr, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coker}(f) \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\ & & C \end{tikzcd}$$

KERNEL OF $f: X \to Y$

$$\begin{tikzcd} K \arrow[d, dashed, "\exists! \gamma'"'] \arrow[dr, "\forall \beta'"] \arrow[drr, bend left, "\vec{0}"] \arrow[d, phantom, "\circlearrowleft", shift left=6] & & \\ \mathrm{ker}(f) \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

↑ RECOVERS THE USUAL NOTION OF (CO)KERNELS IN MANY CATEGORIES

EXERCISE 2.15

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\ & & C \end{tikzcd}$$

↑ IS A SPECIAL CASE OF A (CO)EQUALIZER

EXERCISE 2.13

← THINK HOW TO GET A COKERNEL FROM THIS

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[rd, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[ld, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[rd, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ru, phantom, "\circlearrowleft"] & X
\end{tikzcd}$$

FORM II

g IS A **ZERO MORPH.**

IF $gf = gf' \ \forall f, f'$

& $hg = h'g \ \forall h, h'$

Ex.

$$\vec{0}_{X,Y} : X \xrightarrow{{}_X\vec{0}} 0 \xrightarrow{\vec{0}_Y} Y$$

$$\Downarrow$$

$$f \circ \vec{0} = \vec{0} \ \ \& \ \ \vec{0} \circ f = \vec{0} \quad \forall f \in \mathcal{C}$$

GIVEN A CATEGORY $\mathcal{C}$ W/ ZERO OBJ.:

COKERNEL OF $f: X \to Y$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[rrd, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[rd, "\forall \beta"'] & \mathrm{coker}(f) \arrow[d, dashed, "\exists! \gamma"] \arrow[ld, phantom, "\circlearrowleft"] \\
 & & C
\end{tikzcd}$$

KERNEL OF $f: X \to Y$

$$\begin{tikzcd}
K \arrow[d, dashed, "\exists! \gamma'"'] \arrow[rd, "\forall \beta'"] \arrow[rrd, bend left, "\vec{0}"] \arrow[d, phantom, shift left=4, "\circlearrowleft"] & & \\
\ker(f) \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

EXERCISE 2.13 COKERNELS ARE EPIC, & KERNELS ARE MONIC, ← { RIGHT / LEFT } CANCELLATIVE

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

$\circlearrowright$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

$\circlearrowright$

FORM II

GIVEN A CATEGORY $\mathscr{C}$ W/ ZERO OBJ.:

COKERNEL OF $f: X \to Y$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[drr, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coker}(f) \arrow[d, dashed, "\exists ! \gamma"] \\ & & C \end{tikzcd}$$

$\circlearrowright$

KERNEL OF $f: X \to Y$

$$\begin{tikzcd} K \arrow[d, dashed, "\exists ! \gamma'"'] \arrow[dr, "\forall \beta'"] \arrow[drr, bend left, "\vec{0}"] & & \\ \ker(f) \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

$\circlearrowright$

g IS A ZERO MORPH.

IF $gf = gf' \ \forall f, f'$

& $hg = h'g \ \forall h, h'$

Ex.

$$\vec{0}_{X,Y} : X \xrightarrow{{}_X\vec{0}} 0 \xrightarrow{\vec{0}_Y} Y$$

$\rightsquigarrow$

$$f \circ \vec{0} = \vec{0} \ \& \ \vec{0} \circ f = \vec{0}$$
$$\forall f \in \mathscr{C}$$

EXERCISE 2.13 COKERNELS ARE EPIC ← RIGHT CANCELLATIVE

You do!

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

OPERATION ON MORPHISMS

GIVEN A CATEGORY $\mathcal{C}$ w/ ZERO OBJ.:

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall\beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists!\gamma'_X"'] \arrow[dr, "\forall\beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X
\end{tikzcd}$$

FORM II

COKERNEL OF $f: X \to Y$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[drr, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall\beta"'] & \mathrm{coker}(f) \arrow[d, dashed, "\exists!\gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & & C
\end{tikzcd}$$

KERNEL OF $f: X \to Y$

$$\begin{tikzcd}
K \arrow[d, dashed, "\exists!\gamma'"'] \arrow[dr, "\forall\beta'"] \arrow[drr, bend left, "\vec{0}"] & & \\
\ker(f) \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] \arrow[ur, phantom, "\circlearrowleft"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

g IS A ZERO MORPH.

IF $gf = gf'$ $\forall f, f'$

& $hg = h'g$ $\forall h, h'$

Ex.

$\vec{0}_{X,Y} : X \xrightarrow{{}_X\vec{0}} 0 \xrightarrow{\vec{0}_Y} Y$

$\downarrow$

$f \circ \vec{0} = \vec{0}$ & $\vec{0} \circ f = \vec{0}$

$\forall f \in \mathcal{C}$

EXERCISE 2.13 COKERNELS ARE EPIC ← RIGHT CANCELLATIVE

PF/ TAKE $h, h' : \mathrm{coker}(f) \to Z$ ∋. $h\alpha = h'\alpha$ AS MORPHISMS $Y \to Z$

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

OPERATION ON MORPHISMS

GIVEN A CATEGORY $\mathcal{C}$ W/ZERO OBJ.:

COKERNEL OF $f: X \to Y$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[rrd, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[rd, "\forall \beta"'] \arrow[r, phantom, shift right=6, "\circlearrowright"] & \mathrm{coker}(f) \arrow[d, dashed, "\exists!\,\gamma"] \\
& & C
\end{tikzcd}$$

KERNEL OF $f: X \to Y$

$$\begin{tikzcd}
K \arrow[d, dashed, "\exists!\,\gamma'"'] \arrow[rd, "\forall \beta'"] \arrow[rrd, bend left, "\vec{0}"] \arrow[d, phantom, shift left=6, "\circlearrowright"] & & \\
\ker(f) \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[rd, "\forall \beta_X"'] \arrow[r, phantom, shift right=6, "\circlearrowright"] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\,\gamma_X"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists!\,\gamma'_X"'] \arrow[rd, "\forall \beta'_X"] \arrow[d, phantom, shift left=6, "\circlearrowright"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

g IS A ZERO MORPH. IF $gf = gf'$ $\forall f, f'$ & $hg = h'g$ $\forall h, h'$

Ex.

$\vec{0}_{X,Y} : X \xrightarrow{{}_X\vec{0}} 0 \xrightarrow{\vec{0}_Y} Y$

$\downarrow$

$f \circ \vec{0} = \vec{0}$ & $\vec{0} \circ f = \vec{0}$ $\forall f \in \mathcal{C}$

EXERCISE 2.13 COKERNELS ARE EPIC ← RIGHT CANCELLATIVE

PF/ TAKE $h, h' : \mathrm{coker}(f) \to Z$ ∋. $h\alpha = h'\alpha$ AS MORPHISMS $Y \to Z$

NOW

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] & Y \arrow[r, "\alpha"] \arrow[rd, "h\alpha"'] & \mathrm{coker}(f) \arrow[d, "h"] \\
& & Z
\end{tikzcd}$$

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

← OPERATION ON MORPHISMS

GIVEN A CATEGORY $\mathscr{C}$ W/ ZERO OBJ.:

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowright"] & X
\end{tikzcd}$$

FORM II

COKERNEL OF $f: X \to Y$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[drr, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coker}(f) \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowright"] \\
 & & C
\end{tikzcd}$$

KERNEL OF $f: X \to Y$

$$\begin{tikzcd}
K \arrow[d, dashed, "\exists! \gamma'"'] \arrow[dr, "\forall \beta'"] \arrow[drr, bend left, "\vec{0}"] & & \\
\ker(f) \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] \arrow[ur, phantom, "\circlearrowright"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

g IS A ZERO MORPH.

IF $gf = gf'$ $\forall f, f'$

& $hg = h'g$ $\forall h, h'$

Ex.

$\vec{0}_{X,Y}: X \xrightarrow{{}_X\vec{0}} 0 \xrightarrow{\vec{0}_Y} Y$

⇝

$f \circ \vec{0} = \vec{0}$ & $\vec{0} \circ f = \vec{0}$ $\forall f \in \mathscr{C}$

EXERCISE 2.13 COKERNELS ARE EPIC ← RIGHT CANCELLATIVE

PF/ TAKE $h, h' : \mathrm{coker}(f) \to Z$ ∋. $h\alpha = h'\alpha$ AS MORPHISMS $Y \to Z$

NOW

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[drr, bend right, "\vec{0}"'] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[r, "\alpha"] \arrow[dr, "h\alpha"'] & \mathrm{coker}(f) \arrow[d, "\gamma = h"] \arrow[dl, phantom, "\circlearrowright"] \\
 & & Z
\end{tikzcd}$$

$$\left[(h\alpha)f = h(\alpha f) = h\vec{0} = \vec{0} \right]$$

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X
\end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$ W/ZERO OBJ.:

COKERNEL OF $f: X \to Y$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[drr, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coker}(f) \arrow[d, dashed, "\exists ! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & & C
\end{tikzcd}$$

KERNEL OF $f: X \to Y$

$$\begin{tikzcd}
K \arrow[d, dashed, "\exists ! \gamma'"'] \arrow[dr, "\forall \beta'"] \arrow[drr, bend left, "\vec{0}"] & & \\
\mathrm{ker}(f) \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] \arrow[ur, phantom, "\circlearrowleft"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

g IS A ZERO MORPH.

IF $gf = gf'$ $\forall f, f'$

& $hg = h'g$ $\forall h, h'$

Ex.

$\vec{0}_{X,Y}: X \xrightarrow{{}_X\vec{0}} 0 \xrightarrow{\vec{0}_Y} Y$

⤵

$f \circ \vec{0} = \vec{0}$ & $\vec{0} \circ f = \vec{0}$ $\forall f \in \mathcal{C}$

EXERCISE 2.13 COKERNELS ARE EPIC ← RIGHT CANCELLATIVE

PF/ TAKE $h, h' : \mathrm{coker}(f) \to Z$ ∋. $h\alpha = h'\alpha$ AS MORPHISMS $Y \to Z$

NOW

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[drr, bend right, "\vec{0}"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[r, "\alpha"] \arrow[dr, "h'\alpha = h\alpha"'] & \mathrm{coker}(f) \arrow[d, "\gamma = h \text{ or } h'"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & & Z
\end{tikzcd}$$

$$\left[(h\alpha) f = h(\alpha f) = h\vec{0} = \vec{0} \right]$$

BY UNIQUENESS OF γ, WE MUST HAVE $h = h'$. ///

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

OPERATION ON MORPHISMS

GIVEN A CATEGORY $\mathscr{C}$ W/ZERO OBJ.:

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"'] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X
\end{tikzcd}$$

FORM II

g IS A ZERO MORPH.

IF $gf = gf'$ $\forall f, f'$

& $hg = h'g$ $\forall h, h'$

Ex.

$$\vec{0}_{X,Y} : X \xrightarrow{{}_X\vec{0}} 0 \xrightarrow{\vec{0}_Y} Y$$

$$\rightsquigarrow$$

$$f \circ \vec{0} = \vec{0} \ \ \& \ \ \vec{0} \circ f = \vec{0} \quad \forall f \in \mathscr{C}$$

COKERNEL OF $f: X \to Y$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[drr, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coker}(f) \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & & C
\end{tikzcd}$$

$\Downarrow$

α EPIC

KERNEL OF $f: X \to Y$

$$\begin{tikzcd}
K \arrow[d, dashed, "\exists! \gamma'"'] \arrow[dr, "\forall \beta'"] \arrow[drr, bend left, "\vec{0}"] \arrow[dr, phantom, "\circlearrowleft"'] & & \\
\mathrm{ker}(f) \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

$\Downarrow$

α' MONIC

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[rd, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \\ \arrow[ru, phantom, "\circlearrowleft"] & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[rd, "\forall \beta'_X"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] & X \end{tikzcd}$$

FORM II

g IS A ZERO MORPH. IF $gf = gf'$ $\forall f, f'$ & $hg = h'g$ $\forall h, h'$

Ex. $\vec{0}_{X,Y} : X \xrightarrow{{}_X\vec{0}} 0 \xrightarrow{\vec{0}_Y} Y$

$\Downarrow$

$f \circ \vec{0} = \vec{0}$ & $\vec{0} \circ f = \vec{0}$ $\forall f \in \mathscr{C}$

GIVEN A CATEGORY $\mathscr{C}$ W/ ZERO OBJ.:

COKERNEL OF $f : X \to Y$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[rrd, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[rd, "\forall \beta"'] & \mathrm{coker}(f) \arrow[d, dashed, "\exists! \gamma"] \\ & & C \end{tikzcd}$$

KERNEL OF $f : X \to Y$

$$\begin{tikzcd} K \arrow[d, dashed, "\exists! \gamma'"'] \arrow[rd, "\forall \beta'"] \arrow[rrd, bend left, "\vec{0}"] & & \\ \mathrm{ker}(f) \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

$\Downarrow \Uparrow$

α EPIC

$\Downarrow \Uparrow$

α' MONIC

$\left\{\begin{array}{l}\text{AN EPI}\\ \text{A MONO}\end{array}\right.$ IS CALLED NORMAL IF IT ARISES AS THE $\left\{\begin{array}{l}\text{COKERNEL}\\ \text{KERNEL}\end{array}\right.$ OF A MORPHISM.

I. UNIVERSAL CONSTRUCTIONS : COKERNELS & KERNELS

OPERATION ON MORPHISMS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists ! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists ! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"'] \arrow[ur, phantom, "\circlearrowleft"] & X
\end{tikzcd}$$

FORM II

g IS A ZERO MORPH. IF $gf = gf'$ $\forall f, f'$ & $hg = h'g$ $\forall h, h'$

Ex.

$$\vec{0}_{X,Y} : X \xrightarrow{{}_X\vec{0}} 0 \xrightarrow{\vec{0}_Y} Y$$

$$\Downarrow$$

$$f \circ \vec{0} = \vec{0} \ \& \ \vec{0} \circ f = \vec{0} \quad \forall f \in \mathcal{C}$$

GIVEN A CATEGORY $\mathcal{C}$ W/ ZERO OBJ.:

COKERNEL OF $f : X \to Y$

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "\vec{0}"] \arrow[drr, bend right, "\vec{0}"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] & \mathrm{coker}(f) \arrow[d, dashed, "\exists ! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\
 & & C
\end{tikzcd}$$

KERNEL OF $f : X \to Y$

$$\begin{tikzcd}
K \arrow[d, dashed, "\exists ! \gamma'"'] \arrow[dr, "\forall \beta'"] \arrow[drr, bend left, "\vec{0}"] & & \\
\mathrm{ker}(f) \arrow[r, "\alpha'"] \arrow[rr, bend right, "\vec{0}"'] \arrow[ur, phantom, "\circlearrowleft"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

$\Downarrow \Uparrow$

α EPIC

$\Downarrow \Uparrow$

α' MONIC

$\left\{ \begin{array}{l} \text{AN EPI} \\ \text{A MONO} \end{array} \right.$ IS CALLED NORMAL IF IT ARISES AS THE $\left\{ \begin{array}{l} \text{COKERNEL} \\ \text{KERNEL} \end{array} \right.$ OF A MORPHISM.

≡ NEXT WE DISCUSS CATEGORIES IN WHICH ALL OF THE ABOVE EXISTS ≡

II. ABELIAN CATEGORIES : OVERVIEW

TAKE $\mathcal{C}$ A CATEGORY

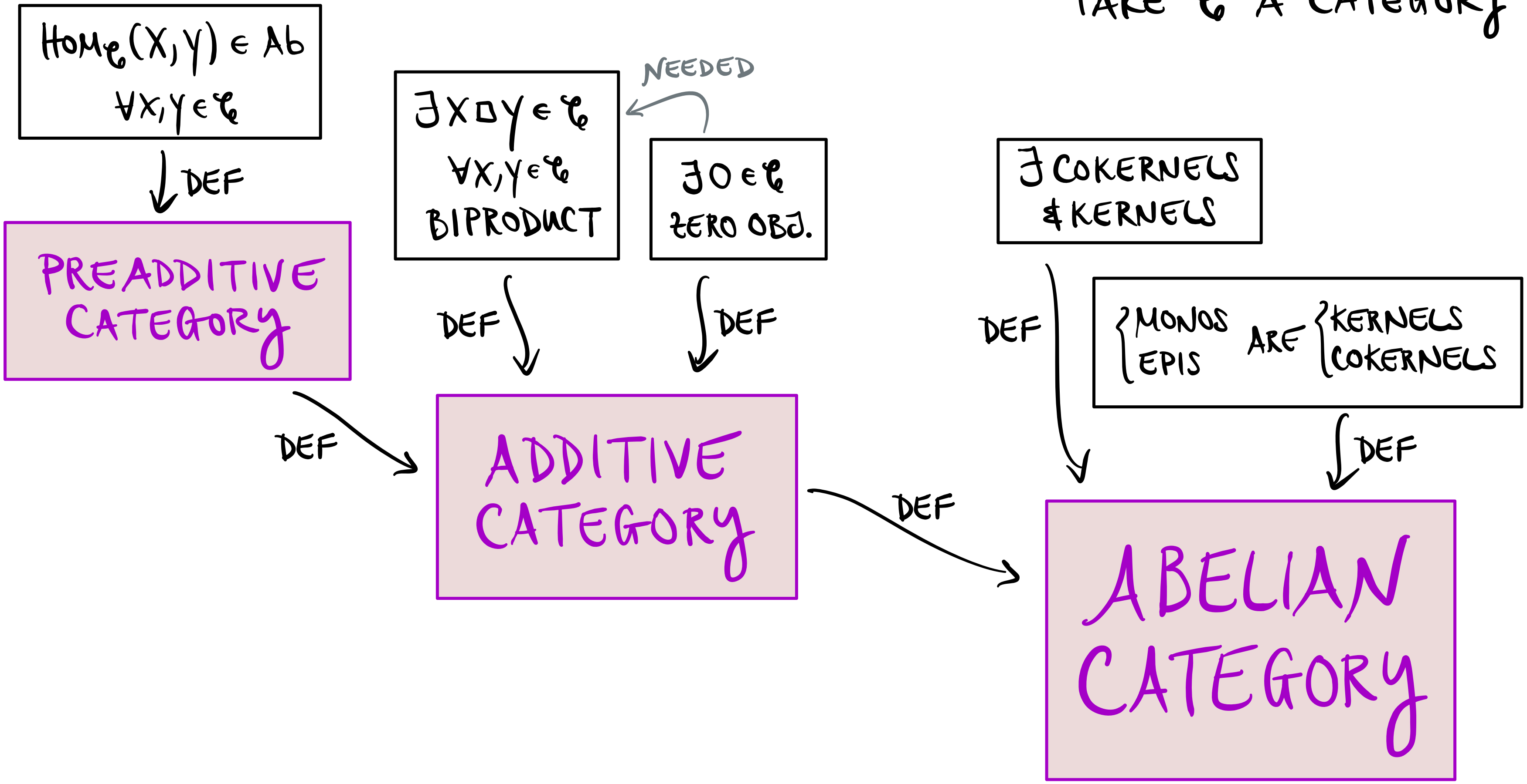
PREADDITIVE CATEGORY

ADDITIVE CATEGORY

ABELIAN CATEGORY

II. ABELIAN CATEGORIES : OVERVIEW

TAKE $\mathcal{C}$ A CATEGORY



$\text{Hom}_{\mathcal{C}}(X,Y) \in Ab$ $\forall X,Y \in \mathcal{C}$

DEF

PREADDITIVE CATEGORY

$\exists X \square Y \in \mathcal{C}$ $\forall X,Y \in \mathcal{C}$ BIPRODUCT

NEEDED

$\exists 0 \in \mathcal{C}$ ZERO OBJ.

