MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #8

LAST TIME

- UNIVERSAL CONSTRUCTIONS
- ABELIAN CATEGORIES

TOPICS:

I. FUNCTORS (§§2.3.1–2.3.2)

II. BIFUNCTORS & MULTIFUNCTORS (§2.3.3)

III. NATURAL TRANSFORMATIONS (§2.3.4)

IV. COMPOSITIONS OF NATURAL TRANSFORMATIONS (§2.3.5)

≡ RECALL ≡

A CATEGORY $\mathcal{C}$ CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ $\forall X, Y \in \mathcal{C}$.

(c) $id_X : X \to X$ $\forall X \in \mathcal{C}$.

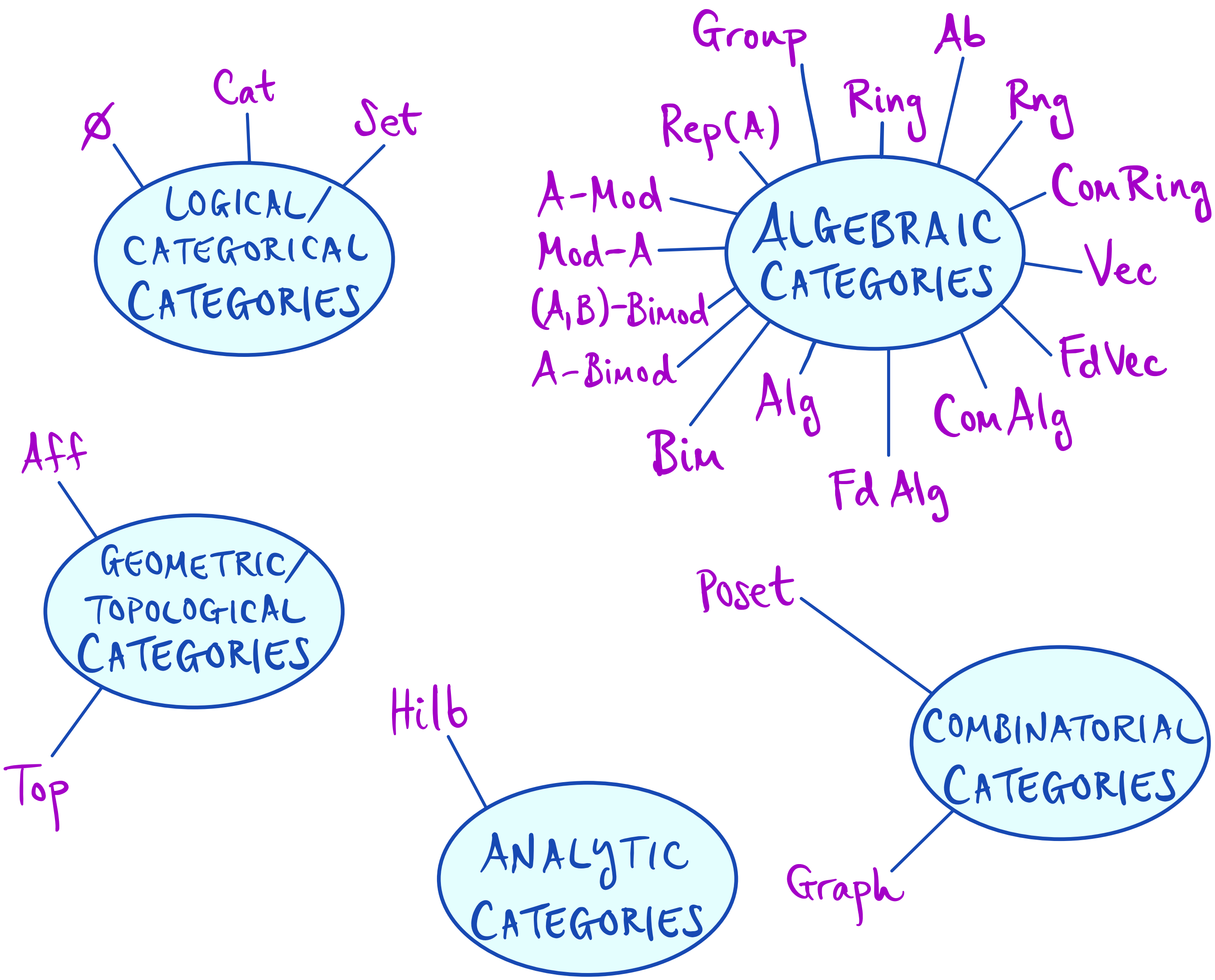
(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY $(hg)f = h(gf)$