DEF

DEF

DEF

ADDITIVE CATEGORY

$\exists$ COKERNELS & KERNELS

$\begin{cases} \text{MONOS} \\ \text{EPIS} \end{cases}$ ARE $\begin{cases} \text{KERNELS} \\ \text{COKERNELS} \end{cases}$

DEF

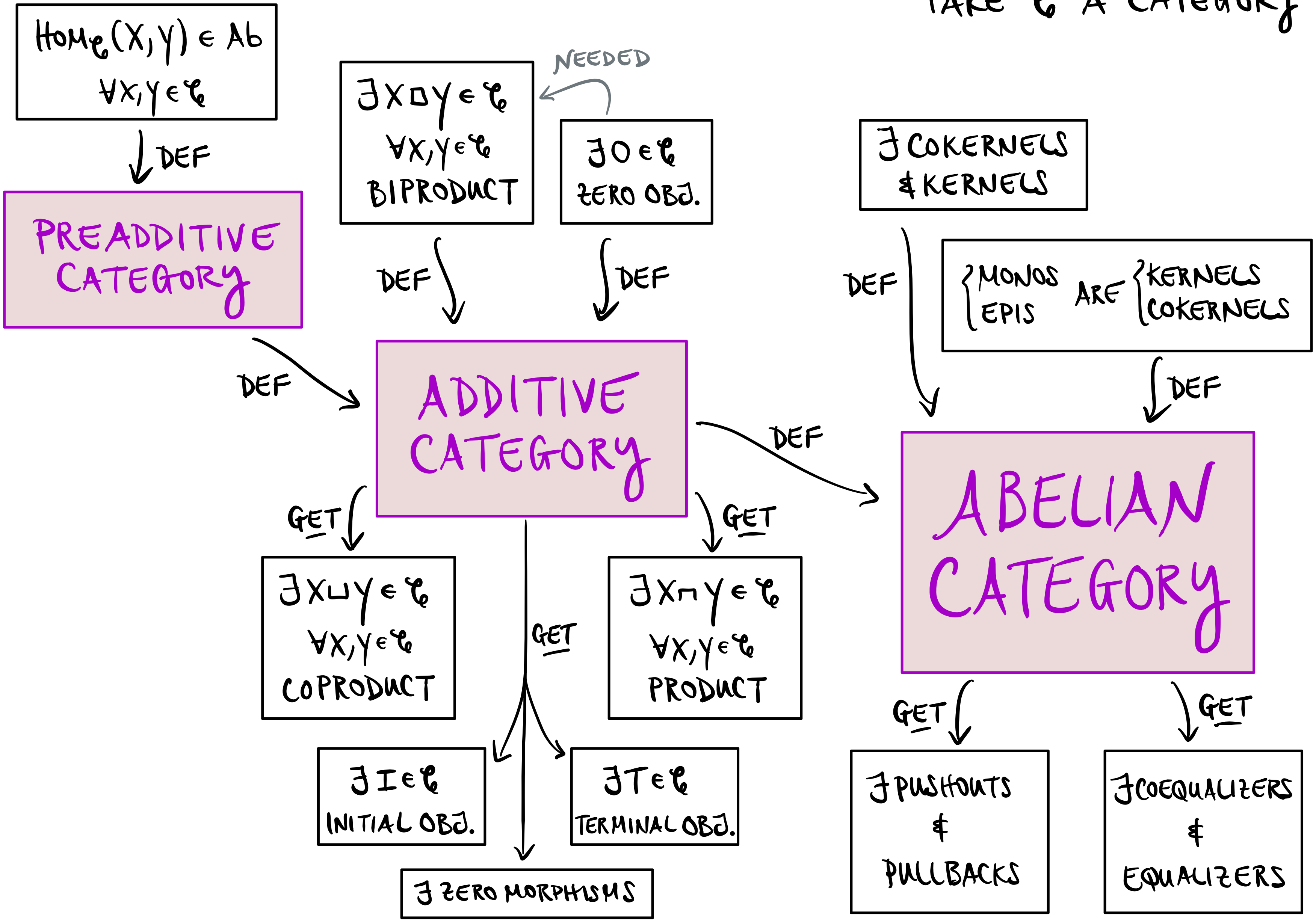
DEF

DEF

ABELIAN CATEGORY

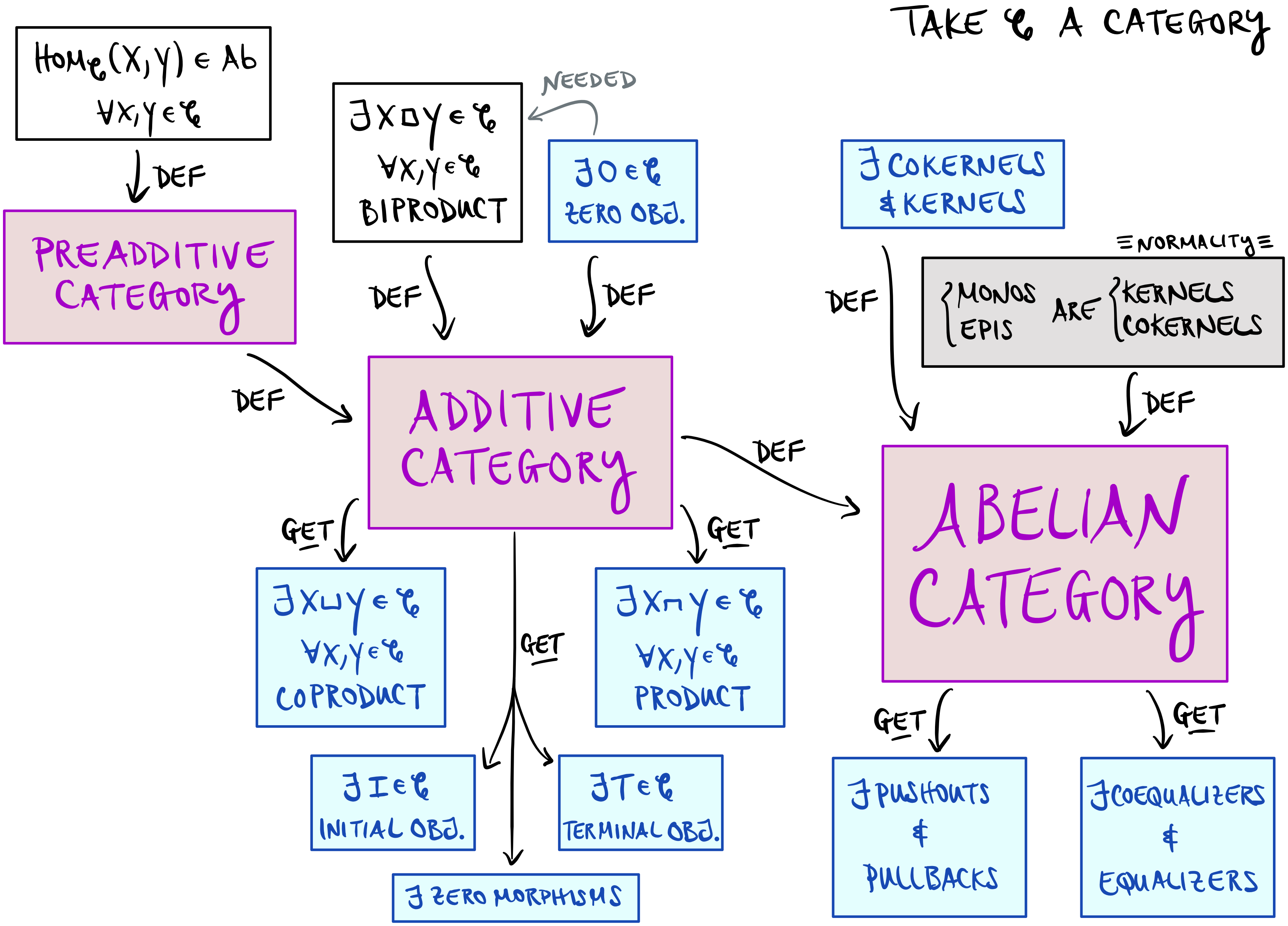
II. ABELIAN CATEGORIES : OVERVIEW

TAKE $\mathcal{C}$ A CATEGORY

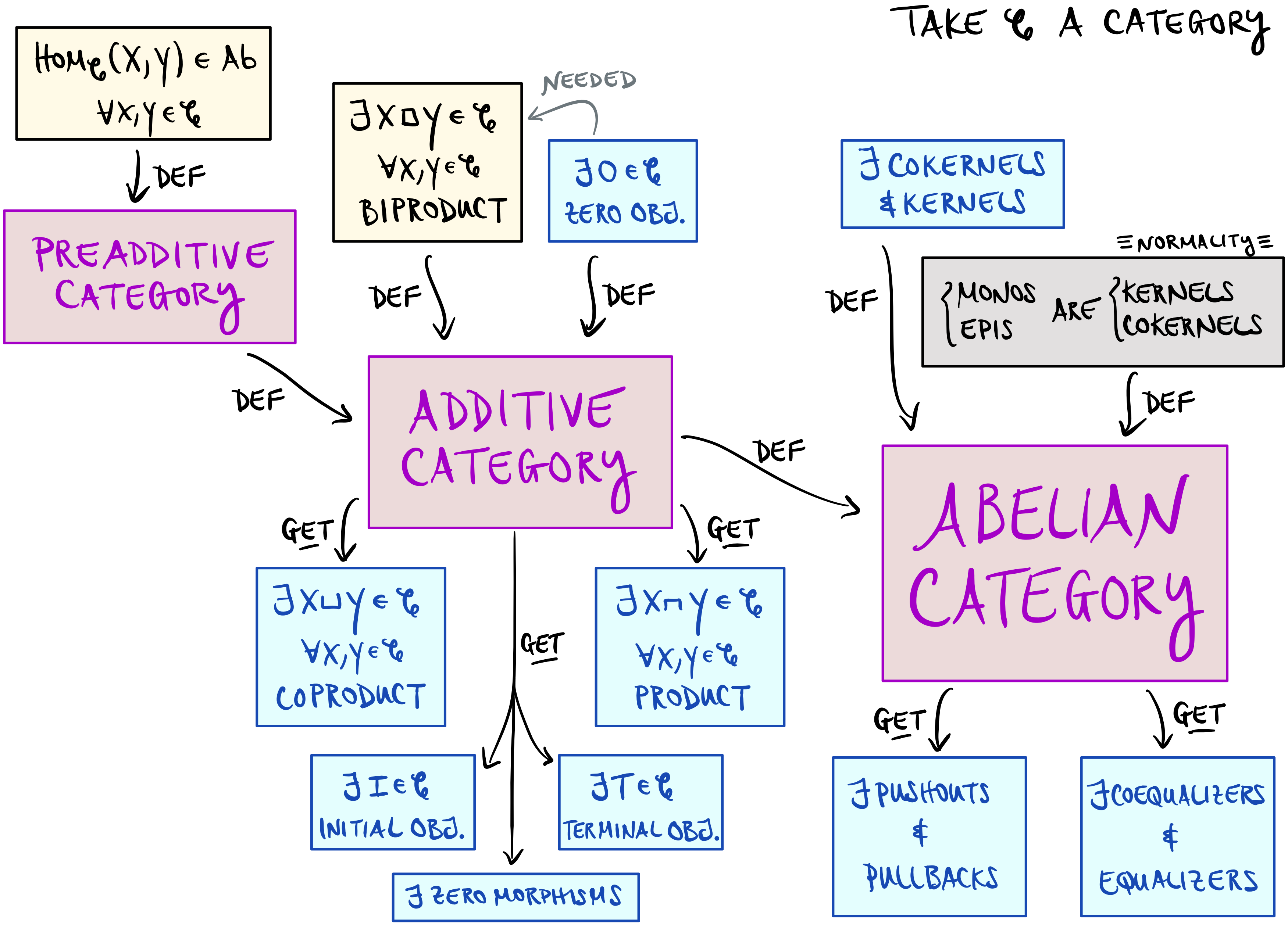


II. ABELIAN CATEGORIES : OVERVIEW

TAKE $\mathcal{C}$ A CATEGORY



II. ABELIAN CATEGORIES : OVERVIEW



II. ABELIAN CATEGORIES : PREADDITIVE & LINEAR CATEGORIES

$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Ab}$
$\forall X,Y \in \mathcal{C}$

↓ DEF

PREADDITIVE CATEGORY

II. ABELIAN CATEGORIES : PREADDITIVE & LINEAR CATEGORIES

$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Ab}$
$\forall X,Y \in \mathcal{C}$

↓ DEF

PREADDITIVE CATEGORY

A CATEGORY $\mathcal{C}$ IS **PREADDITIVE**
(OR AN **Ab-CATEGORY**,
OR **ENRICHED OVER Ab**)

IF $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ IS AN ABELIAN GROUP $\forall X,Y \in \mathcal{C}$

w/
- OPERATION $+$
- ADDITIVE IDENTITY $\vec{0}$ (WHEN $\exists\, 0 \in \mathcal{C}$)
- ADDITIVE INVERSE OF $f: X \to Y$ DENOTED BY $-f: X \to Y$

ALSO REQUIRE $\circ$ DISTRIBUTES OVER $+$

II. ABELIAN CATEGORIES : PREADDITIVE & LINEAR CATEGORIES

$\text{Hom}_{\mathcal{C}}(X,Y) \in \text{Ab}$
$\forall X,Y \in \mathcal{C}$

$\downarrow$ DEF

PREADDITIVE CATEGORY

Ex. Ab

A CATEGORY $\mathcal{C}$ IS PREADDITIVE
(OR AN Ab-CATEGORY,
OR ENRICHED OVER Ab)

IF $\text{Hom}_{\mathcal{C}}(X,Y)$ IS AN ABELIAN GROUP $\forall X,Y \in \mathcal{C}$

W/
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II. ABELIAN CATEGORIES : PREADDITIVE & LINEAR CATEGORIES

$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Ab}$
$\forall X,Y \in \mathcal{C}$

$\downarrow$ DEF

PREADDITIVE CATEGORY

Ex. Ab

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(OR AN **Ab-CATEGORY**,
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IF $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ IS AN ABELIAN GROUP $\forall X,Y \in \mathcal{C}$

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- OPERATION $+$
- ADDITIVE IDENTITY $\vec{0}$ (WHEN $\exists\, 0 \in \mathcal{C}$)
- ADDITIVE INVERSE OF $f: X \to Y$ DENOTED BY $-f: X \to Y$

ALSO REQUIRE $\circ$ DISTRIBUTES OVER $+$

FIX GROUND FIELD $\Bbbk$ ⤷

A CATEGORY $\mathcal{C}$ IS **LINEAR** (OR **ENRICHED OVER Vec**)
IF $\mathcal{C}$ IS PREADDITIVE
& $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ IS A VECTOR SPACE $\forall X,Y \in \mathcal{C}$
& $\circ$ DISTRIBUTES OVER $*$.

II. ABELIAN CATEGORIES : PREADDITIVE & LINEAR CATEGORIES

$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Ab}$
$\forall X,Y \in \mathcal{C}$

↓ DEF

PREADDITIVE CATEGORY

Ex. Ab

Ex. $\mathrm{Vec}_{\Bbbk}$
A-Mod

A CATEGORY $\mathcal{C}$ IS **PREADDITIVE**
(OR AN **Ab-CATEGORY**,
OR **ENRICHED OVER Ab**)

IF $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ IS AN ABELIAN GROUP $\forall X,Y \in \mathcal{C}$

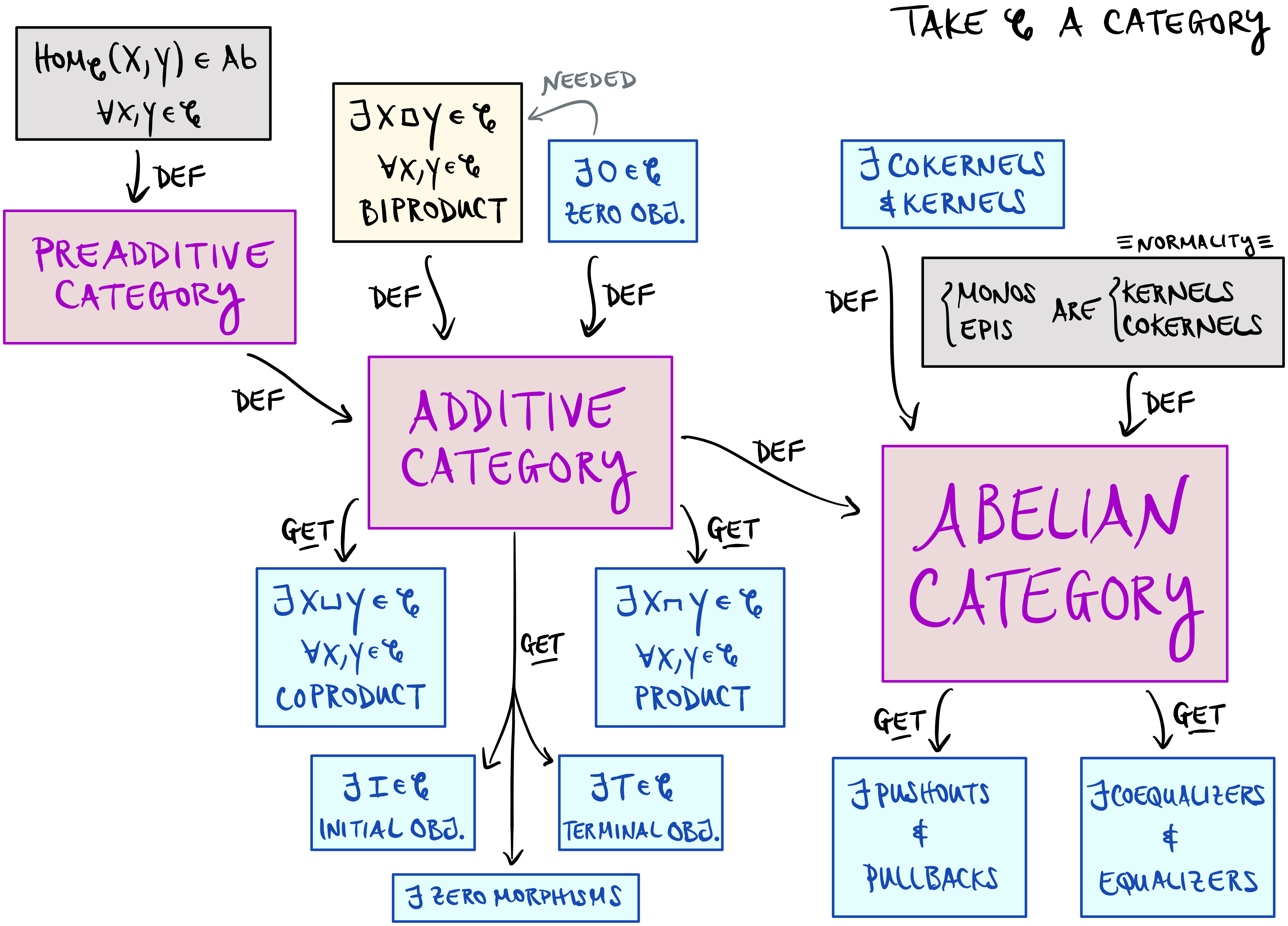
W/
- OPERATION $+$
- ADDITIVE IDENTITY $\vec{0}$ (WHEN $\exists\, 0 \in \mathcal{C}$)
- ADDITIVE INVERSE OF $f: X \to Y$ DENOTED BY $-f: X \to Y$

ALSO REQUIRE $\circ$ DISTRIBUTES OVER $+$

FIX GROUND FIELD $\Bbbk$

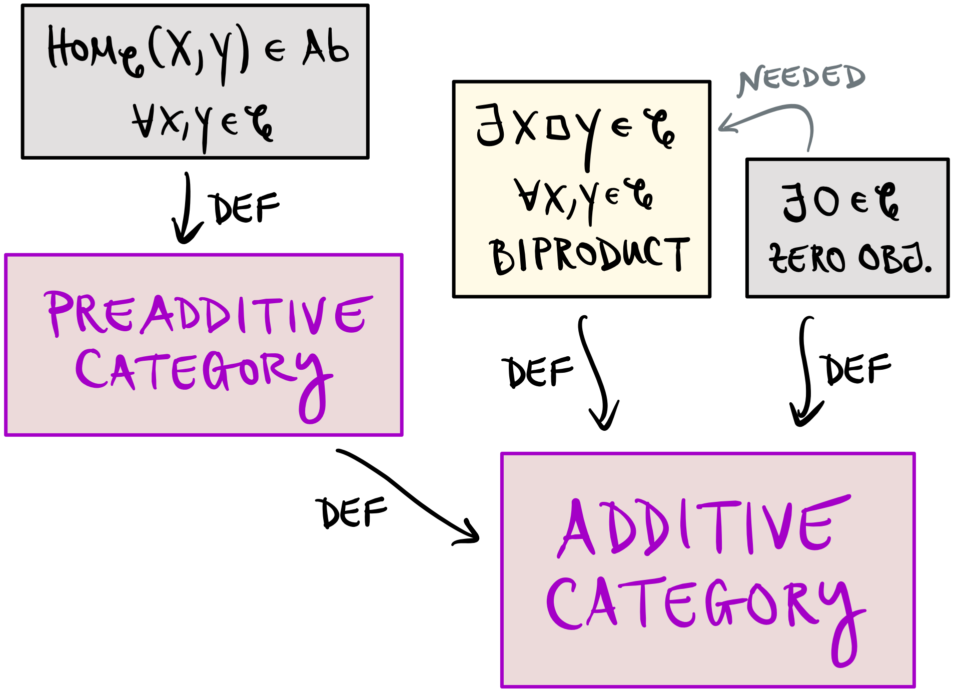
A CATEGORY $\mathcal{C}$ IS **LINEAR** (OR **ENRICHED OVER Vec**)
IF $\mathcal{C}$ IS PREADDITIVE
& $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ IS A VECTOR SPACE $\forall X,Y \in \mathcal{C}$
& $\circ$ DISTRIBUTES OVER $*$.

II. ABELIAN CATEGORIES : OVERVIEW



II. ABELIAN CATEGORIES : OVERVIEW

TAKE $\mathcal{C}$ A CATEGORY



$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Ab}$
$\forall X,Y \in \mathcal{C}$

DEF

PREADDITIVE CATEGORY

$\exists X \square Y \in \mathcal{C}$
$\forall X,Y \in \mathcal{C}$
BIPRODUCT

NEEDED

$\exists 0 \in \mathcal{C}$
ZERO OBJ.

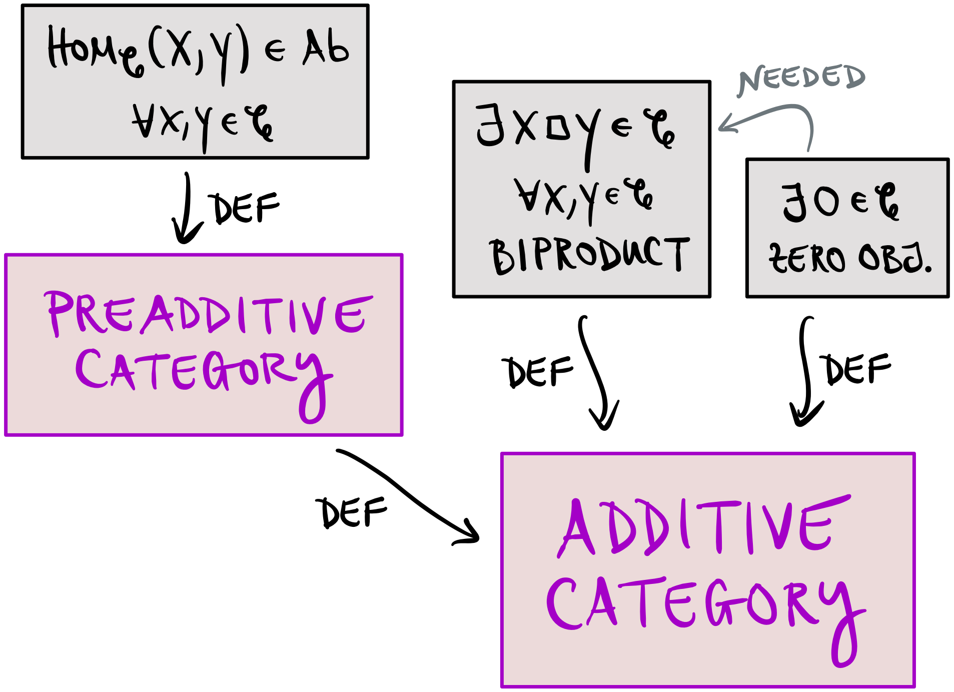
DEF

DEF

DEF

ADDITIVE CATEGORY

II. ABELIAN CATEGORIES : ADDITIVE CATEGORIES



$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in Ab$ $\forall X,Y \in \mathcal{C}$

DEF

PREADDITIVE CATEGORY

$\exists X\square Y \in \mathcal{C}$ $\forall X,Y\in\mathcal{C}$ BIPRODUCT

NEEDED

$\exists 0 \in \mathcal{C}$ ZERO OBJ.