UNITALITY $id_X f = f$, $g\, id_X = g$

EXAMPLES...



≡ QUESTIONS ≡

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

A CATEGORY $\mathcal{C}$

CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ $\forall X,Y \in \mathcal{C}$.

(c) $\mathrm{id}_X : X \to X$ $\forall X \in \mathcal{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

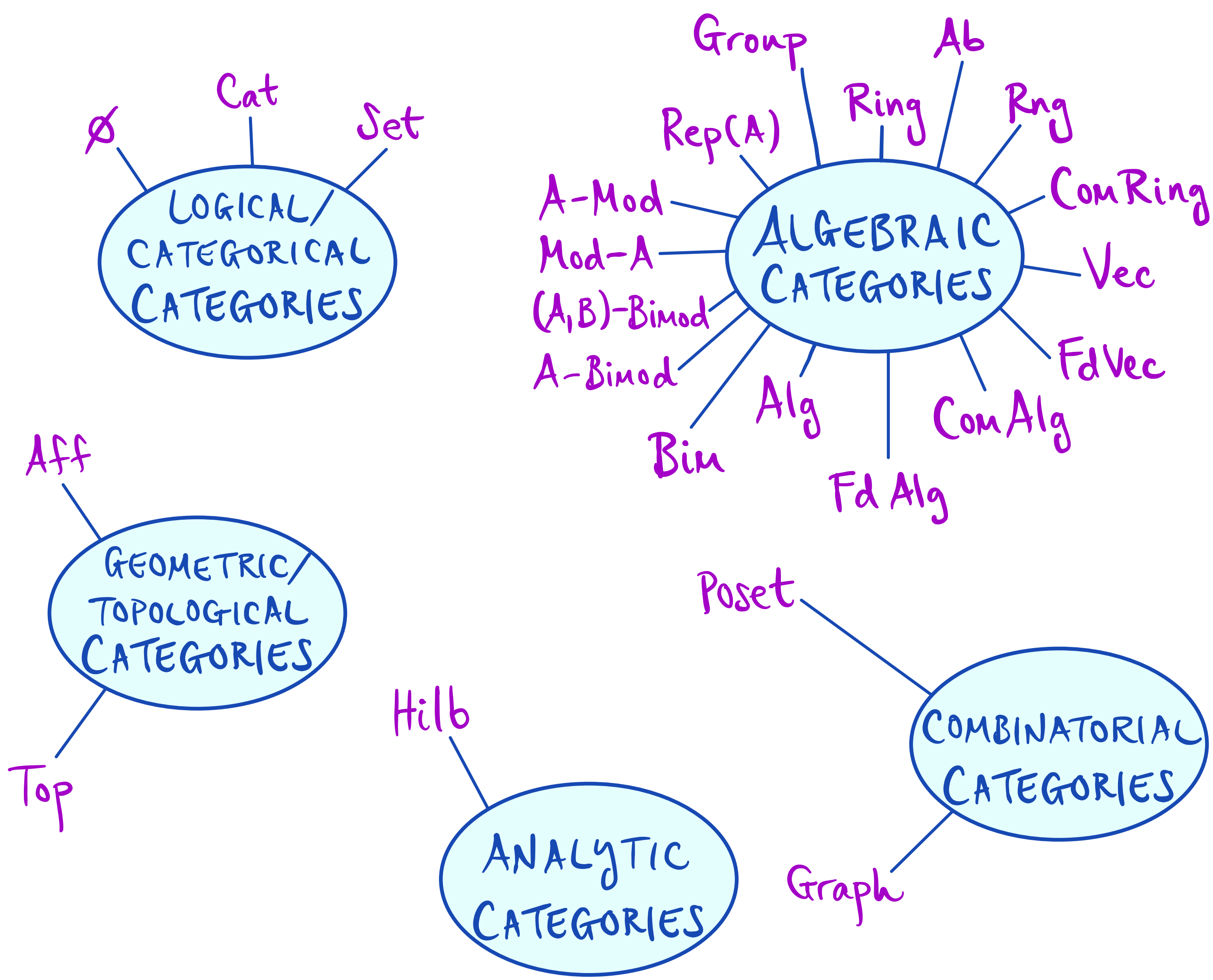
ASSOCIATIVITY

$(hg)f = h(gf)$

UNITALITY

$\mathrm{id}_X f = f,\ g\,\mathrm{id}_X = g$

EXAMPLES...



≡ QUESTIONS ≡

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

HOW DO WE MOVE ONE FROM CATEGORY TO ANOTHER?

A CATEGORY $\mathscr{C}$

CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ $\forall X, Y \in \mathscr{C}$.

(c) $id_X : X \to X$ $\forall X \in \mathscr{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