DEF

DEF

DEF

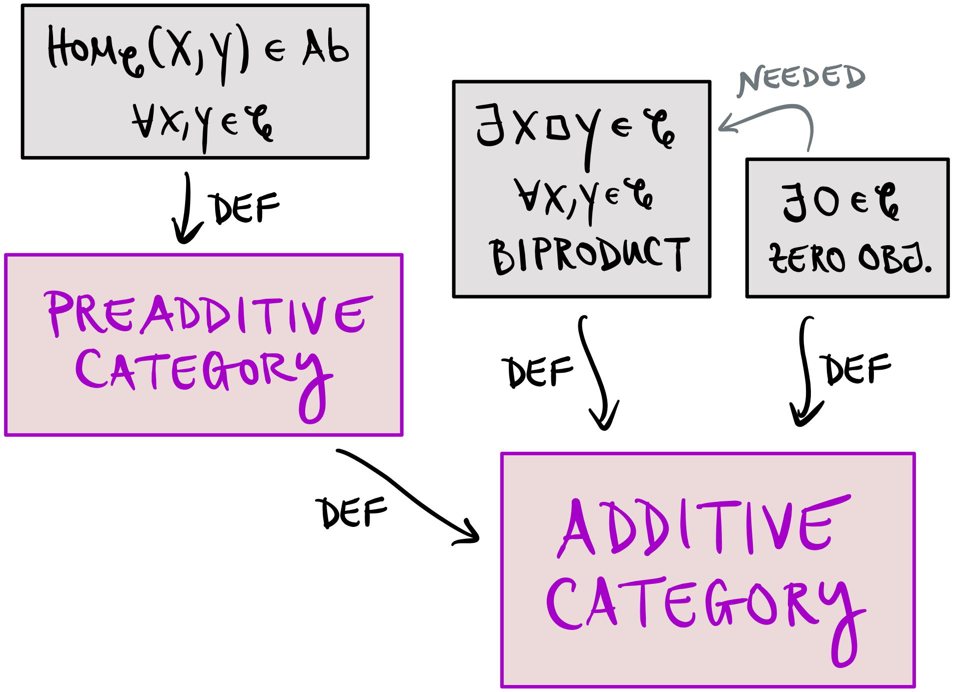
ADDITIVE CATEGORY

TAKE $\mathcal{C}$ A CATEGORY WITH ZERO OBJECT

A BIPRODUCT OF $X,Y\in\mathcal{C}$ IS A TUPLE

$$\Big(\underset{\substack{\text{OBJECT}\\ \text{IN }\mathcal{C}}}{X\square Y},\ \overbrace{\underset{\substack{\vdots\\ X\\ \downarrow\\ X\square Y}}{\alpha_X},\ \underset{\substack{\vdots\\ Y\\ \downarrow\\ X\square Y}}{\alpha_Y},\ \underset{\substack{\vdots\\ X\square Y\\ \downarrow\\ X}}{\alpha'_X},\ \underset{\substack{\vdots\\ X\square Y\\ \downarrow\\ Y}}{\alpha'_Y}}^{\text{MORPHISMS}}\Big)$$

II. ABELIAN CATEGORIES : ADDITIVE CATEGORIES



$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Ab}$ $\forall X,Y \in \mathcal{C}$

DEF

PREADDITIVE CATEGORY

$\exists X \square Y \in \mathcal{C}$ $\forall X,Y \in \mathcal{C}$ BIPRODUCT

NEEDED

$\exists 0 \in \mathcal{C}$ ZERO OBJ.

DEF

DEF

DEF

ADDITIVE CATEGORY

TAKE $\mathcal{C}$ A CATEGORY WITH ZERO OBJECT

A **BIPRODUCT** OF $X,Y \in \mathcal{C}$ IS A TUPLE

$$\Big(\underset{\text{OBJECT IN }\mathcal{C}}{X \square Y},\ \overbrace{\underset{X \to X\square Y}{\alpha_X},\ \underset{Y \to X\square Y}{\alpha_Y},\ \underset{X\square Y \to X}{\alpha'_X},\ \underset{X\square Y \to Y}{\alpha'_Y}}^{\text{MORPHISMS}}\Big)$$

SUCH THAT

$$(X \square Y, \alpha_X, \alpha_Y) = \underset{\text{COPRODUCT}}{X \sqcup Y}$$

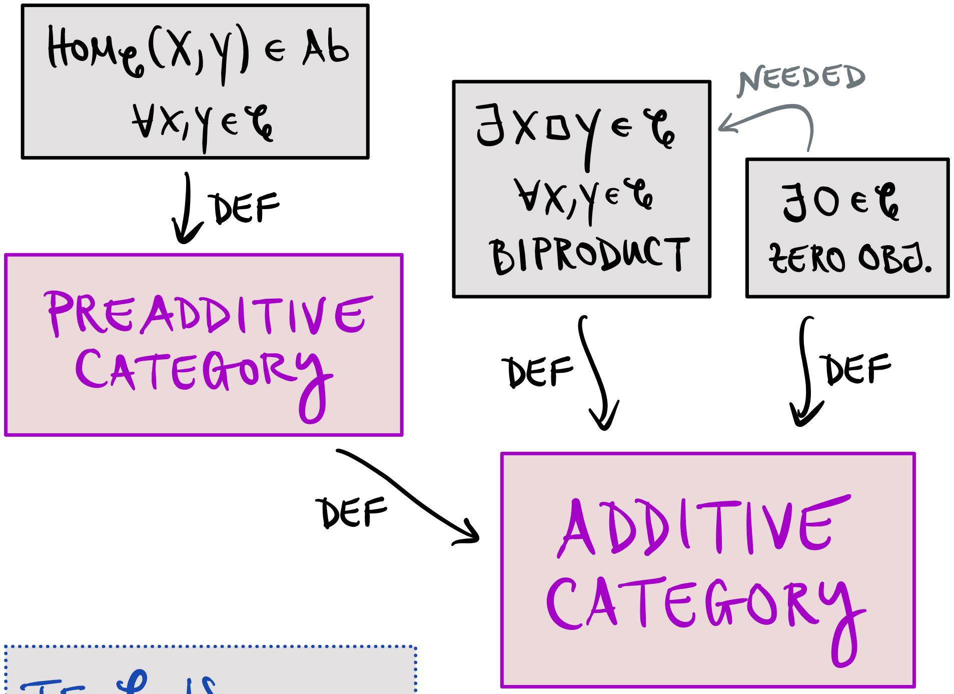
$$(X \square Y, \alpha'_X, \alpha'_Y) = \underset{\text{PRODUCT}}{X \sqcap Y}$$

&

$$\alpha'_X \alpha_X = \mathrm{id}_X, \quad \alpha'_Y \alpha_Y = \mathrm{id}_Y$$

$$\alpha'_Y \alpha_X = \vec{0}_{X,Y}, \quad \alpha'_X \alpha_Y = \vec{0}_{Y,X}$$

II. ABELIAN CATEGORIES : ADDITIVE CATEGORIES



TAKE $\mathcal{C}$ A CATEGORY WITH ZERO OBJECT

A **BIPRODUCT** OF $X,Y \in \mathcal{C}$ IS A TUPLE

$$(X \square Y, \overbrace{\alpha_X, \alpha_Y, \alpha'_X, \alpha'_Y}^{\text{MORPHISMS}})$$

$X \square Y$: OBJECT IN $\mathcal{C}$

$\alpha_X : X \to X \square Y$

$\alpha_Y : Y \to X \square Y$

$\alpha'_X : X \square Y \to X$

$\alpha'_Y : X \square Y \to Y$

SUCH THAT

① $(X \square Y, \alpha_X, \alpha_Y) = X \sqcup Y$ COPRODUCT

② $(X \square Y, \alpha'_X, \alpha'_Y) = X \sqcap Y$ PRODUCT

&

★ $\begin{cases} \alpha'_X \alpha_X = \mathrm{id}_X, & \alpha'_Y \alpha_Y = \mathrm{id}_Y \\ \alpha'_Y \alpha_X = \vec{0}_{X,Y}, & \alpha'_X \alpha_Y = \vec{0}_{Y,X} \end{cases}$

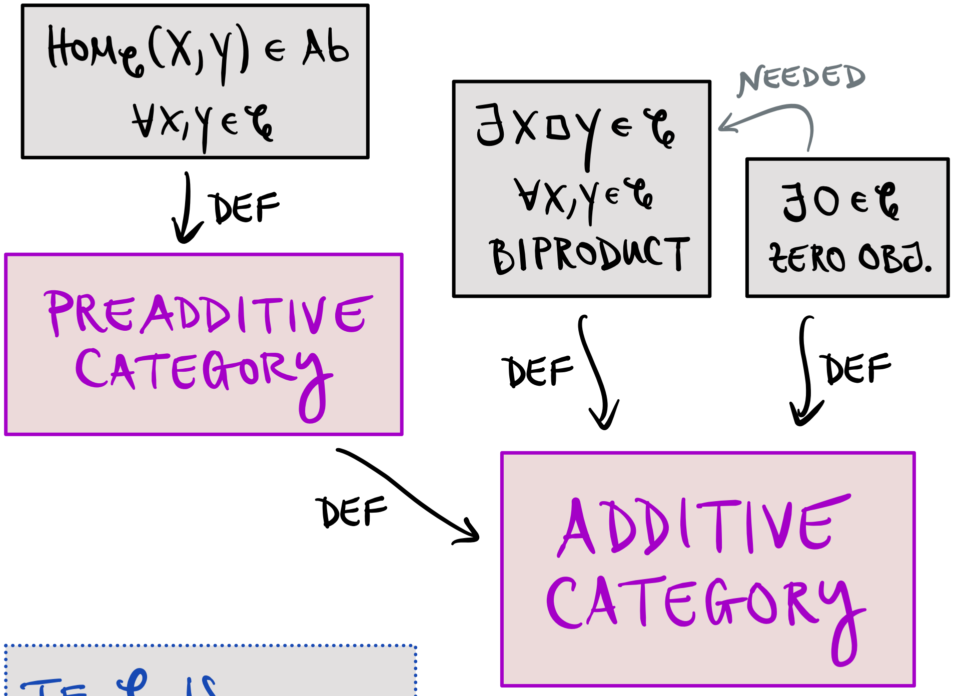
IF $\mathcal{C}$ IS PREADDITIVE, THEN ★ IMPLIES THAT TFAE:

①

②

&③ $\alpha_X \alpha'_X + \alpha_Y \alpha'_Y = \mathrm{id}_{X \square Y}$

II. ABELIAN CATEGORIES : ADDITIVE CATEGORIES



$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Ab}$ $\forall X,Y \in \mathcal{C}$

DEF

PREADDITIVE CATEGORY

$\exists X\square Y \in \mathcal{C}$ $\forall X,Y\in\mathcal{C}$ BIPRODUCT

NEEDED

$\exists 0 \in \mathcal{C}$ ZERO OBJ.

DEF

DEF

DEF

ADDITIVE CATEGORY

TAKE $\mathcal{C}$ A CATEGORY WITH ZERO OBJECT

A **BIPRODUCT** OF $X,Y\in\mathcal{C}$ IS A TUPLE

$$(X\square Y, \overbrace{\alpha_X, \alpha_Y, \alpha'_X, \alpha'_Y}^{\text{MORPHISMS}})$$

OBJECT IN $\mathcal{C}$

$\alpha_X : X \to X\square Y$, $\alpha_Y : Y \to X\square Y$, $\alpha'_X : X\square Y \to X$, $\alpha'_Y : X\square Y \to Y$

SUCH THAT

① $(X\square Y, \alpha_X, \alpha_Y) = X \sqcup Y$ COPRODUCT

② $(X\square Y, \alpha'_X, \alpha'_Y) = X \sqcap Y$ PRODUCT

&

★ $\begin{cases} \alpha'_X\alpha_X = \mathrm{id}_X, & \alpha'_Y\alpha_Y = \mathrm{id}_Y \\ \alpha'_Y\alpha_X = \vec{0}_{X,Y}, & \alpha'_X\alpha_Y = \vec{0}_{Y,X} \end{cases}$

IF $\mathcal{C}$ IS PREADDITIVE, THEN ★ IMPLIES THAT TFAE:

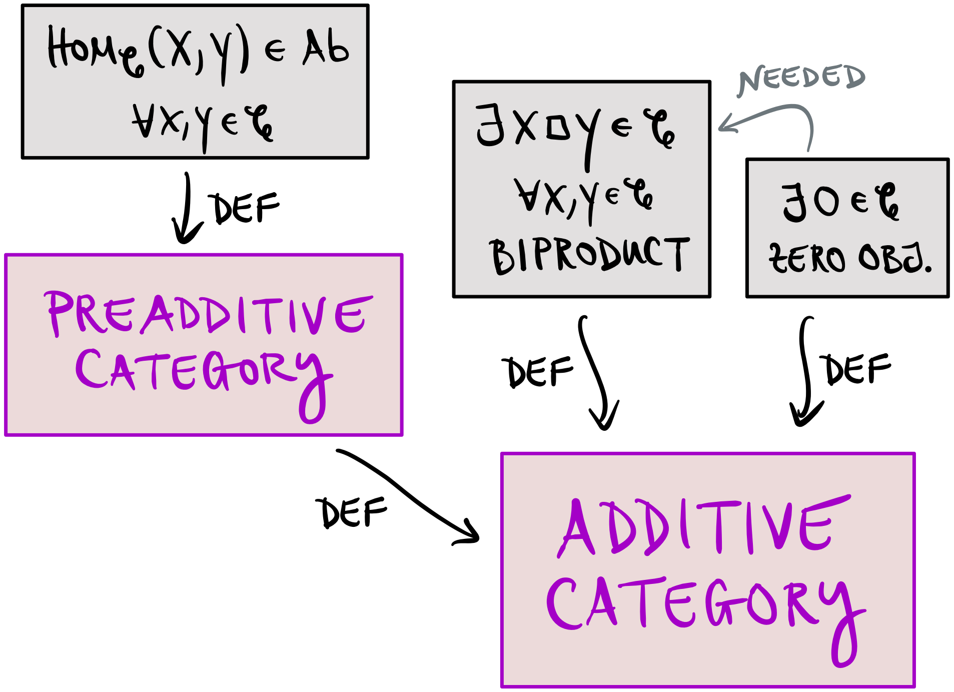
①

②

& ③ $\alpha_X\alpha'_X + \alpha_Y\alpha'_Y = \mathrm{id}_{X\square Y}$

A CATEGORY $\mathcal{C}$ IS **ADDITIVE** IF IT IS PREADDITIVE, $\exists$ ZERO OBJECT $\in\mathcal{C}$ & $\exists X\square Y \in \mathcal{C}$ $\forall X,Y\in\mathcal{C}$.

II. ABELIAN CATEGORIES : ADDITIVE CATEGORIES



$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Ab}$ $\forall X,Y \in \mathcal{C}$

DEF

PREADDITIVE CATEGORY

$\exists X \square Y \in \mathcal{C}$ $\forall X,Y \in \mathcal{C}$ BIPRODUCT

NEEDED

$\exists 0 \in \mathcal{C}$ ZERO OBJ.

DEF

DEF

DEF

ADDITIVE CATEGORY

TAKE $\mathcal{C}$ A CATEGORY WITH ZERO OBJECT

A **BIPRODUCT** OF $X,Y \in \mathcal{C}$ IS A TUPLE

$$(X \square Y, \alpha_X, \alpha_Y, \alpha'_X, \alpha'_Y) \ni.$$

$$(X \square Y, \alpha_X, \alpha_Y) = X \sqcup Y$$

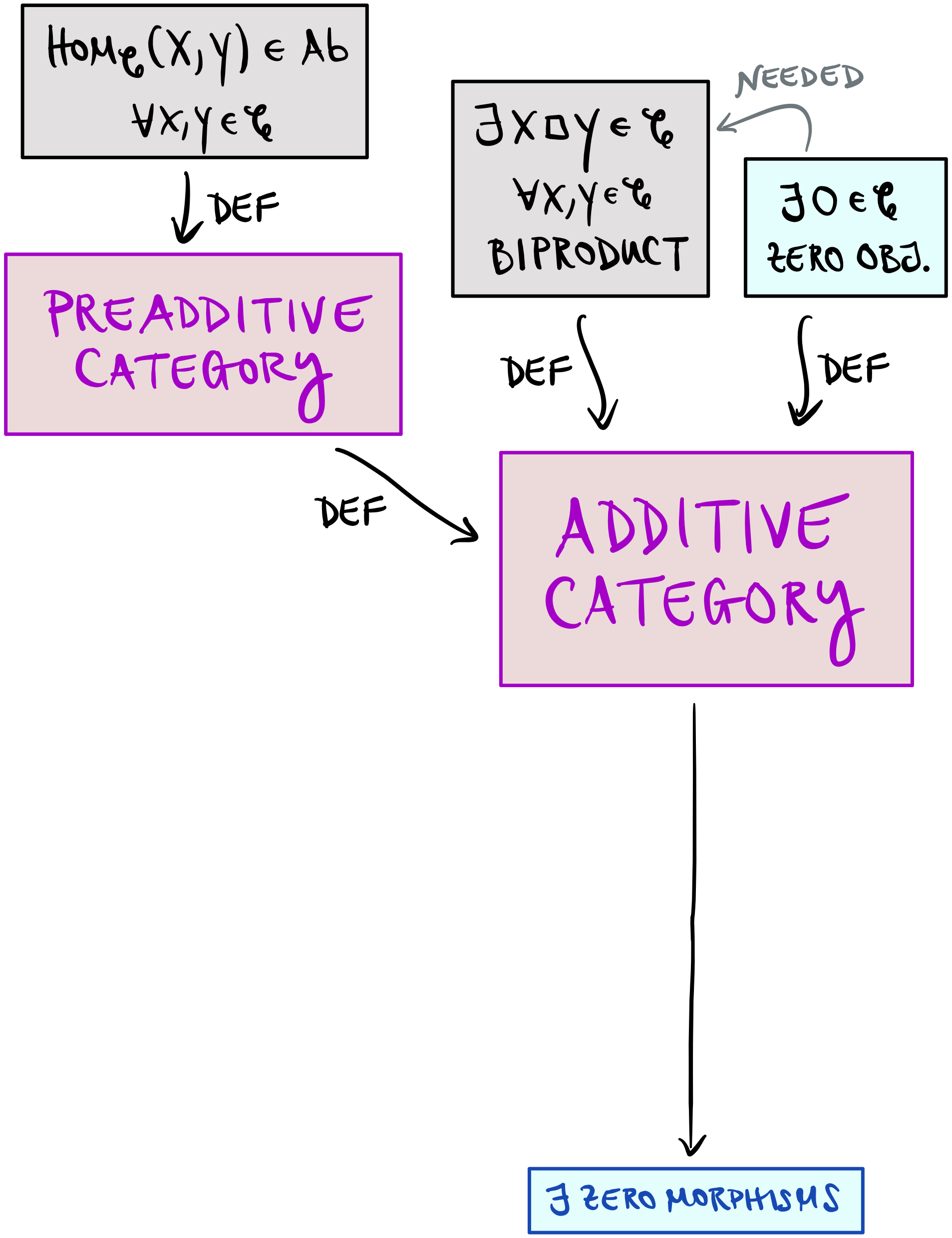
$$(X \square Y, \alpha'_X, \alpha'_Y) = X \sqcap Y$$

&

$$\alpha'_X \alpha_X = \mathrm{id}_X, \quad \alpha'_Y \alpha_Y = \mathrm{id}_Y$$

$$\alpha'_Y \alpha_X = \vec{0}_{X,Y}, \quad \alpha'_X \alpha_Y = \vec{0}_{Y,X}$$

II. ABELIAN CATEGORIES : ADDITIVE CATEGORIES



$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in Ab$
$\forall X,Y \in \mathcal{C}$

DEF

PREADDITIVE CATEGORY

$\exists X \square Y \in \mathcal{C}$
$\forall X,Y \in \mathcal{C}$
BIPRODUCT

NEEDED

$\exists 0 \in \mathcal{C}$
ZERO OBJ.

DEF

DEF

DEF

ADDITIVE CATEGORY

$\exists$ ZERO MORPHISMS

TAKE $\mathcal{C}$ A CATEGORY
WITH ZERO OBJECT

A **BIPRODUCT** OF $X,Y \in \mathcal{C}$
IS A TUPLE

$$(X \square Y, \alpha_X, \alpha_Y, \alpha'_X, \alpha'_Y)$$

$\ni.$

$$(X \square Y, \alpha_X, \alpha_Y) = X \sqcup Y$$

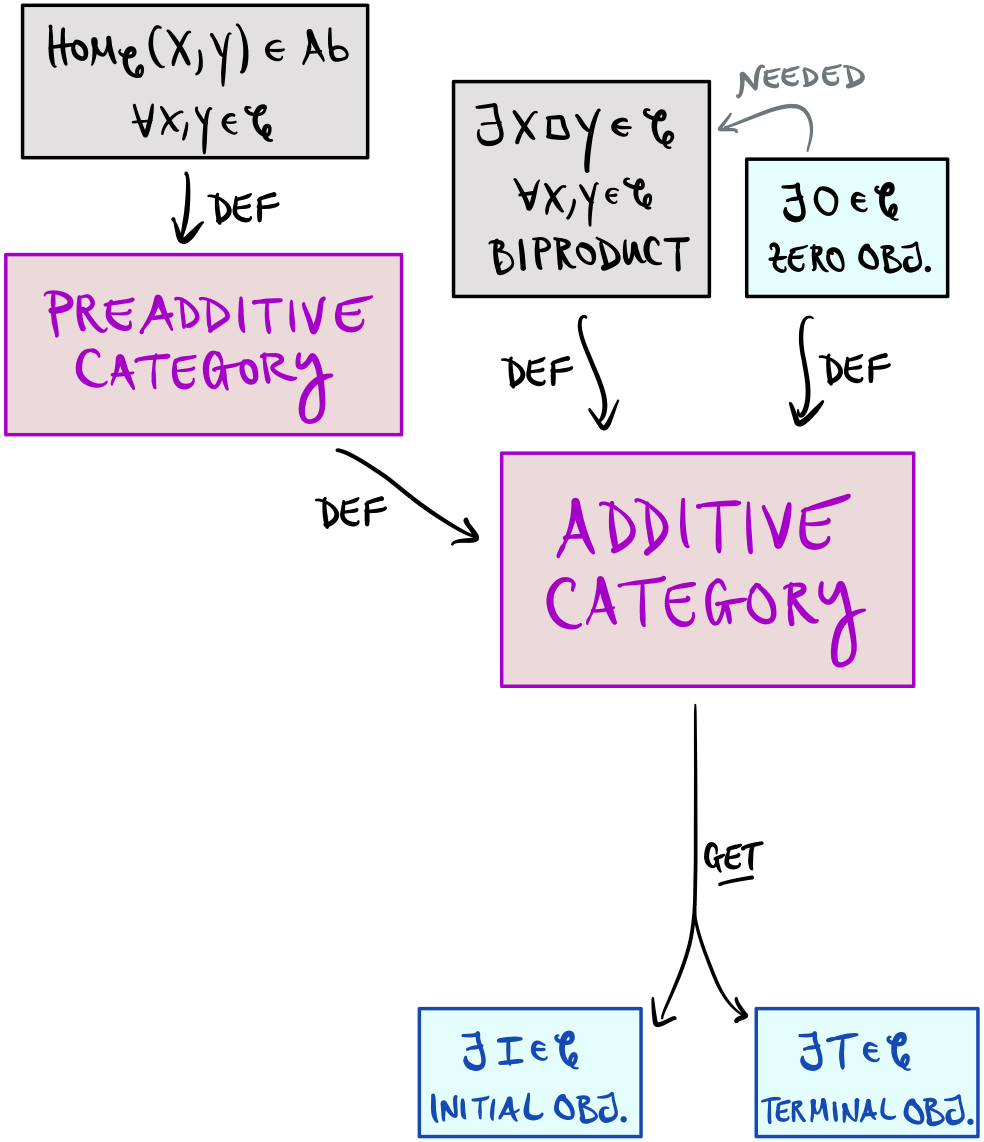
$$(X \square Y, \alpha'_X, \alpha'_Y) = X \sqcap Y$$

&

$$\alpha'_X \alpha_X = id_X, \quad \alpha'_Y \alpha_Y = id_Y$$

$$\alpha'_Y \alpha_X = \vec{0}_{X,Y}, \quad \alpha'_X \alpha_Y = \vec{0}_{Y,X}$$

II. ABELIAN CATEGORIES : ADDITIVE CATEGORIES



TAKE $\mathcal{C}$ A CATEGORY WITH ZERO OBJECT

A **BIPRODUCT** OF $X,Y \in \mathcal{C}$ IS A TUPLE

$$(X \square Y, \alpha_X, \alpha_Y, \alpha'_X, \alpha'_Y)$$

$\ni$.

$$(X \square Y, \alpha_X, \alpha_Y) = X \sqcup Y$$

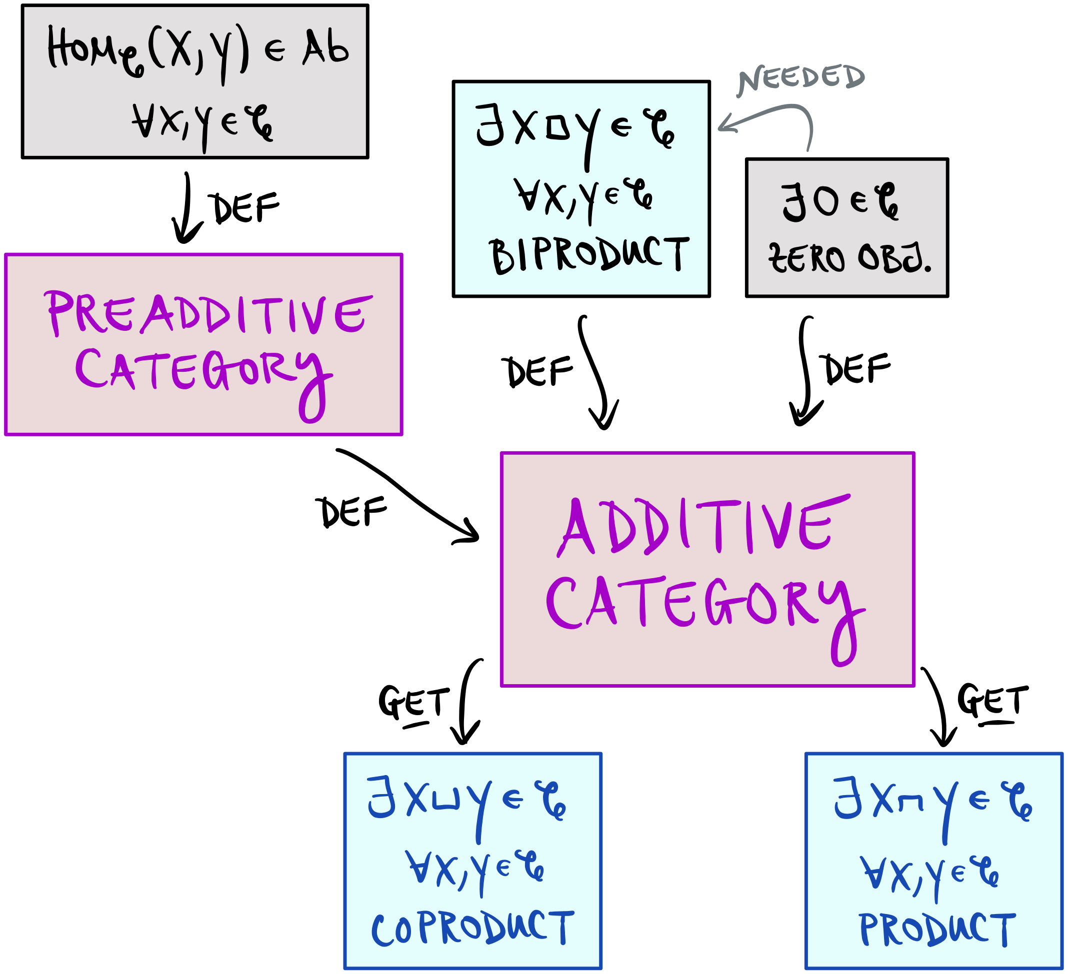
$$(X \square Y, \alpha'_X, \alpha'_Y) = X \sqcap Y$$

&

$$\alpha'_X \alpha_X = id_X, \quad \alpha'_Y \alpha_Y = id_Y$$

$$\alpha'_Y \alpha_X = \vec{0}_{X,Y}, \quad \alpha'_X \alpha_Y = \vec{0}_{Y,X}$$

II. ABELIAN CATEGORIES : ADDITIVE CATEGORIES



$\mathrm{HOM}_{\mathscr{C}}(X,Y) \in Ab$ $\forall X,Y \in \mathscr{C}$

DEF

PREADDITIVE CATEGORY

$\exists X \square Y \in \mathscr{C}$ $\forall X,Y \in \mathscr{C}$ BIPRODUCT

NEEDED

$\exists 0 \in \mathscr{C}$ ZERO OBJ.

DEF

DEF

DEF

ADDITIVE CATEGORY

GET

GET

$\exists X \sqcup Y \in \mathscr{C}$ $\forall X,Y \in \mathscr{C}$ COPRODUCT

$\exists X \sqcap Y \in \mathscr{C}$ $\forall X,Y \in \mathscr{C}$ PRODUCT

TAKE $\mathscr{C}$ A CATEGORY WITH ZERO OBJECT

A BIPRODUCT OF $X,Y \in \mathscr{C}$ IS A TUPLE

$$(X \square Y, \alpha_X, \alpha_Y, \alpha'_X, \alpha'_Y)$$

∋.

$$(X \square Y, \alpha_X, \alpha_Y) = X \sqcup Y$$

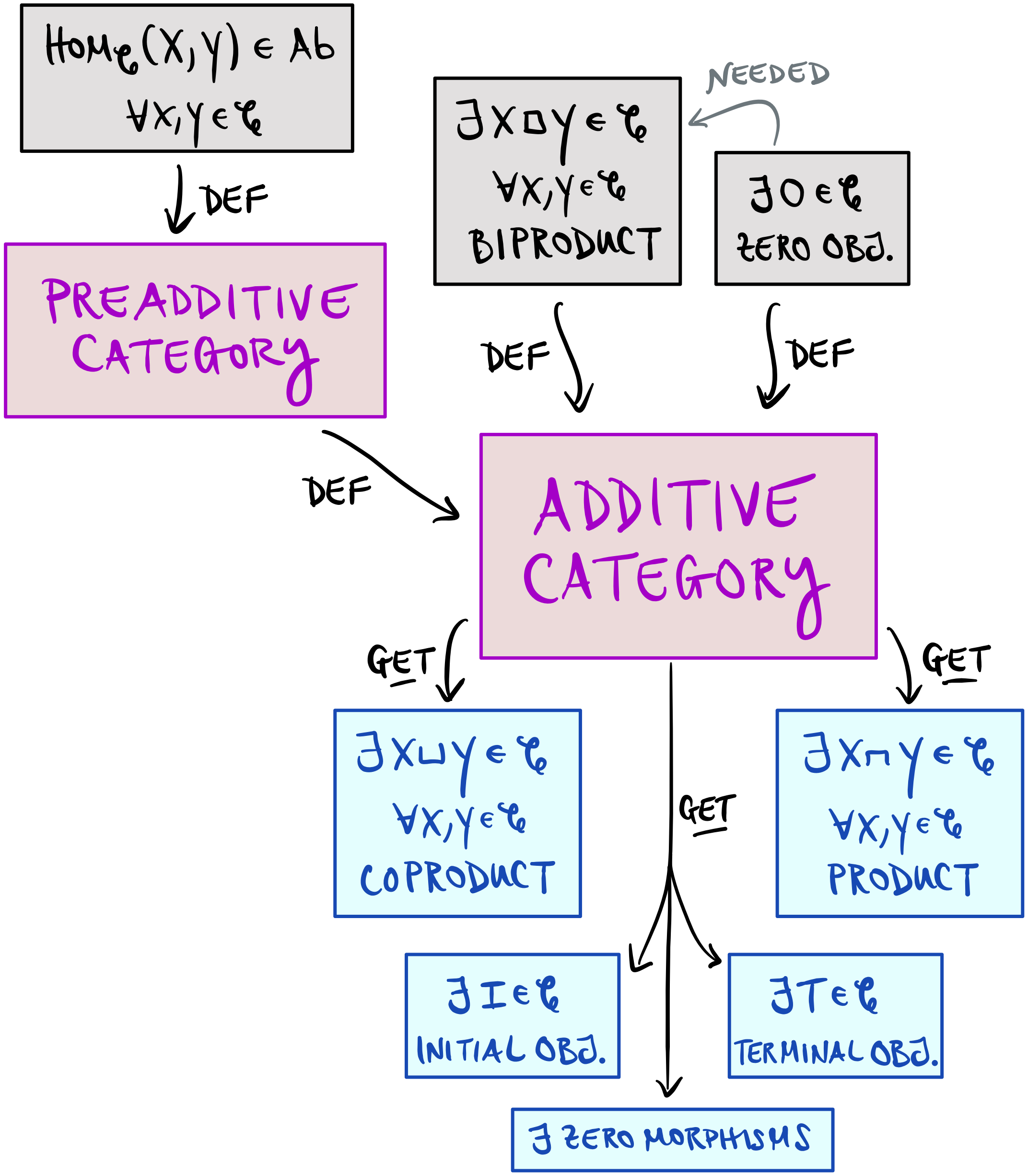
$$(X \square Y, \alpha'_X, \alpha'_Y) = X \sqcap Y$$

&

$$\alpha'_X \alpha_X = id_X, \quad \alpha'_Y \alpha_Y = id_Y$$

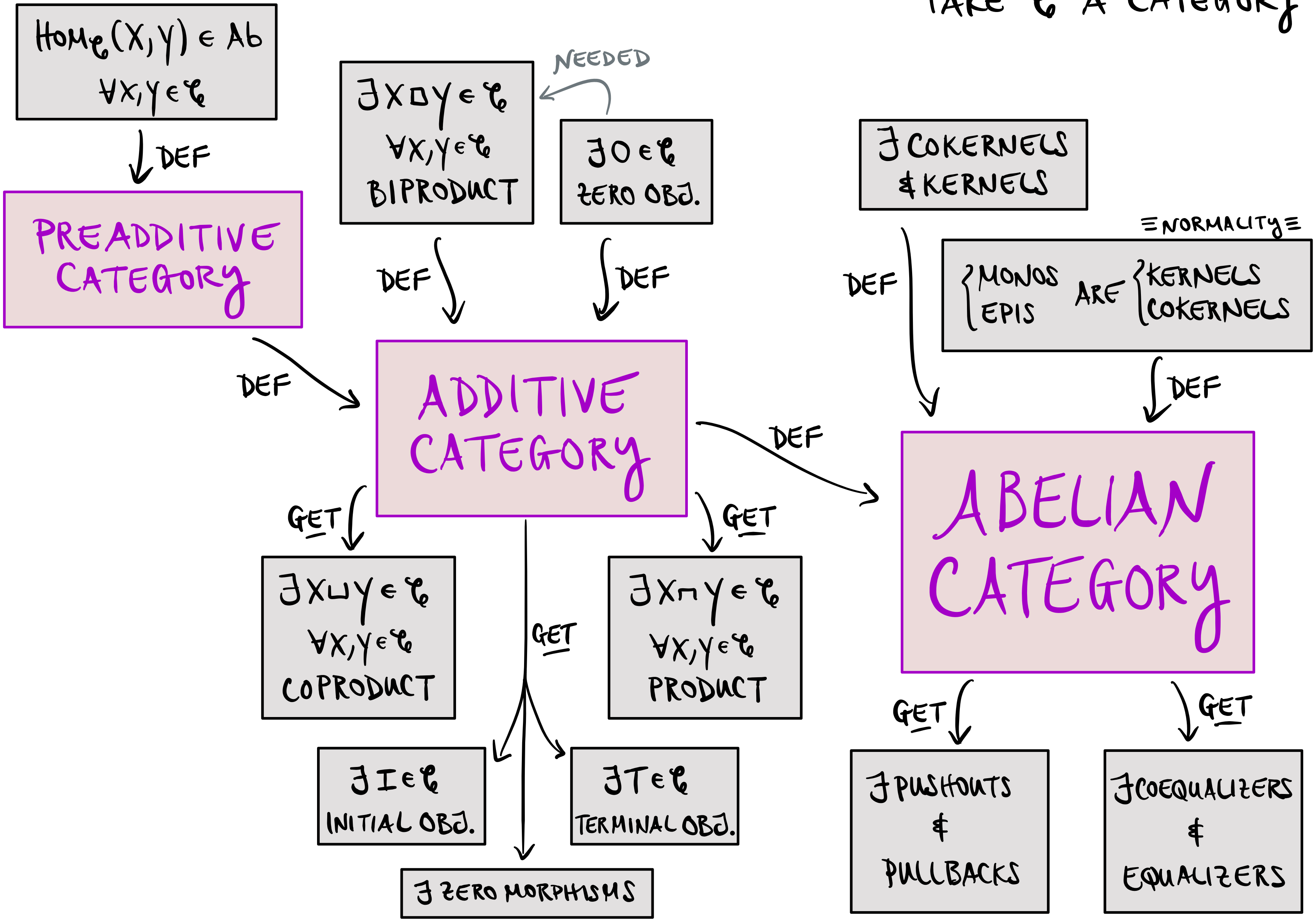
$$\alpha'_Y \alpha_X = \vec{0}_{X,Y}, \quad \alpha'_X \alpha_Y = \vec{0}_{Y,X}$$

II. ABELIAN CATEGORIES : ADDITIVE CATEGORIES



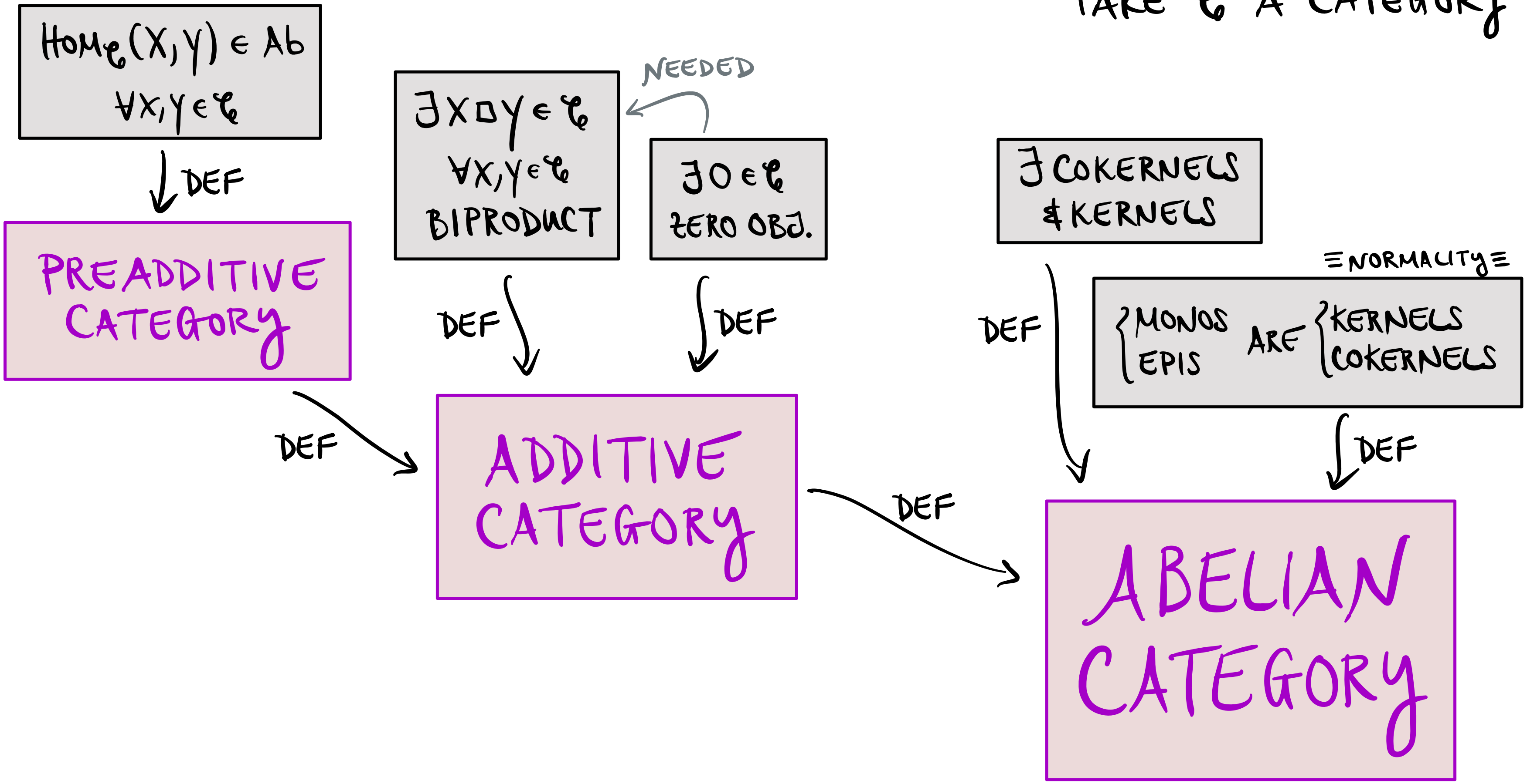
II. ABELIAN CATEGORIES

TAKE $\mathcal{C}$ A CATEGORY



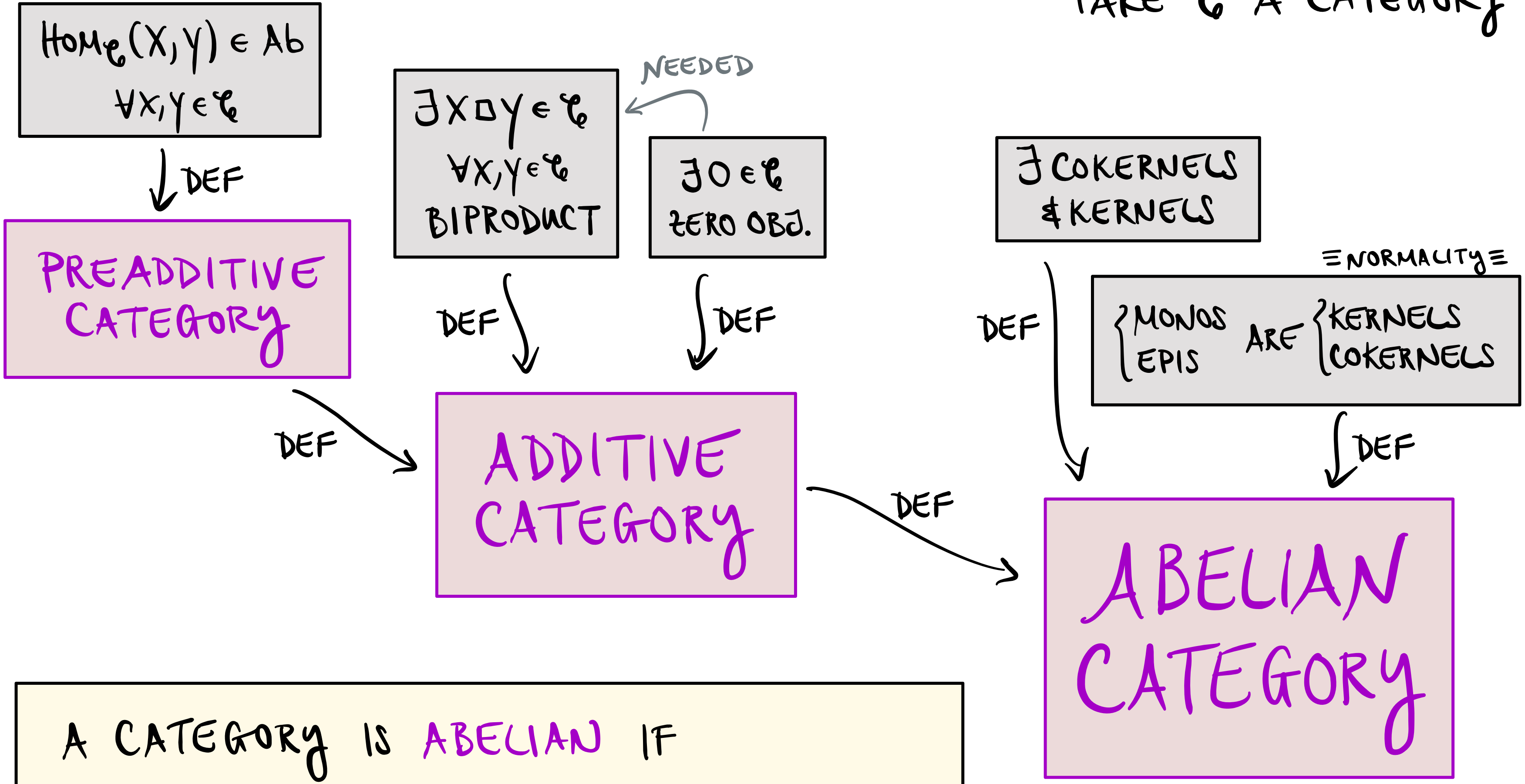
II. ABELIAN CATEGORIES

TAKE $\mathcal{C}$ A CATEGORY



II. ABELIAN CATEGORIES

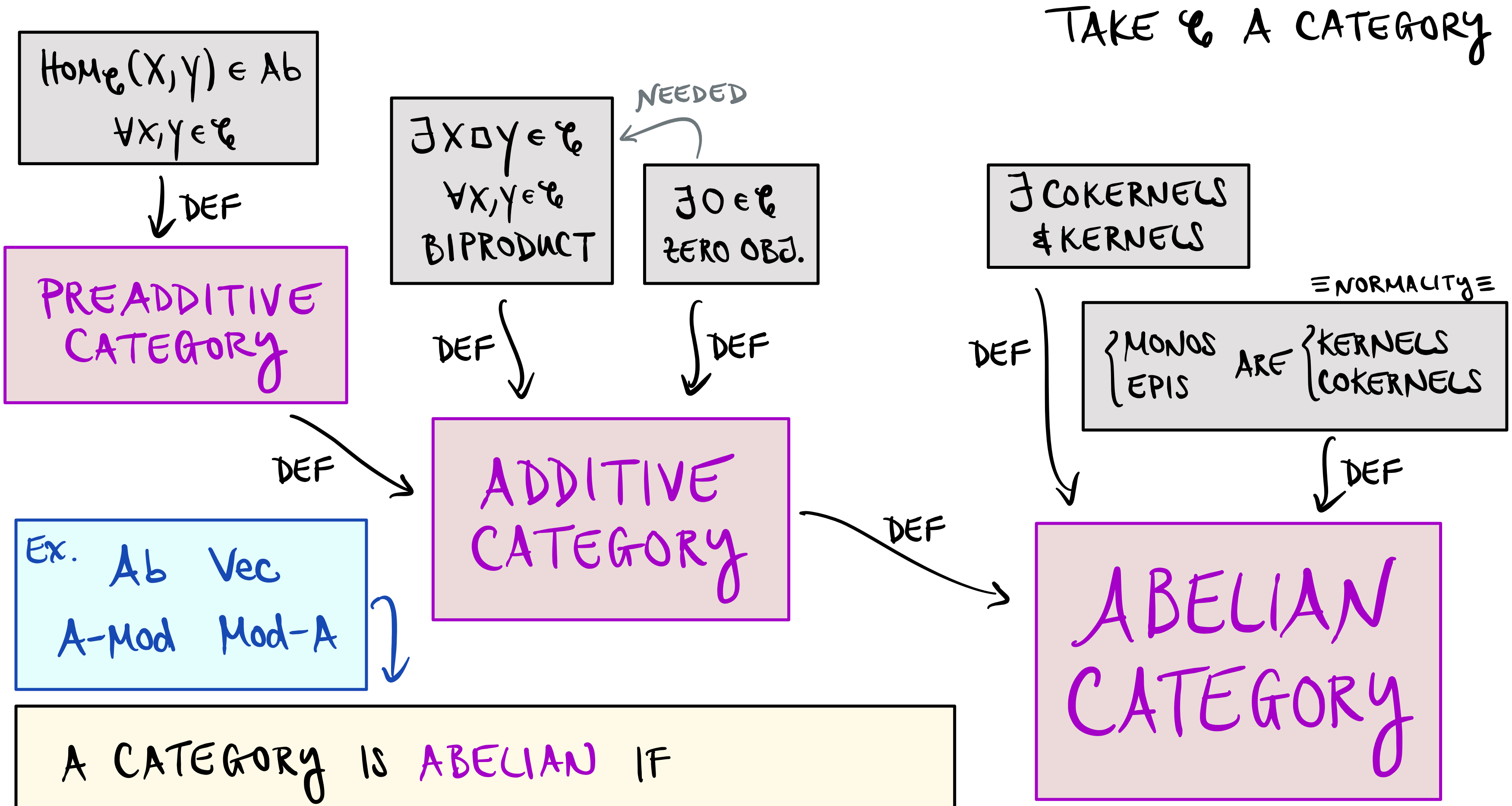
TAKE $\mathcal{C}$ A CATEGORY



A CATEGORY IS ABELIAN IF

- IT IS ADDITIVE
- ∃ COKERNEL & KERNEL $\forall$ MORPHISM $f \in \mathcal{C}$
  coker(f) ker(f)
- ALL MONOS & EPIS ARE NORMAL