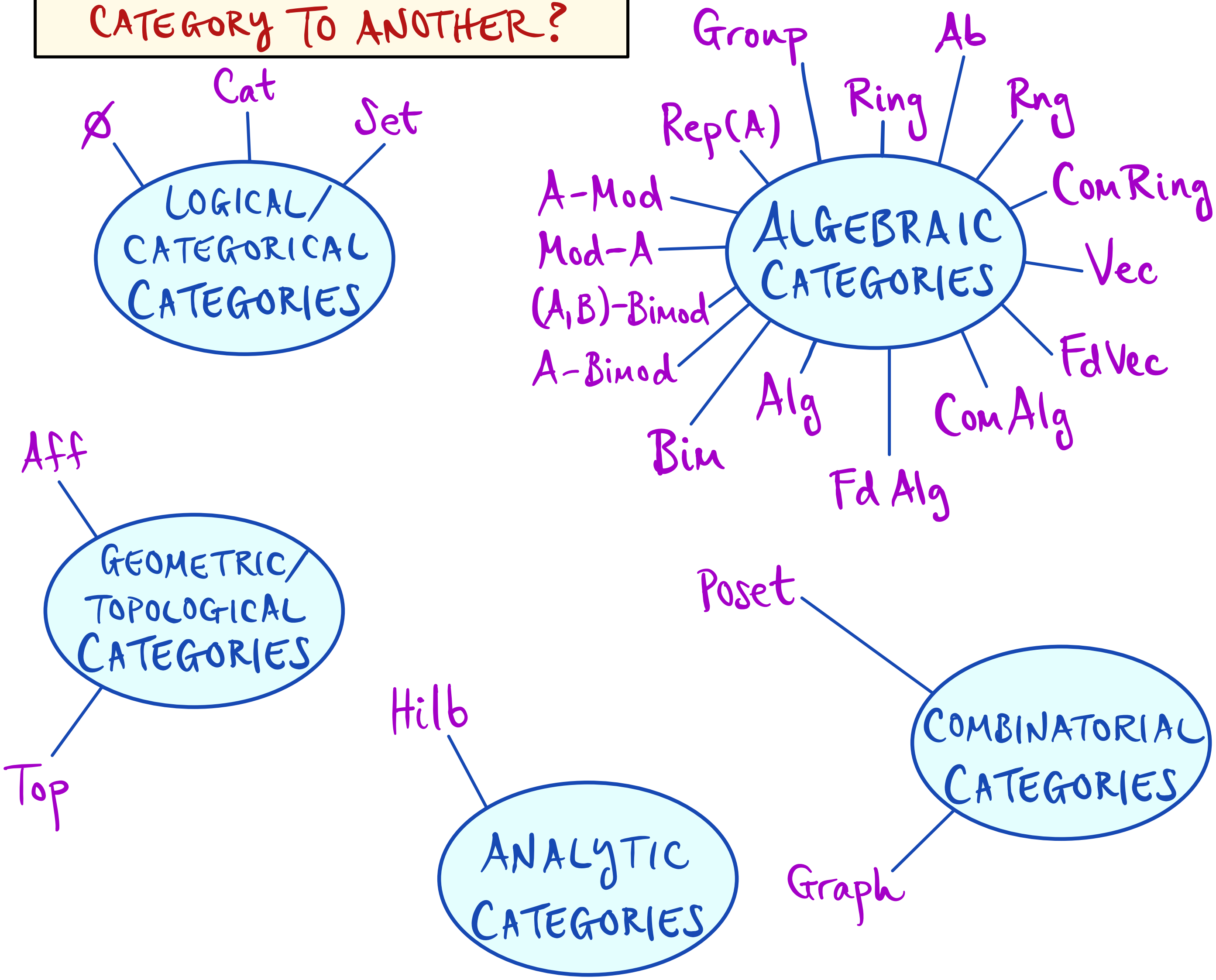
SATISFYING

ASSOCIATIVITY

$(hg)f = h(gf)$

UNITALITY

$id_X f = f$, $g \, id_X = g$



I. FUNCTORS

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

HOW DO WE MOVE ONE FROM CATEGORY TO ANOTHER?

A CATEGORY $\mathcal{C}$

CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ $\forall X,Y \in \mathcal{C}$.

(c) $\mathrm{id}_X : X \to X$ $\forall X \in \mathcal{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