II. ABELIAN CATEGORIES



TAKE $\mathcal{C}$ A CATEGORY

$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in Ab$ $\forall X,Y \in \mathcal{C}$

DEF

PREADDITIVE CATEGORY

$\exists X \oplus Y \in \mathcal{C}$ $\forall X,Y \in \mathcal{C}$ BIPRODUCT

NEEDED

$\exists 0 \in \mathcal{C}$ ZERO OBJ.

DEF

DEF

DEF

ADDITIVE CATEGORY

$\exists$ COKERNELS & KERNELS

=NORMALITY=

$\left\{\begin{array}{l}\text{MONOS}\\ \text{EPIS}\end{array}\right.$ ARE $\left\{\begin{array}{l}\text{KERNELS}\\ \text{COKERNELS}\end{array}\right.$

DEF

DEF

DEF

ABELIAN CATEGORY

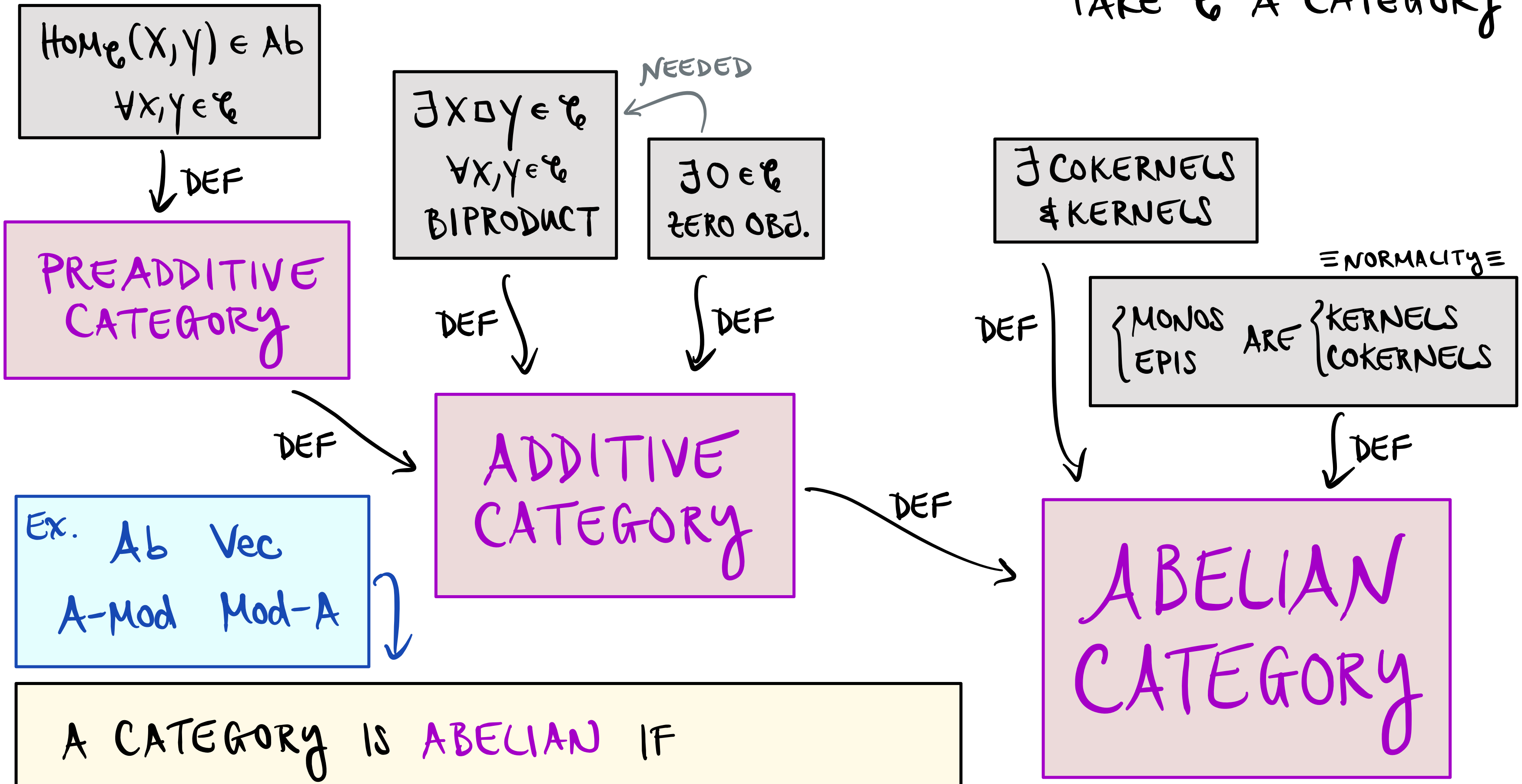
Ex. Ab Vec A-Mod Mod-A

A CATEGORY IS ABELIAN IF

- IT IS ADDITIVE
- $\exists$ $\underset{\mathrm{coker}(f)}{\text{COKERNEL}}$ & $\underset{\ker(f)}{\text{KERNEL}}$ $\forall$ MORPHISM $f \in \mathcal{C}$
- ALL MONOS & EPIS ARE NORMAL

II. ABELIAN CATEGORIES

TAKE $\mathcal{C}$ A CATEGORY



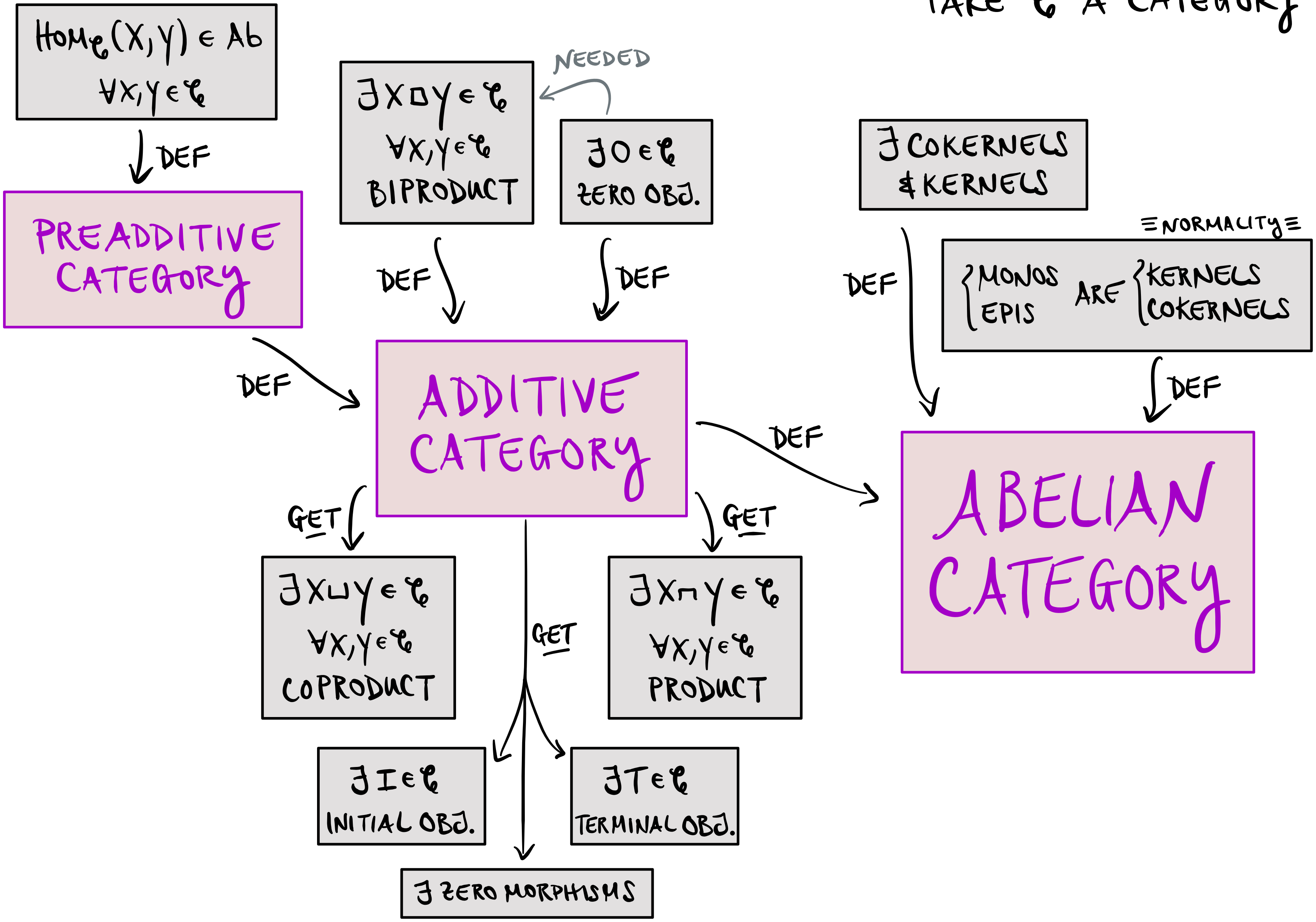
A CATEGORY IS ABELIAN IF

- IT IS ADDITIVE
- $\exists$ COKERNEL & KERNEL $\forall$ MORPHISM $f \in \mathcal{C}$
  $\mathrm{coker}(f)$ $\ker(f)$
- ALL MONOS & EPIS ARE NORMAL

NON-EXAMPLES VIA RESULTS/TOOLS LATER

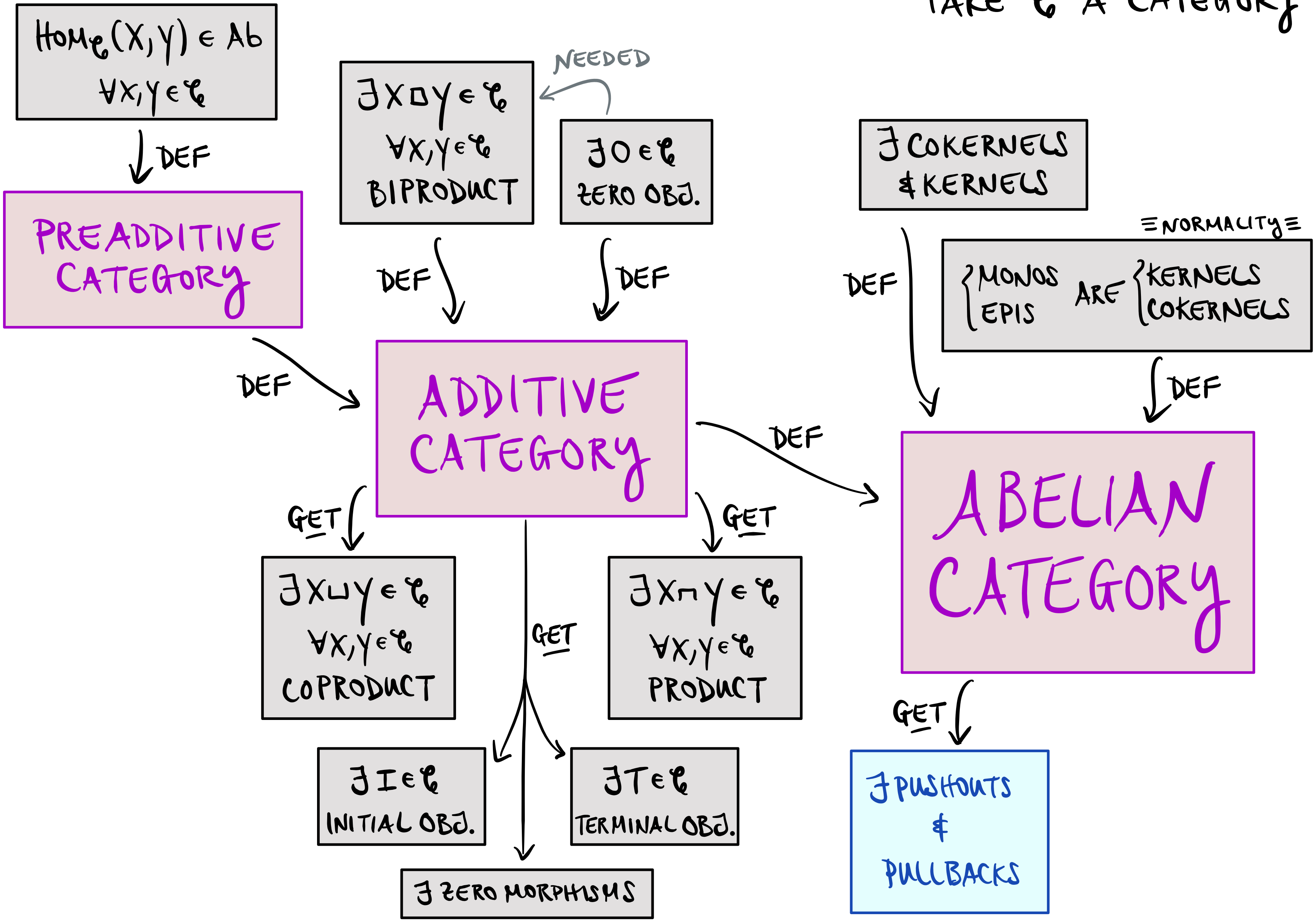
II. ABELIAN CATEGORIES

TAKE $\mathcal{C}$ A CATEGORY

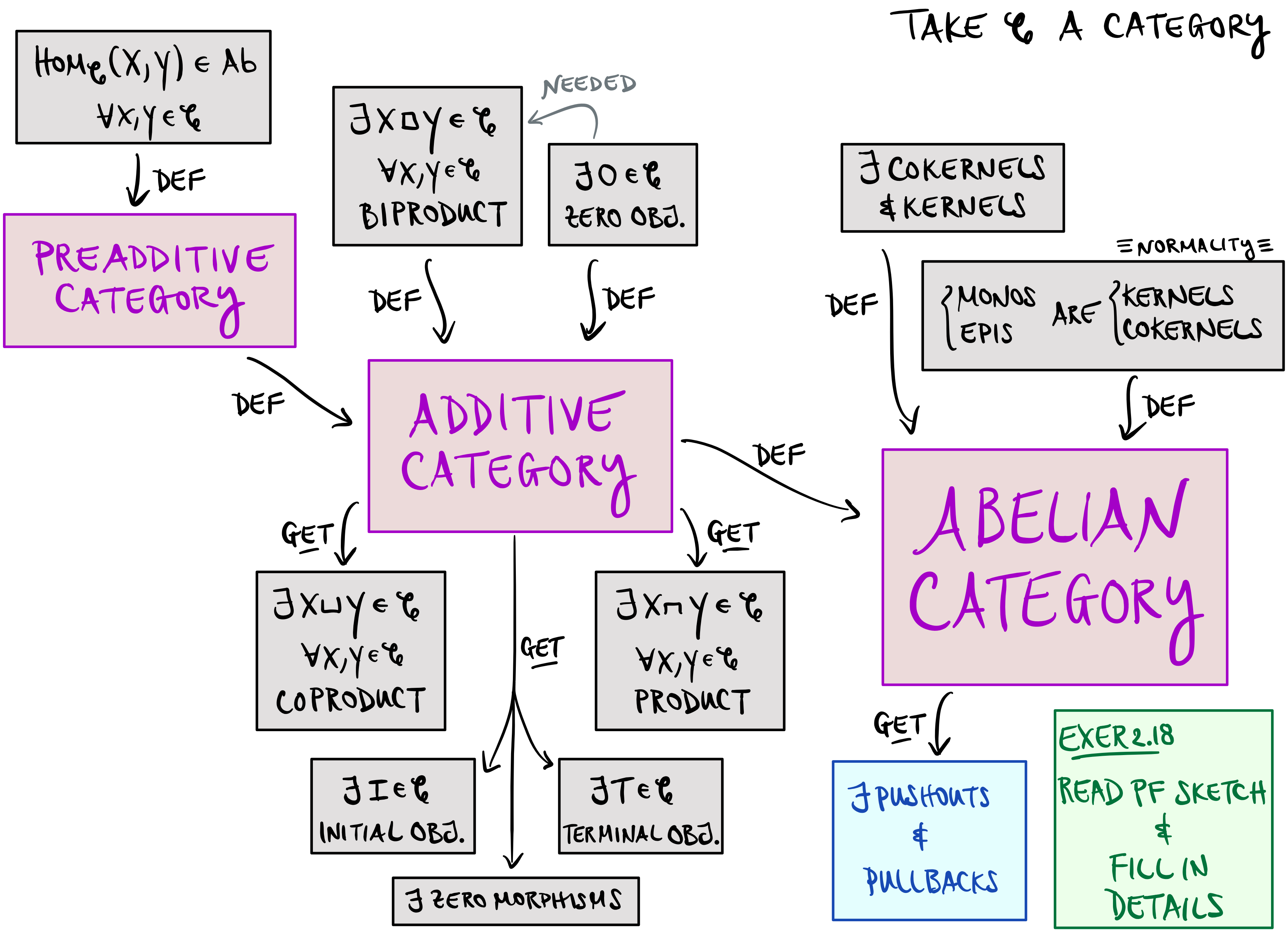


II. ABELIAN CATEGORIES

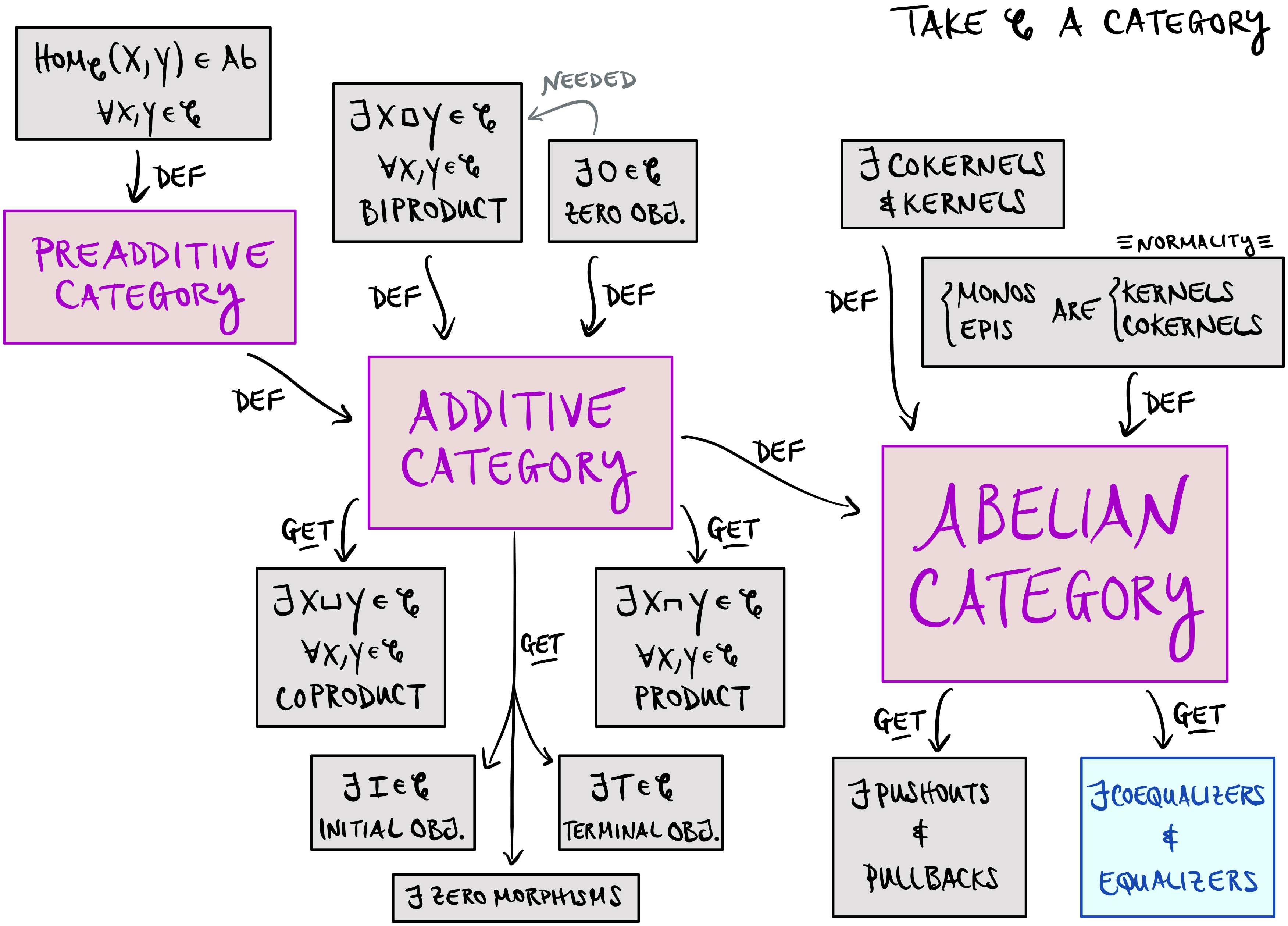
TAKE $\mathcal{C}$ A CATEGORY



II. ABELIAN CATEGORIES



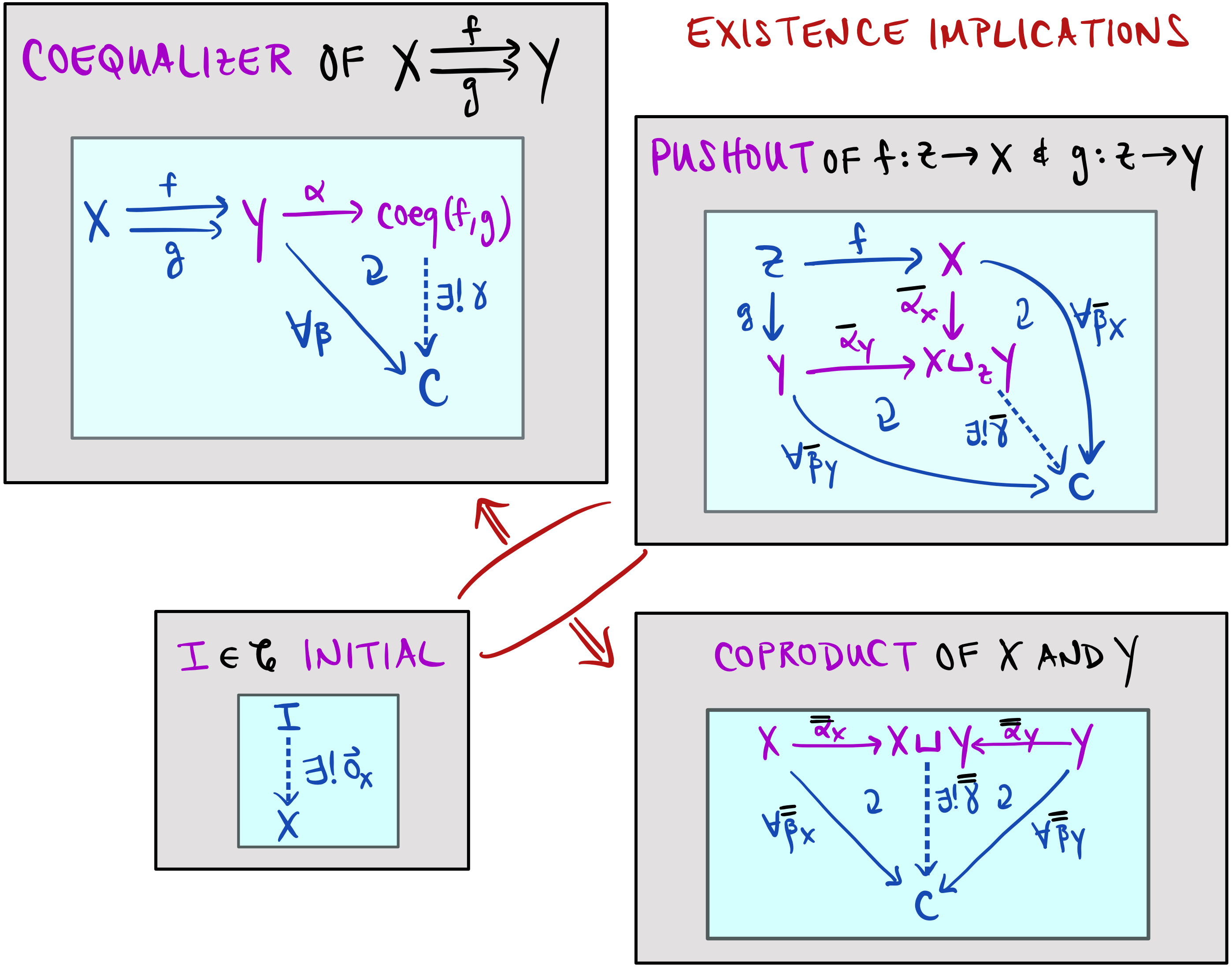
II. ABELIAN CATEGORIES



≡ RECALL ≡

GIVEN A CATEGORY $\mathcal{C}$:

HAVE THE FOLLOWING EXISTENCE IMPLICATIONS



COEQUALIZER OF $X \overset{f}{\underset{g}{\rightrightarrows}} Y$

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\alpha"] \arrow[dr, "\forall \beta"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & \mathrm{coeq}(f,g) \arrow[d, dashed, "\exists! \gamma"] \\
 & & C
\end{tikzcd}$$

PUSHOUT OF $f: Z \to X$ & $g: Z \to Y$

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "\overline{\alpha}_X"'] \arrow[ddr, bend left, "\forall \overline{\beta}_X"] \arrow[ddr, phantom, "\circlearrowleft"] & \\
Y \arrow[r, "\overline{\alpha}_Y"] \arrow[drr, bend right, "\forall \overline{\beta}_Y"'] \arrow[drr, phantom, "\circlearrowleft"] & X \sqcup_Z Y \arrow[dr, dashed, "\exists! \overline{\gamma}"'] & \\
 & & C
\end{tikzcd}$$

$I \in \mathcal{C}$ **INITIAL**

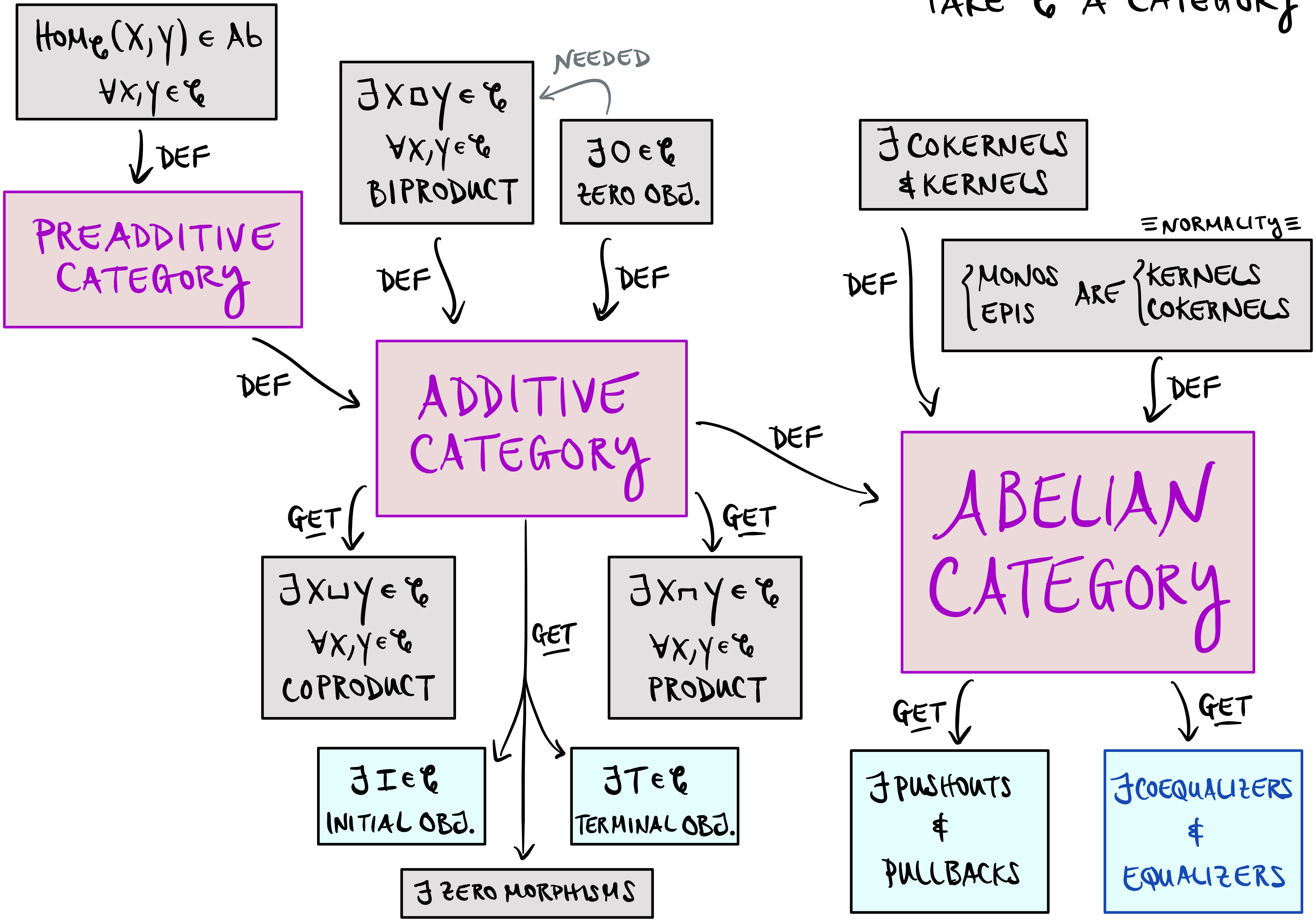
$$\begin{tikzcd}
I \arrow[d, dashed, "\exists! \vec{0}_X"] \\
X
\end{tikzcd}$$

COPRODUCT OF X AND Y

$$\begin{tikzcd}
X \arrow[r, "\overline{\overline{\alpha}}_X"] \arrow[dr, "\forall \overline{\overline{\beta}}_X"'] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] & X \sqcup Y \arrow[d, dashed, "\exists! \overline{\overline{\gamma}}"] & Y \arrow[l, "\overline{\overline{\alpha}}_Y"'] \arrow[dl, "\forall \overline{\overline{\beta}}_Y"] \arrow[dl, phantom, shift right=3, "\circlearrowleft"] \\
 & C &
\end{tikzcd}$$

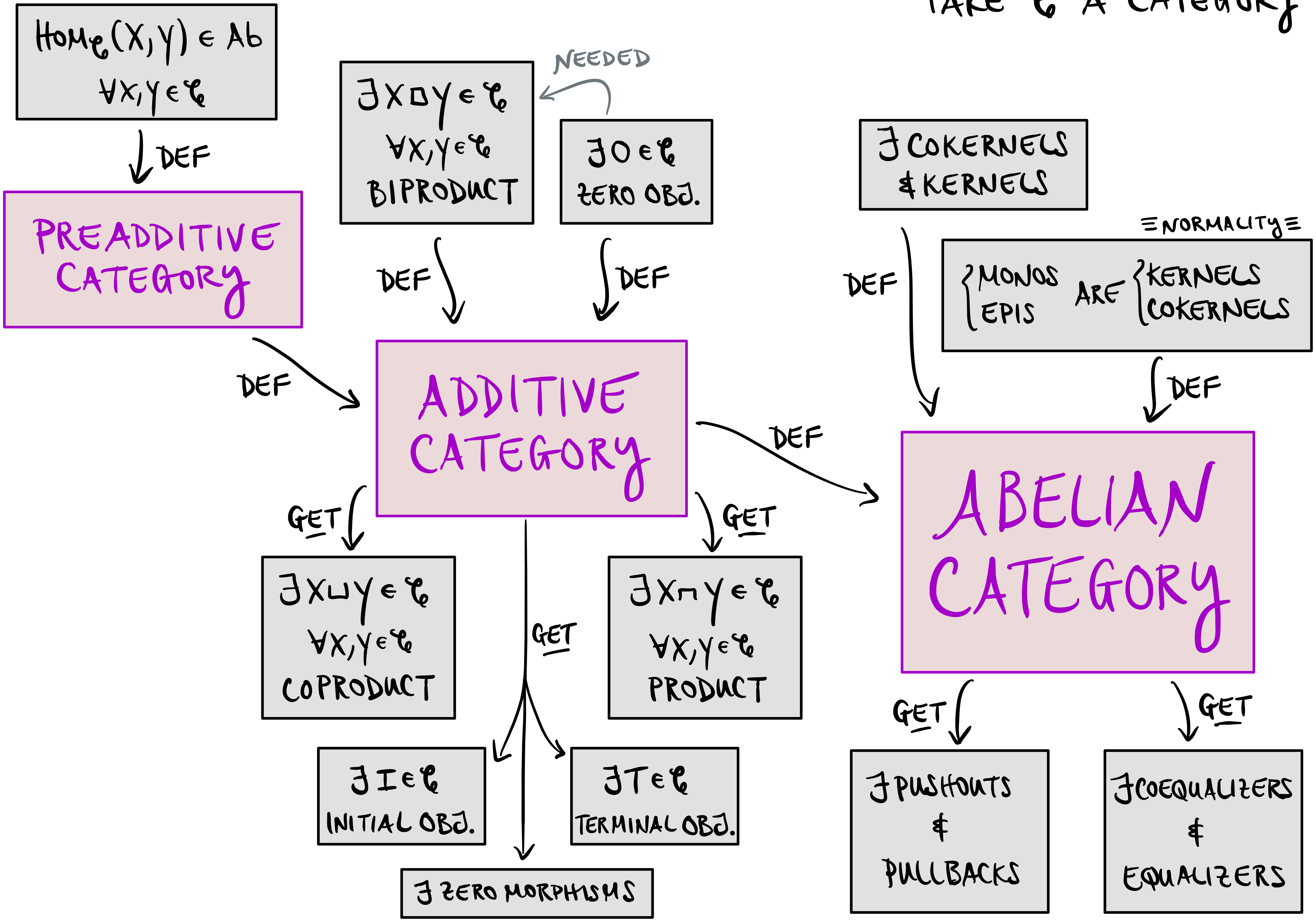
II. ABELIAN CATEGORIES

TAKE $\mathcal{C}$ A CATEGORY



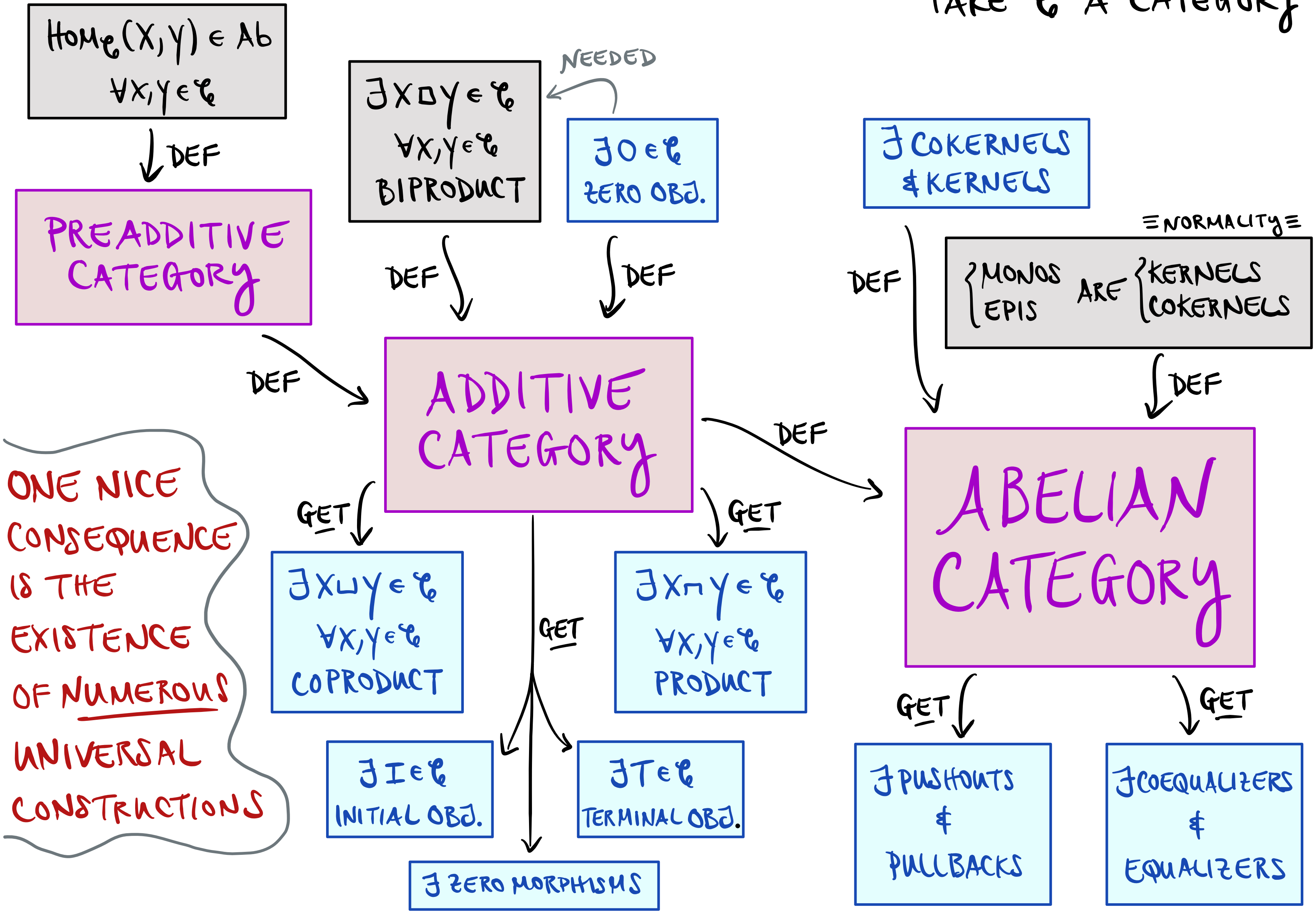
II. ABELIAN CATEGORIES

TAKE $\mathscr{C}$ A CATEGORY



II. ABELIAN CATEGORIES

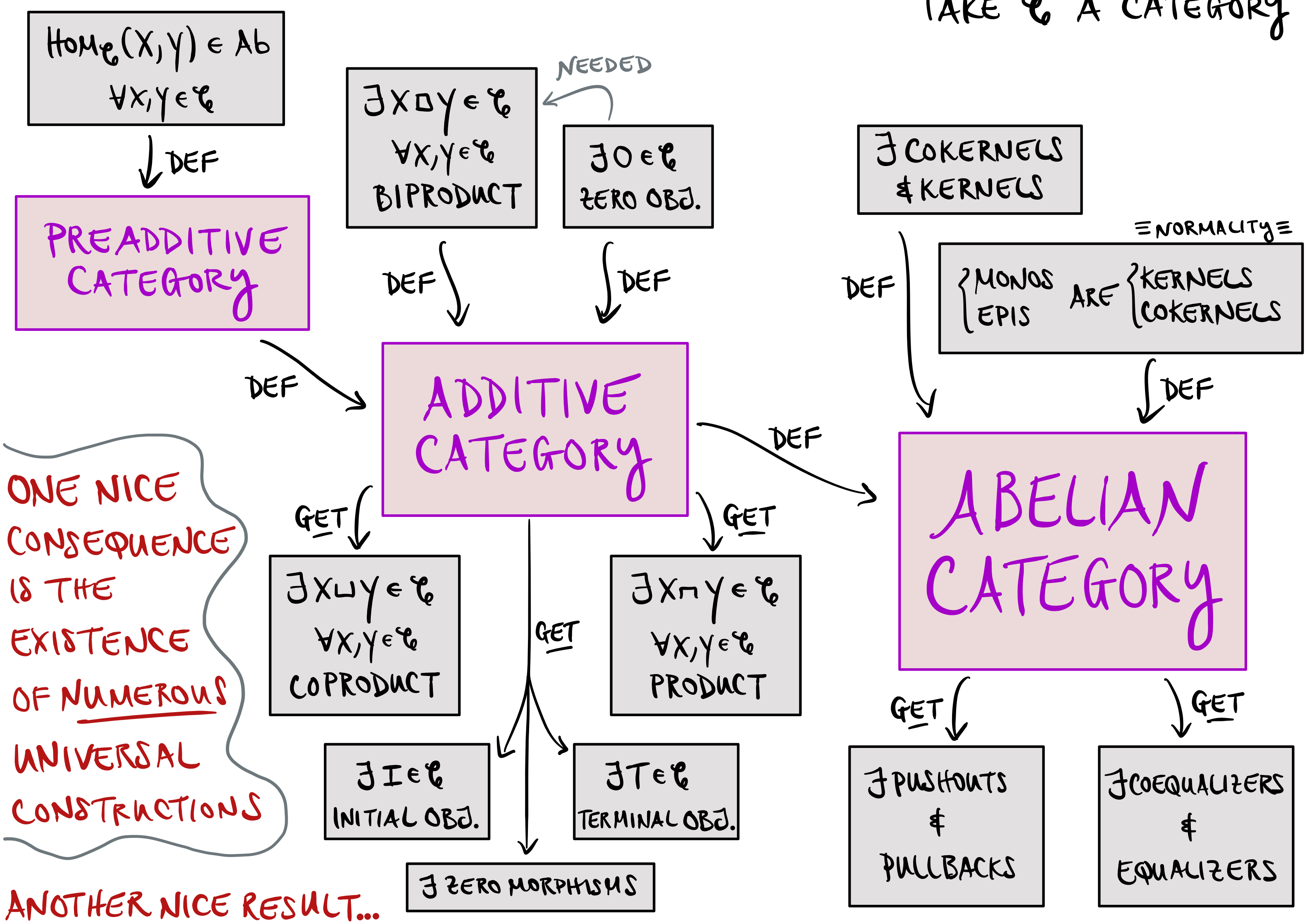
TAKE $\mathcal{C}$ A CATEGORY



ONE NICE CONSEQUENCE IS THE EXISTENCE OF NUMEROUS UNIVERSAL CONSTRUCTIONS

II. ABELIAN CATEGORIES

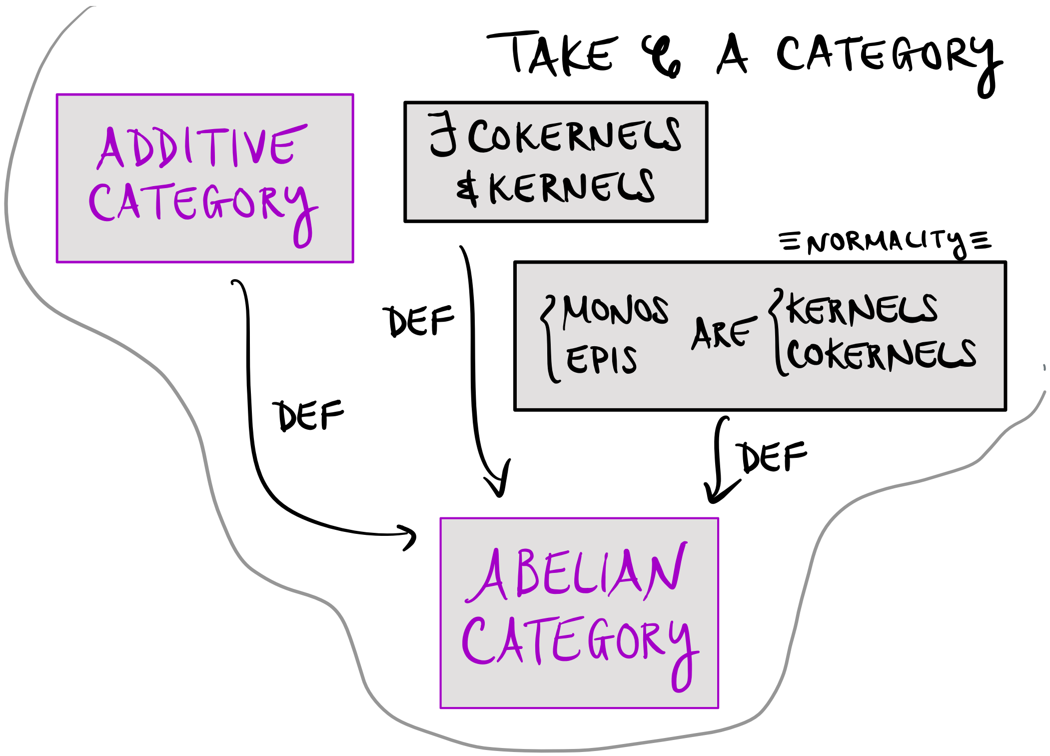
TAKE $\mathcal{C}$ A CATEGORY



ONE NICE CONSEQUENCE IS THE EXISTENCE OF NUMEROUS UNIVERSAL CONSTRUCTIONS

ANOTHER NICE RESULT...