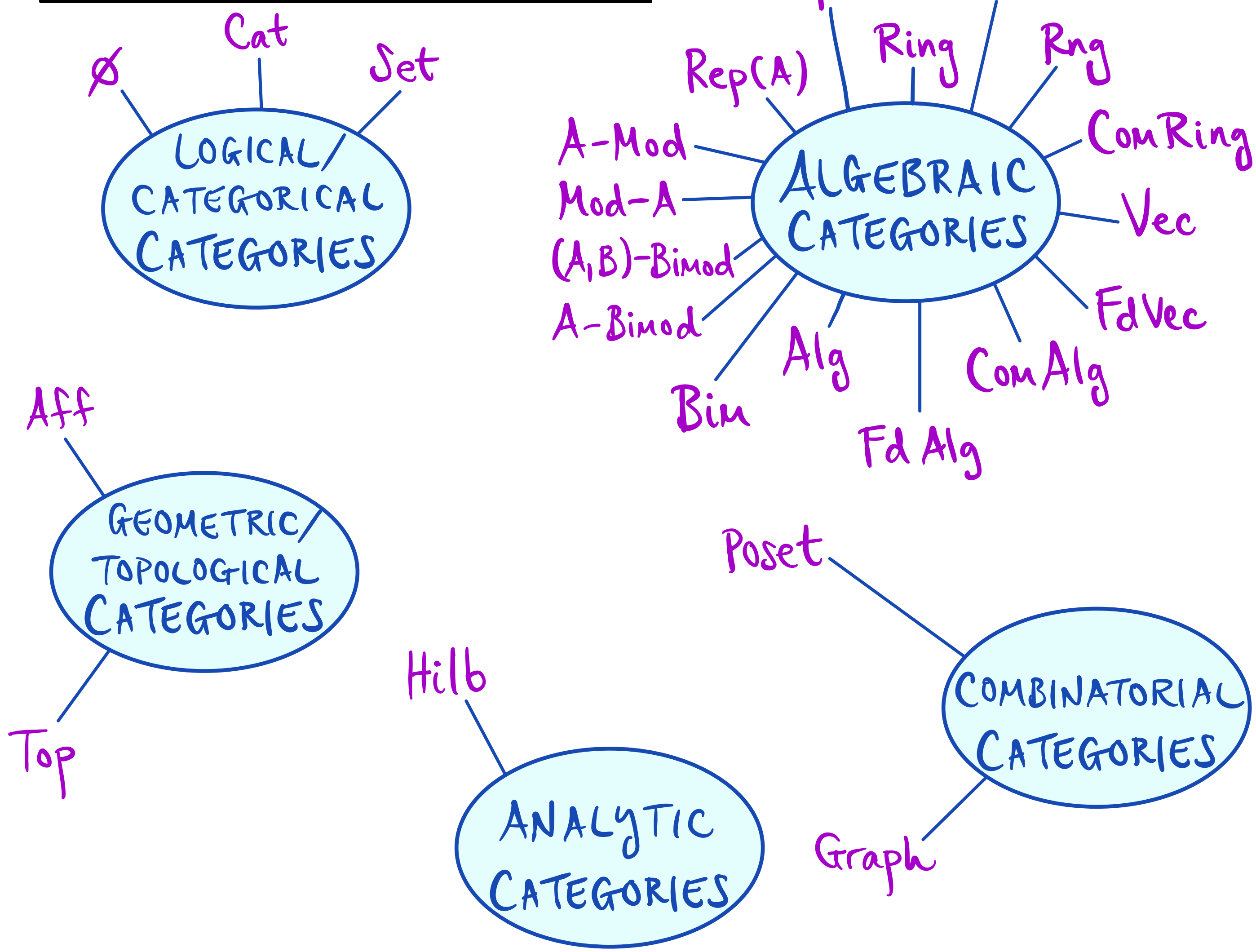
SATISFYING

ASSOCIATIVITY

$(hg)f = h(gf)$

UNITALITY

$\mathrm{id}_X f = f, \; g\,\mathrm{id}_X = g$



I. FUNCTORS

A **CATEGORY** $\mathscr{C}$ CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ $\forall X, Y \in \mathscr{C}$.

(c) $id_X : X \to X$ $\forall X \in \mathscr{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY
$(hg)f = h(gf)$

UNITALITY
$id_X f = f$, $g\, id_X = g$

TAKE TWO CATEGORIES $\mathscr{C}$ AND $\mathscr{D}$

I. FUNCTORS

TAKE TWO CATEGORIES $\mathcal{C}$ AND $\mathcal{D}$

A CATEGORY $\mathcal{C}$
CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS
$\mathrm{Hom}_{\mathcal{C}}(X,Y)$
$\forall X,Y \in \mathcal{C}$.

(c) $id_X : X \to X$
$\forall X \in \mathcal{C}$.

(d) $gf : W \to Y$
$\forall f : W \to X$
$g : X \to Y$.

SATISFYING
ASSOCIATIVITY
$(hg)f = h(gf)$

UNITALITY
$id_X f = f,\ g\, id_X = g$

A (COVARIANT) FUNCTOR
$F : \mathcal{C} \longrightarrow \mathcal{D}$
CONSISTS OF

(a) AN OBJECT $F(X) \in \mathcal{D}$
FOR EACH $X \in \mathcal{C}$.

(b) A MORPHISM
$F(g) : F(X) \to F(Y) \in \mathcal{D}$
FOR EACH $g : X \to Y \in \mathcal{C}$.

I. FUNCTORS

TAKE TWO CATEGORIES $\mathcal{C}$ AND $\mathcal{D}$

A CATEGORY $\mathcal{C}$

CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ $\forall X,Y \in \mathcal{C}$.

(c) $\mathrm{id}_X : X \to X$ $\forall X \in \mathcal{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY

$(hg)f = h(gf)$

UNITALITY

$\mathrm{id}_X f = f, \; g\,\mathrm{id}_X = g$

A (COVARIANT) FUNCTOR $F : \mathcal{C} \longrightarrow \mathcal{D}$

CONSISTS OF

(a) AN OBJECT $F(X) \in \mathcal{D}$ FOR EACH $X \in \mathcal{C}$.

(b) A MORPHISM $F(g) : F(X) \to F(Y) \in \mathcal{D}$ FOR EACH $g : X \to Y \in \mathcal{C}$.

RESPECTING:

- IDENTITY MORPHISMS

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$$

- COMPOSITION OF MORPHISMS

$$F(hg) = F(h)F(g)$$

$$\forall g : X \to Y, \; h : Y \to Z \in \mathcal{C}$$

I. FUNCTORS

TAKE TWO CATEGORIES $\mathscr{C}$ AND $\mathscr{D}$

A CATEGORY $\mathscr{C}$

CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ $\forall X,Y \in \mathscr{C}$.

(c) $id_X : X \to X$ $\forall X \in \mathscr{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY

$(hg)f = h(gf)$

UNITALITY

$id_X f = f$, $g\, id_X = g$

A (COVARIANT) FUNCTOR

$F : \mathscr{C} \longrightarrow \mathscr{D}$

CONSISTS OF

(a) AN OBJECT $F(X) \in \mathscr{D}$ FOR EACH $X \in \mathscr{C}$.

(b) A MORPHISM $F(g) : F(X) \to F(Y) \in \mathscr{D}$ FOR EACH $g : X \to Y \in \mathscr{C}$.

RESPECTING:

- IDENTITY MORPHISMS

$$F(id_X) = id_{F(X)} \quad \forall X \in \mathscr{C}$$

- COMPOSITION OF MORPHISMS

$$F(hg) = F(h)F(g)$$
$$\forall g : X \to Y,\ h : Y \to Z \in \mathscr{C}$$

A CONTRAVARIANT FUNCTOR

$F : \mathscr{C} \longrightarrow \mathscr{D}$

CONSISTS OF

(a) AN OBJECT $F(X) \in \mathscr{D}$ FOR EACH $X \in \mathscr{C}$.

(b) A MORPHISM $F(g) : F(Y) \to F(X) \in \mathscr{D}$ FOR EACH $g : X \to Y \in \mathscr{C}$.

DIRECTION OF MORPHISMS REVERSED

I. FUNCTORS

TAKE TWO CATEGORIES $\mathscr{C}$ AND $\mathscr{D}$

A CATEGORY $\mathscr{C}$ CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ $\forall X,Y \in \mathscr{C}$.

(c) $id_X : X \to X$ $\forall X \in \mathscr{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY $(hg)f = h(gf)$

UNITALITY $id_X f = f$, $g\, id_X = g$

A (COVARIANT) FUNCTOR $F : \mathscr{C} \to \mathscr{D}$ CONSISTS OF

(a) AN OBJECT $F(X) \in \mathscr{D}$ FOR EACH $X \in \mathscr{C}$.

(b) A MORPHISM $F(g) : F(X) \to F(Y) \in \mathscr{D}$ FOR EACH $g : X \to Y \in \mathscr{C}$.

RESPECTING:

- IDENTITY MORPHISMS $F(id_X) = id_{F(X)}$ $\forall X \in \mathscr{C}$
- COMPOSITION OF MORPHISMS $F(hg) = F(h)F(g)$ $\forall g : X \to Y,\ h : Y \to Z \in \mathscr{C}$

A CONTRAVARIANT FUNCTOR $F : \mathscr{C} \to \mathscr{D}$ CONSISTS OF

(a) AN OBJECT $F(X) \in \mathscr{D}$ FOR EACH $X \in \mathscr{C}$.