II. ABELIAN CATEGORIES

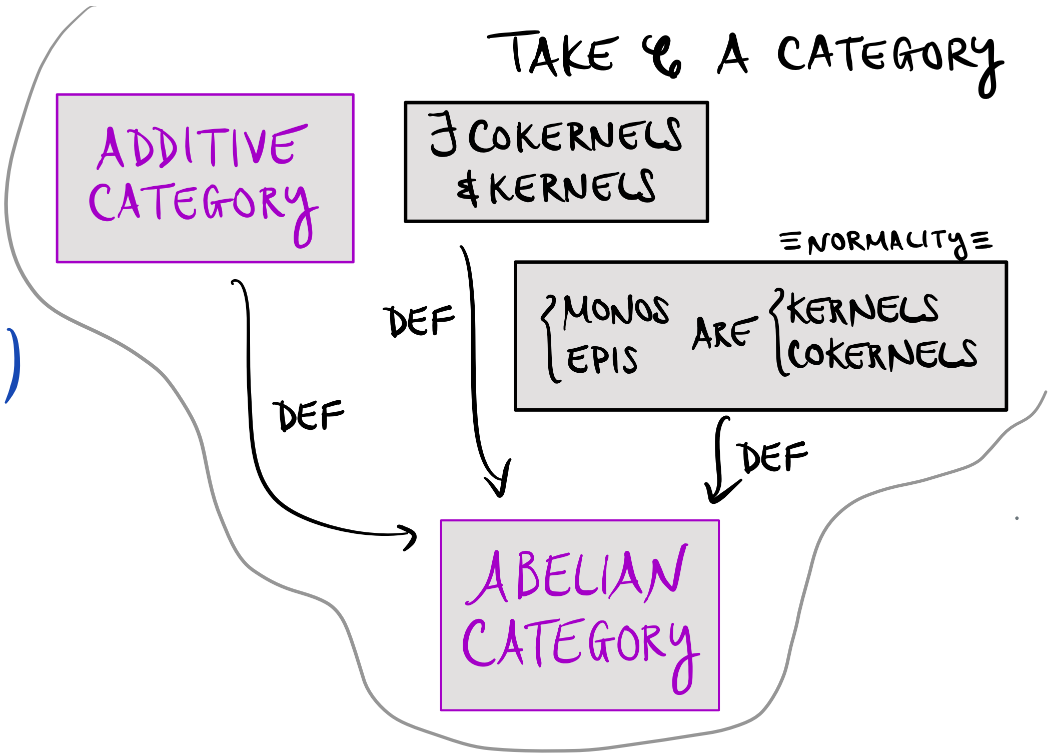


II. ABELIAN CATEGORIES

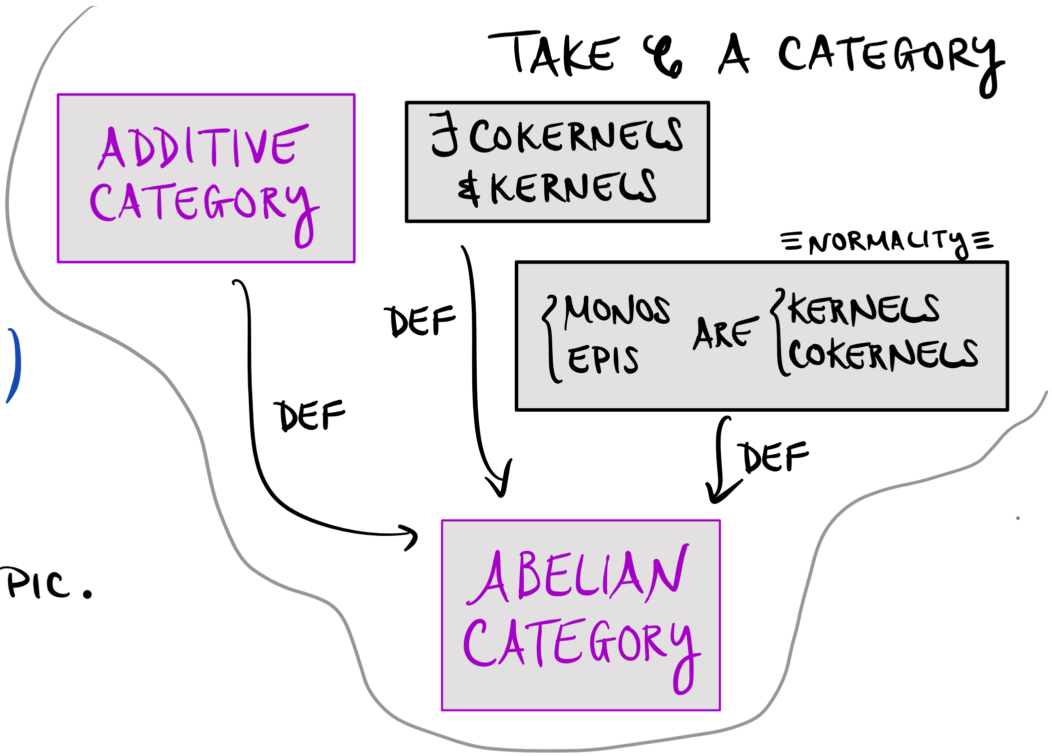
PROP:

IN AN ABELIAN CATEGORY $\mathcal{C}$:

ISO $\Leftrightarrow$ MONIC EPI ($\Leftrightarrow$ EPI MONO)



II. ABELIAN CATEGORIES



PROP:

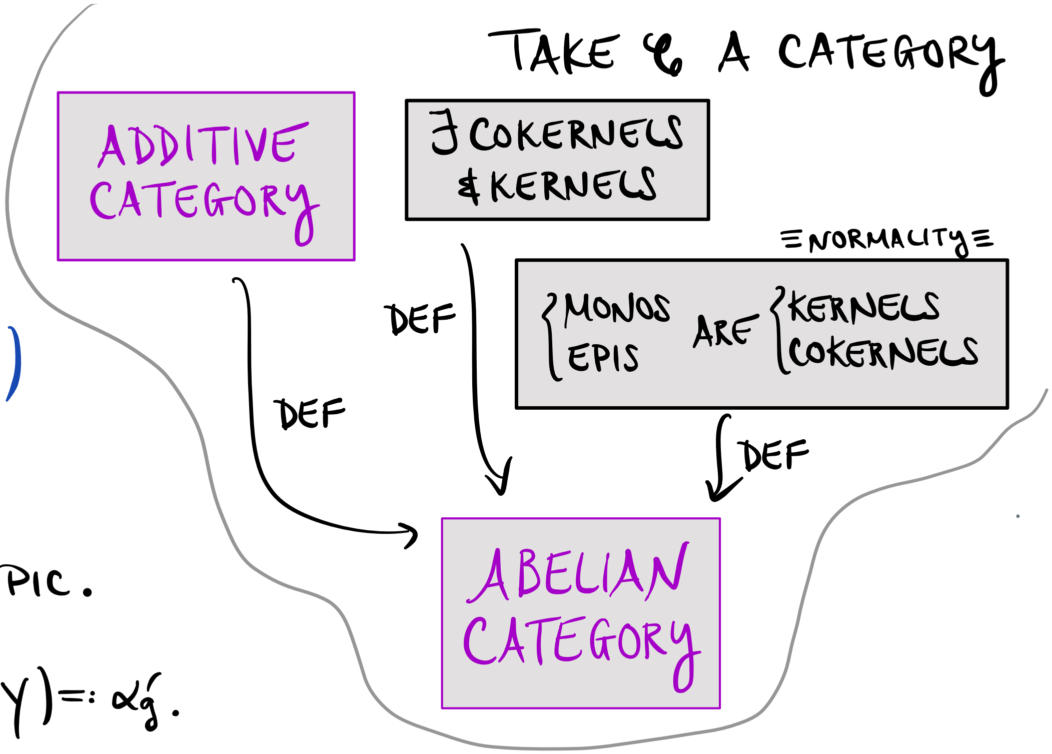
IN AN ABELIAN CATEGORY $\mathcal{C}$:

ISO $\Leftrightarrow$ MONIC EPI ($\Leftrightarrow$ EPI MONO)

PF/ ($\Rightarrow$) ✓

($\Leftarrow$) LET $f: W \to X$ BE MONIC EPIC.

II. ABELIAN CATEGORIES



PROP:

IN AN ABELIAN CATEGORY $\mathcal{C}$:

ISO ⟺ MONIC EPI (⟺ EPI MONO)

PF/ (⇒) ✓

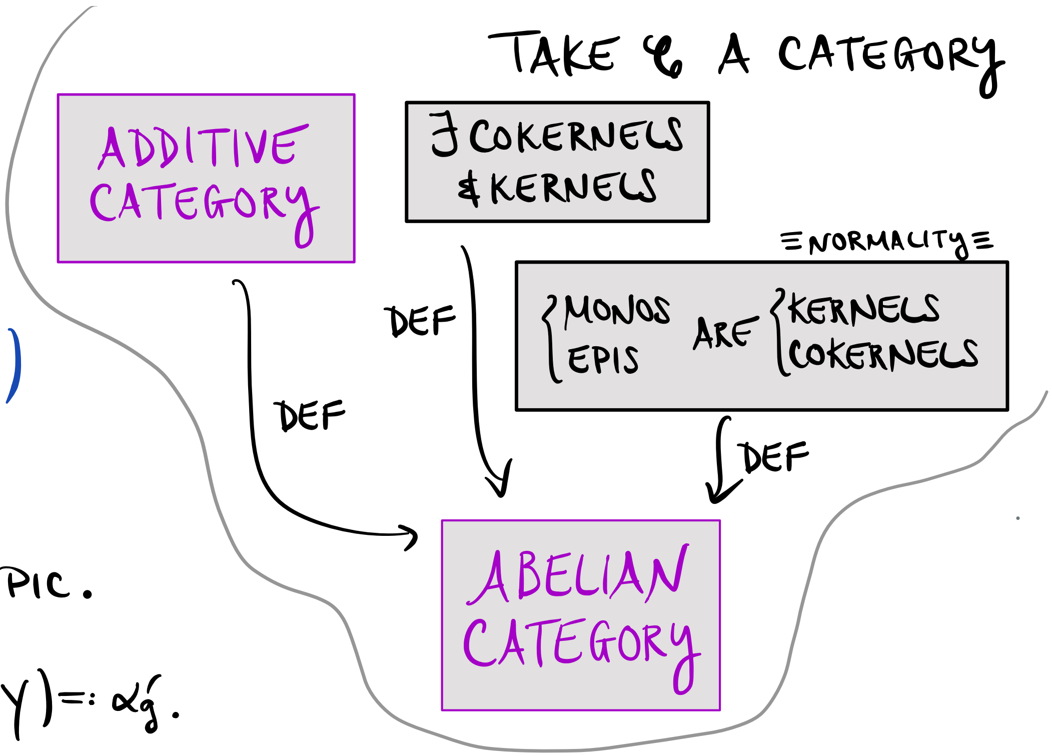
(⇐) LET $f: W \to X$ BE MONIC EPIC.

NORMALITY ⇒ $f = \ker(g: X \to Y) =: \alpha_g$.

GET
$$\begin{tikzcd} \ker(g) \arrow[r, "\alpha_g = f"] \arrow[rr, bend right, "\vec{0}"'] \arrow[r, phantom, shift right=3, "\circlearrowright"] & X \arrow[r, "g"] & Y \end{tikzcd}.$$

II. ABELIAN CATEGORIES

TAKE $\mathcal{C}$ A CATEGORY



PROP :

IN AN ABELIAN CATEGORY $\mathcal{C}$:

ISO $\Leftrightarrow$ MONIC EPI ($\Leftrightarrow$ EPI MONO)

PF/ ($\Rightarrow$) ✓

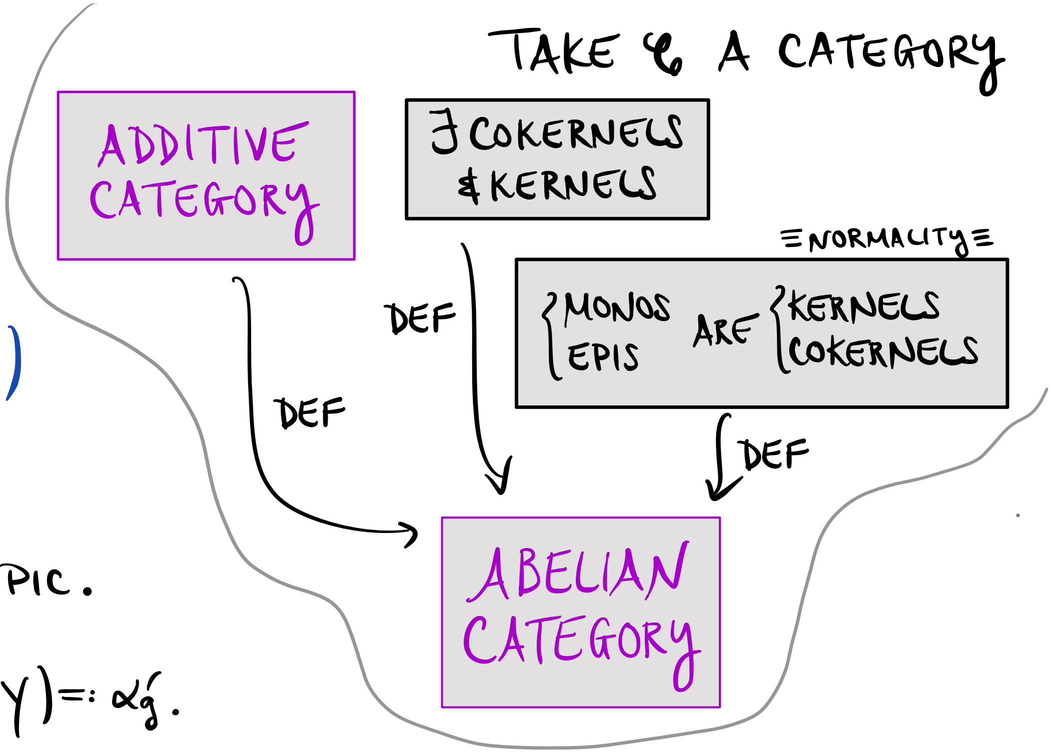
($\Leftarrow$) LET $f : W \to X$ BE MONIC EPIC.

NORMALITY $\Rightarrow$ $f = \ker(g : X \to Y) =: \alpha_g$.

GET
$$\begin{tikzcd} \ker(g) \arrow[r, "\alpha_g = f", "\circlearrowright"'] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "g"] & Y \end{tikzcd}$$
. ALSO
$$\begin{tikzcd} \ker(g) \arrow[r, "f", "\circlearrowright"'] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "\vec{0}"] & Y \end{tikzcd}$$
. $\therefore gf = \vec{0} f$.

II. ABELIAN CATEGORIES

TAKE $\mathscr{C}$ A CATEGORY



PROP:

IN AN ABELIAN CATEGORY $\mathscr{C}$:

ISO $\Leftrightarrow$ MONIC EPI ($\Leftrightarrow$ EPI MONO)

PF/ ($\Rightarrow$) ✓

($\Leftarrow$) LET $f: W \to X$ BE MONIC EPIC.

NORMALITY $\Rightarrow$ $f = \ker(g: X \to Y) =: \alpha_g$.

GET

$$\begin{tikzcd}
\ker(g) \arrow[r, "\alpha_g = f"] \arrow[rr, bend right, "\vec{0}"'] \arrow[r, phantom, shift right=4, "\circlearrowright"] & X \arrow[r, "g"] & Y
\end{tikzcd}$$

. ALSO

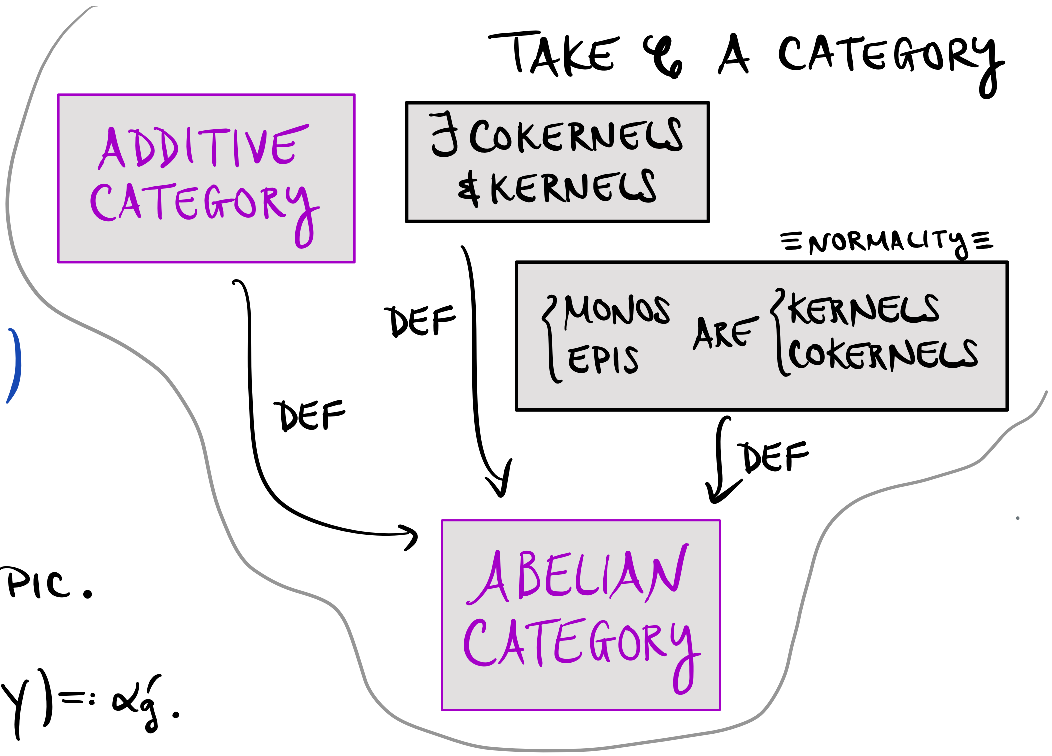
$$\begin{tikzcd}
\ker(g) \arrow[r, "f"] \arrow[rr, bend right, "\vec{0}"'] \arrow[r, phantom, shift right=4, "\circlearrowright"] & X \arrow[r, "\vec{0}"] & Y
\end{tikzcd}$$

. $\therefore$ $gf = \vec{0} f$.

NOW f EPIC $\Rightarrow$ $g = \vec{0}$. $\therefore$ WE GET:

$$\begin{tikzcd}
\ker(\vec{0}) \arrow[r, "f"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "\vec{0} = g"] & Y
\end{tikzcd}$$

II. ABELIAN CATEGORIES



PROP:

IN AN ABELIAN CATEGORY $\mathcal{C}$:

ISO $\Leftrightarrow$ MONIC EPI ($\Leftrightarrow$ EPI MONO)

PF/ ($\Rightarrow$) ✓

($\Leftarrow$) LET $f: W \to X$ BE MONIC EPIC.

NORMALITY $\Rightarrow$ $f = \ker(g: X \to Y) =: \alpha'_g$.