(b) A MORPHISM $F(g) : F(Y) \to F(X) \in \mathscr{D}$ FOR EACH $g : X \to Y \in \mathscr{C}$.

RESPECTING:

- IDENTITY MORPHISMS $F(id_X) = id_{F(X)}$ $\forall X \in \mathscr{C}$
- COMPOSITION OF MORPHISMS $F(gf) = F(f)F(g)$ $\forall f : W \to X,\ g : X \to Y \in \mathscr{C}$

I. FUNCTORS

TAKE CATEGORIES $\mathcal{C}, \mathcal{D}, \mathcal{E}$

A CATEGORY $\mathcal{C}$ CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ $\forall X, Y \in \mathcal{C}$.

(c) $\mathrm{id}_X : X \to X$ $\forall X \in \mathcal{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY $(hg)f = h(gf)$

UNITALITY $\mathrm{id}_X f = f$, $g\, \mathrm{id}_X = g$

A (COVARIANT) FUNCTOR $F : \mathcal{C} \longrightarrow \mathcal{D}$ CONSISTS OF

(a) AN OBJECT $F(X) \in \mathcal{D}$ FOR EACH $X \in \mathcal{C}$.

(b) A MORPHISM $F(g) : F(X) \to F(Y) \in \mathcal{D}$ FOR EACH $g : X \to Y \in \mathcal{C}$.

RESPECTING:

- IDENTITY MORPHISMS

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$$

- COMPOSITION OF MORPHISMS

$$F(hg) = F(h)F(g)$$

$$\forall g : X \to Y,\ h : Y \to Z \in \mathcal{C}$$

I. FUNCTORS

TAKE CATEGORIES $\mathcal{C}, \mathcal{D}, \mathcal{E}$

A CATEGORY $\mathcal{C}$ CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ $\forall X,Y \in \mathcal{C}$.

(c) $id_X : X \to X$ $\forall X \in \mathcal{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY $(hg)f = h(gf)$

UNITALITY $id_X f = f$, $g\, id_X = g$

A (COVARIANT) FUNCTOR $F : \mathcal{C} \to \mathcal{D}$ CONSISTS OF

(a) AN OBJECT $F(X) \in \mathcal{D}$ FOR EACH $X \in \mathcal{C}$.

(b) A MORPHISM $F(g) : F(X) \to F(Y) \in \mathcal{D}$ FOR EACH $g : X \to Y \in \mathcal{C}$.

RESPECTING:

- IDENTITY MORPHISMS $F(id_X) = id_{F(X)}$ $\forall X \in \mathcal{C}$
- COMPOSITION OF MORPHISMS $F(hg) = F(h)F(g)$ $\forall g : X \to Y,\ h : Y \to Z \in \mathcal{C}$

$\mathcal{C}$ IS THE DOMAIN OF F

I. FUNCTORS

TAKE CATEGORIES $\mathcal{C}, \mathcal{D}, \mathcal{E}$

A CATEGORY $\mathcal{C}$ CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ $\forall X, Y \in \mathcal{C}$.

(c) $id_X : X \to X$ $\forall X \in \mathcal{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY $(hg)f = h(gf)$

UNITALITY $id_X f = f$, $g\, id_X = g$

A (COVARIANT) FUNCTOR $F : \mathcal{C} \to \mathcal{D}$ CONSISTS OF

(a) AN OBJECT $F(X) \in \mathcal{D}$ FOR EACH $X \in \mathcal{C}$.

(b) A MORPHISM $F(g) : F(X) \to F(Y) \in \mathcal{D}$ FOR EACH $g : X \to Y \in \mathcal{C}$.