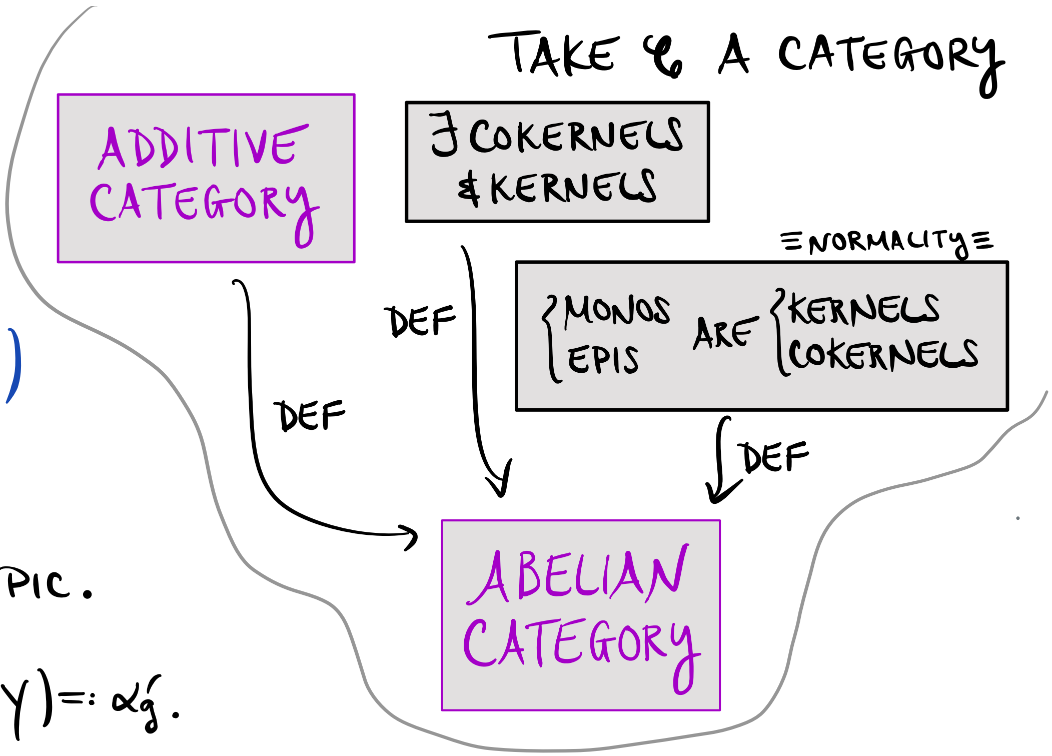
GET
$$\begin{tikzcd} \ker(g) \arrow[r, "\alpha'_g = f"] \arrow[rr, bend right, "\vec{0}"'] \arrow[r, phantom, shift right=3, "\circlearrowright"] & X \arrow[r, "g"] & Y \end{tikzcd}.$$
ALSO
$$\begin{tikzcd} \ker(g) \arrow[r, "f"] \arrow[rr, bend right, "\vec{0}"'] \arrow[r, phantom, shift right=3, "\circlearrowright"] & X \arrow[r, "\vec{0}"] & Y \end{tikzcd}.$$
$\therefore gf = \vec{0} f$.

NOW f EPIC $\Rightarrow$ $g = \vec{0}$. $\therefore$ WE GET:

$$\begin{tikzcd} X \arrow[dr, "\mathrm{id}_X"] \arrow[drr, bend left, "\vec{0}"] & \circlearrowright & \\ \ker(\vec{0}) \arrow[r, "f"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "\vec{0}"] & Y \end{tikzcd}$$

(PICK $\beta'_X = \mathrm{id}_X$)

II. ABELIAN CATEGORIES



PROP:

IN AN ABELIAN CATEGORY $\mathscr{C}$:

ISO $\Longleftrightarrow$ MONIC EPI ($\Longleftrightarrow$ EPI MONO)

PF/ ($\Longrightarrow$) ✓

($\Longleftarrow$) LET $f: W \to X$ BE MONIC EPIC.

NORMALITY $\Rightarrow$ $f = \ker(g: X \to Y) =: \alpha_g'$.

GET
$$\begin{tikzcd}
\ker(g) \arrow[r, "\alpha_g' = f"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "g"] \arrow[d, phantom, "\circlearrowleft"] & Y \\
& {} &
\end{tikzcd}$$
. ALSO
$$\begin{tikzcd}
\ker(g) \arrow[r, "f"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "\vec{0}"] \arrow[d, phantom, "\circlearrowleft"] & Y \\
& {} &
\end{tikzcd}$$
. $\therefore gf = \vec{0} f$.

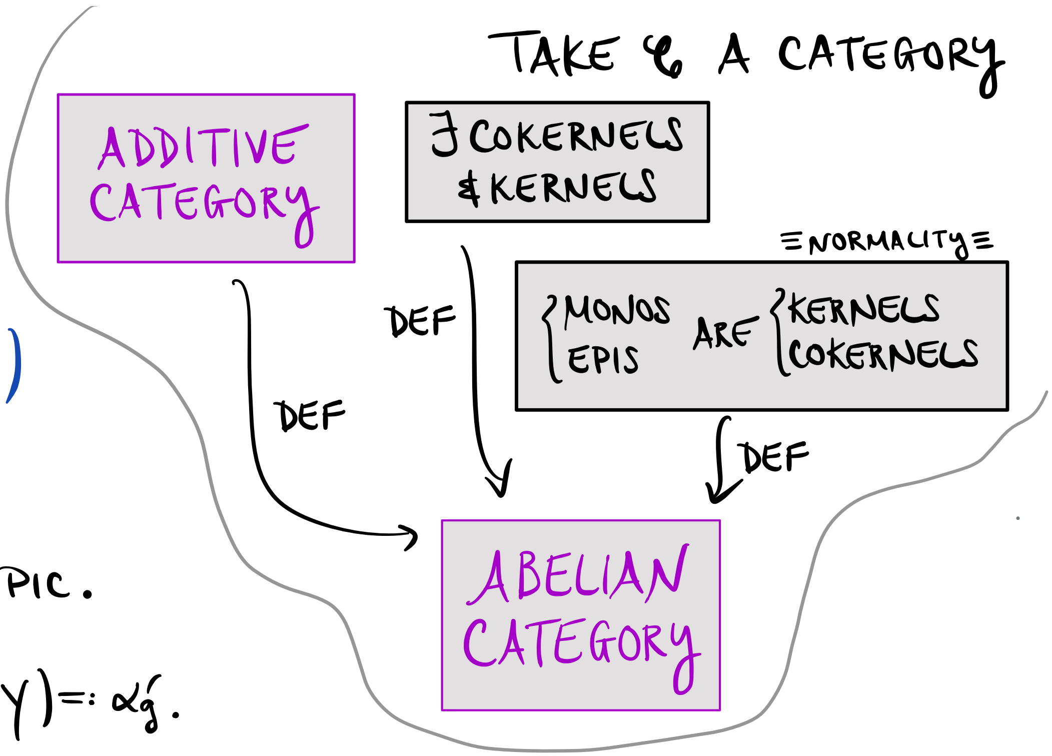
NOW f EPIC $\Rightarrow$ $g = \vec{0}$. $\therefore$ WE GET:

$$\begin{tikzcd}
X \arrow[d, dashed, "\exists! \gamma"'] \arrow[dr, "\mathrm{id}_X"] \arrow[drr, bend left, "\vec{0}"] & \circlearrowleft & \circlearrowleft \\
\ker(\vec{0}) \arrow[r, "f"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "\vec{0}"] & Y
\end{tikzcd}$$

(PICK $\beta_X' = \mathrm{id}_X$)

$\therefore f\gamma = \mathrm{id}_X$

II. ABELIAN CATEGORIES



PROP :

IN AN ABELIAN CATEGORY $\mathcal{C}$:

ISO ⟺ MONIC EPI (⟺ EPI MONO)

PF/ (⟹) ✓

(⟸) LET $f: W \to X$ BE MONIC EPIC.

NORMALITY ⟹ $f = \ker(g: X \to Y) =: \alpha_g'$.

GET
$$\begin{tikzcd}
\ker(g) \arrow[r, "\alpha_g' = f"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "g"] & Y
\end{tikzcd}$$
. ALSO
$$\begin{tikzcd}
\ker(g) \arrow[r, "f"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "\vec{0}"] & Y
\end{tikzcd}$$
. ∴ $gf = \vec{0} f$.

NOW f EPIC ⟹ $g = \vec{0}$. ∴ WE GET:

$$\begin{tikzcd}
X \arrow[d, dashed, "\exists! \gamma"'] \arrow[dr, "id_X"] \arrow[drr, bend left, "\vec{0}"] & & \\
\ker(\vec{0}) \arrow[r, "f"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "\vec{0}"] & Y
\end{tikzcd}$$

(PICK $\beta_X' = id_X$)

∴ $f\gamma = id_X$ ∴ $f\gamma f = f$

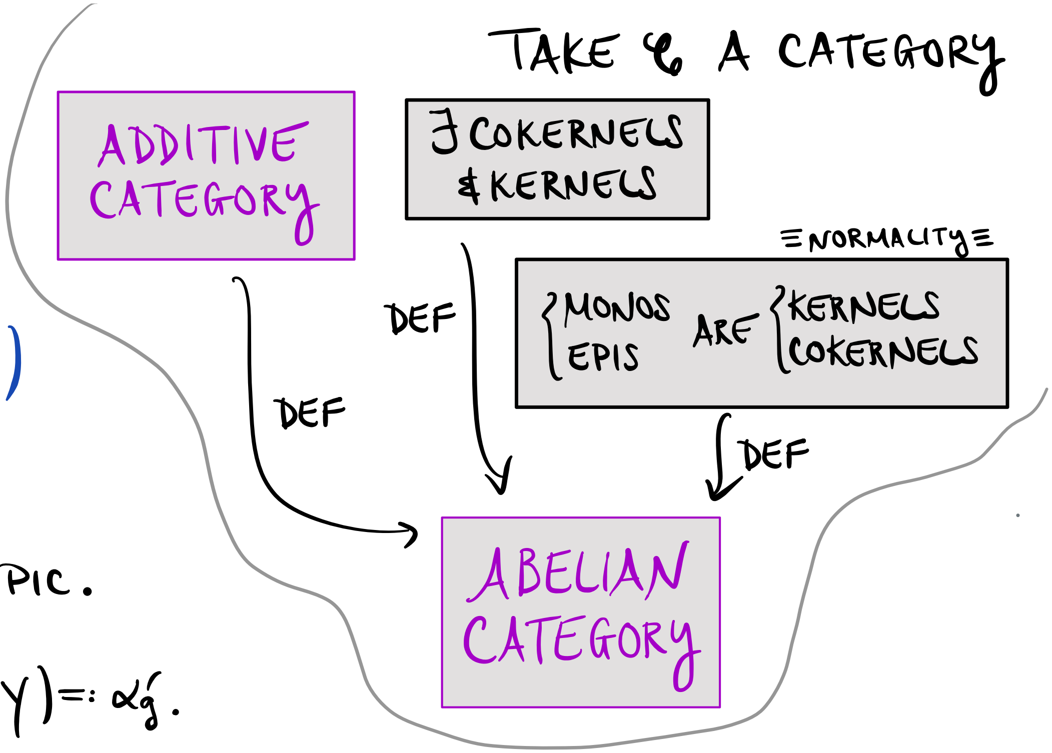
NOW f MONIC ⟹ $\gamma f = id_{\ker(\vec{0})}$

II. ABELIAN CATEGORIES

PROP :

IN AN ABELIAN CATEGORY $\mathcal{C}$:

ISO $\Longleftrightarrow$ MONIC EPI ($\Longleftrightarrow$ EPI MONO)



TAKE $\mathcal{C}$ A CATEGORY

ADDITIVE CATEGORY

$\exists$ COKERNELS & KERNELS

≡NORMALITY≡

$\left\{\begin{matrix}\text{MONOS}\\ \text{EPIS}\end{matrix}\right.$ ARE $\left\{\begin{matrix}\text{KERNELS}\\ \text{COKERNELS}\end{matrix}\right.$

DEF

DEF

DEF

ABELIAN CATEGORY

PF/ ($\Rightarrow$) ✓

($\Leftarrow$) LET $f: W \to X$ BE MONIC EPIC.

NORMALITY $\Rightarrow$ $f = \ker(g: X \to Y) =: \alpha_g'$.

GET
$$\begin{tikzcd} \ker(g) \arrow[r, "\alpha_g' = f"] \arrow[rr, bend right, "\vec{0}"'] \arrow[r, phantom, "\circlearrowright"', shift right=3] & X \arrow[r, "g"] & Y \end{tikzcd}.$$
ALSO
$$\begin{tikzcd} \ker(g) \arrow[r, "f"] \arrow[rr, bend right, "\vec{0}"'] \arrow[r, phantom, "\circlearrowright"', shift right=3] & X \arrow[r, "\vec{0}"] & Y \end{tikzcd}.$$
$\therefore\ gf = \vec{0}f$.

NOW f EPIC $\Rightarrow$ $g = \vec{0}$.

$\therefore$ WE GET:

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!\gamma"'] \arrow[dr, "\mathrm{id}_X"] \arrow[drr, bend left, "\vec{0}"] \arrow[dr, phantom, "\circlearrowright"', near start, shift right=2] \arrow[drr, phantom, "\circlearrowright", near end, shift left=3] & & \\ \ker(\vec{0}) \arrow[r, "f"] \arrow[rr, bend right, "\vec{0}"'] & X \arrow[r, "\vec{0}"] & Y \end{tikzcd}$$

(PICK $\beta_X' = \mathrm{id}_X$)

$\therefore\ f\gamma = \mathrm{id}_X$ $\qquad$ $\therefore\ f\gamma f = f$

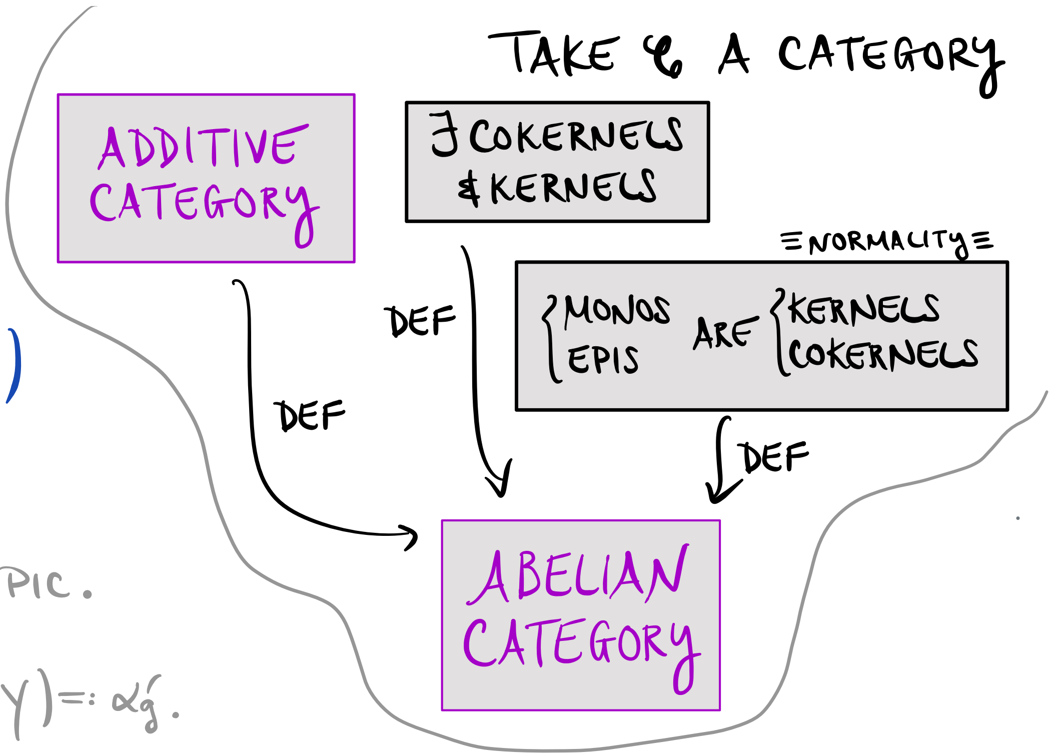
NOW f MONIC $\Rightarrow$ $\gamma f = \mathrm{id}_{\ker(\vec{0})}$ $\qquad$ $\therefore$ f IS AN ISO W/ $f^{-1} = \gamma$. ///

II. ABELIAN CATEGORIES

PROP: IN AN ABELIAN CATEGORY $\mathscr{C}$:

ISO $\Leftrightarrow$ MONIC EPI ($\Leftrightarrow$ EPI MONO)

Ex. Ring IS NOT ABELIAN (EXER 2.2)



TAKE $\mathscr{C}$ A CATEGORY

ADDITIVE CATEGORY

$\exists$ COKERNELS & KERNELS

≡NORMALITY≡

{MONOS / EPIS} ARE {KERNELS / COKERNELS}

DEF

DEF

DEF

ABELIAN CATEGORY

PF/ $(\Rightarrow)$ ✓

$(\Leftarrow)$ LET $f: W \to X$ BE MONIC EPIC.

NORMALITY $\Rightarrow$ $f = \ker(g: X \to Y) =: \alpha'_g$.

GET
$$\begin{tikzcd} \ker(g) \arrow[r, "\alpha'_g = f"] \arrow[rr, bend right, "\vec{0}"'] \arrow[r, phantom, shift right=3, "\circlearrowright"] & X \arrow[r, "g"] & Y \end{tikzcd}$$
. ALSO
$$\begin{tikzcd} \ker(g) \arrow[r, "f"] \arrow[rr, bend right, "\vec{0}"'] \arrow[r, phantom, shift right=3, "\circlearrowright"] & X \arrow[r, "\vec{0}"] & Y \end{tikzcd}$$
. $\therefore gf = \vec{0}f$.

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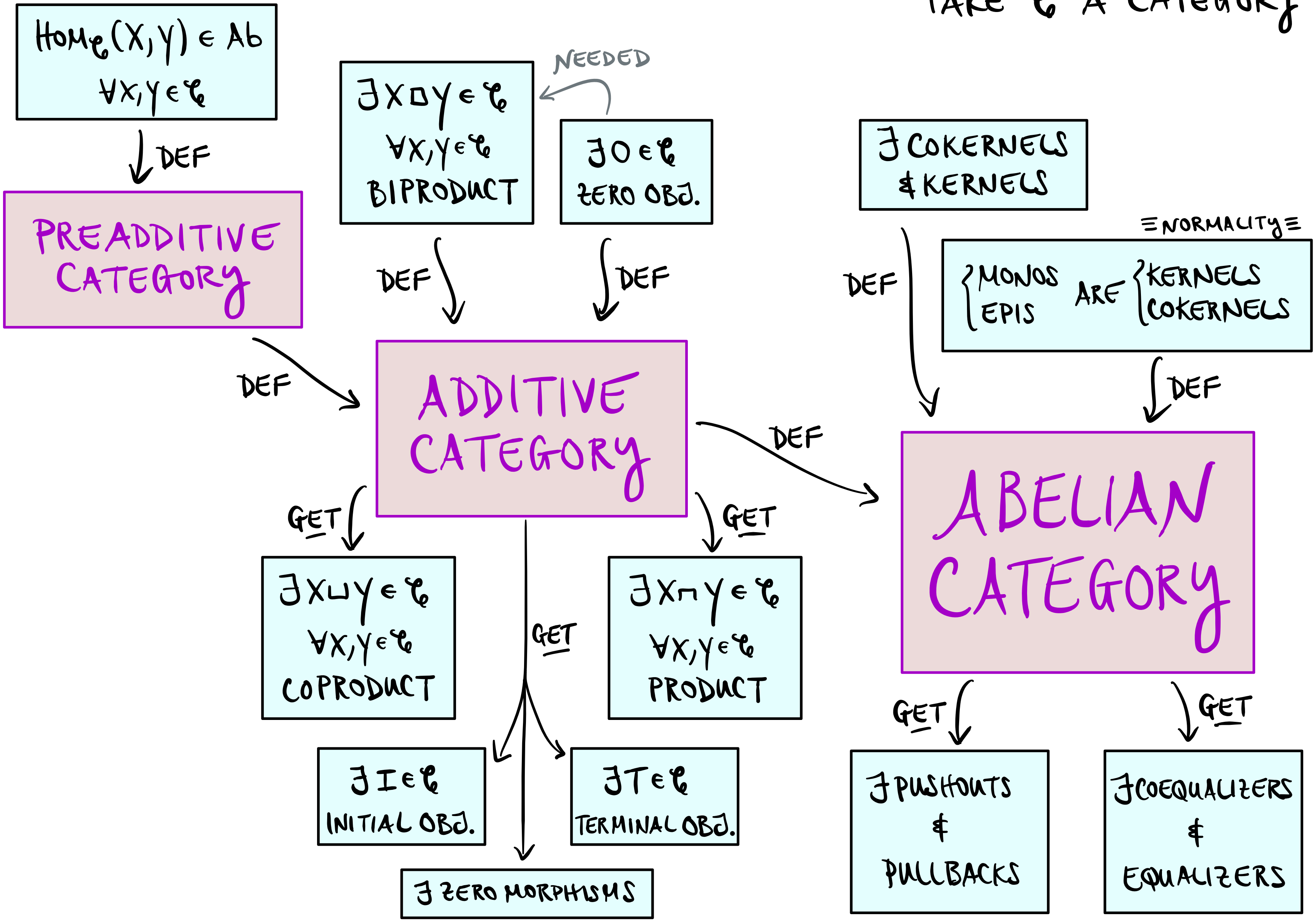
(PICK $\beta'_X = \mathrm{id}_X$)

$\therefore f\gamma = \mathrm{id}_X$ $\quad$ $\therefore f\gamma f = f$

NOW f MONIC $\Rightarrow$ $\gamma f = \mathrm{id}_{\ker(\vec{0})}$ $\quad$ $\therefore f$ IS AN ISO w/ $f^{-1} = \gamma$. ///

II. ABELIAN CATEGORIES

TAKE $\mathscr{C}$ A CATEGORY



MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

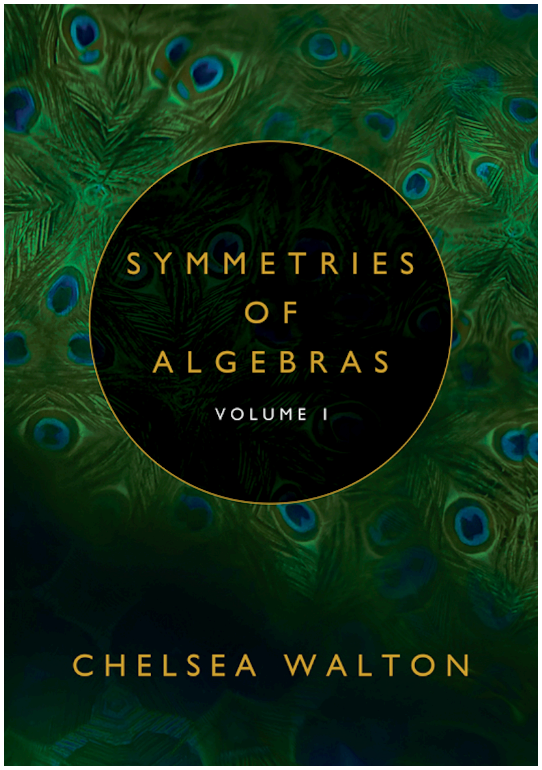
LECTURE #7

TOPICS:

✓ I. UNIVERSAL CONSTRUCTIONS (§2.2.1)

✓ II. ABELIAN CATEGORIES (§2.2.2)

NEXT: FUNCTORS & NATURAL TRANSFORMATIONS



Lecture #7 keywords: additive category, abelian category, biproduct of objects, coequalizer of morphisms, cokernel, equalizer of morphisms, kernel, preadditive category, pullback of morphisms, pushout of morphisms