RESPECTING:

- IDENTITY MORPHISMS
$$F(id_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$$
- COMPOSITION OF MORPHISMS
$$F(hg) = F(h)F(g) \quad \forall g : X \to Y,\ h : Y \to Z \in \mathcal{C}$$

$\mathcal{C}$ IS THE DOMAIN OF F

$\mathcal{D}$ IS THE CODOMAIN OF F

I. FUNCTORS

TAKE CATEGORIES $\mathcal{C}, \mathcal{D}, \mathcal{E}$

A CATEGORY $\mathcal{C}$ CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ $\forall X,Y \in \mathcal{C}$.

(c) $id_X : X \to X$ $\forall X \in \mathcal{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY $(hg)f = h(gf)$

UNITALITY $id_X f = f$, $g\, id_X = g$

A (COVARIANT) FUNCTOR $F : \mathcal{C} \longrightarrow \mathcal{D}$ CONSISTS OF

(a) AN OBJECT $F(X) \in \mathcal{D}$ FOR EACH $X \in \mathcal{C}$.

(b) A MORPHISM $F(g) : F(X) \to F(Y) \in \mathcal{D}$ FOR EACH $g : X \to Y \in \mathcal{C}$.

RESPECTING:

- IDENTITY MORPHISMS $F(id_X) = id_{F(X)}$ $\forall X \in \mathcal{C}$
- COMPOSITION OF MORPHISMS $F(hg) = F(h)F(g)$ $\forall g : X \to Y,\ h : Y \to Z \in \mathcal{C}$

$\mathcal{C}$ IS THE DOMAIN OF F

$\mathcal{D}$ IS THE CODOMAIN OF F

F IS AN ENDOFUNCTOR WHEN $\mathcal{C} = \mathcal{D}$.

I. FUNCTORS

TAKE CATEGORIES $\mathcal{C}, \mathcal{D}, \mathcal{E}$

A CATEGORY $\mathcal{C}$ CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ $\forall X,Y \in \mathcal{C}$.

(c) $id_X : X \to X$ $\forall X \in \mathcal{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY $(hg)f = h(gf)$

UNITALITY $id_X f = f$, $g\, id_X = g$

A (COVARIANT) FUNCTOR $F : \mathcal{C} \to \mathcal{D}$ CONSISTS OF

(a) AN OBJECT $F(X) \in \mathcal{D}$ FOR EACH $X \in \mathcal{C}$.

(b) A MORPHISM $F(g) : F(X) \to F(Y) \in \mathcal{D}$ FOR EACH $g : X \to Y \in \mathcal{C}$.

RESPECTING:

- IDENTITY MORPHISMS $$F(id_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$$
- COMPOSITION OF MORPHISMS $$F(hg) = F(h)F(g)$$ $$\forall g : X \to Y,\ h : Y \to Z \in \mathcal{C}$$

$\mathcal{C}$ IS THE DOMAIN OF F

$\mathcal{D}$ IS THE CODOMAIN OF F

F IS AN ENDOFUNCTOR WHEN $\mathcal{C} = \mathcal{D}$.

Ex. $$Id_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$$ $$X \mapsto X$$ $$f \mapsto f$$

IDENTITY FUNCTOR

I. FUNCTORS

TAKE CATEGORIES $\mathscr{C}, \mathscr{D}, \mathscr{E}$

A CATEGORY $\mathscr{C}$ CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ $\forall X,Y \in \mathscr{C}$.

(c) $id_X : X \to X$ $\forall X \in \mathscr{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY $(hg)f = h(gf)$

UNITALITY $id_X f = f$, $g\, id_X = g$

A (COVARIANT) FUNCTOR $F : \mathscr{C} \to \mathscr{D}$ CONSISTS OF

(a) AN OBJECT $F(X) \in \mathscr{D}$ FOR EACH $X \in \mathscr{C}$.

(b) A MORPHISM $F(g) : F(X) \to F(Y) \in \mathscr{D}$ FOR EACH $g : X \to Y \in \mathscr{C}$.

RESPECTING:

- IDENTITY MORPHISMS $F(id_X) = id_{F(X)}$ $\forall X \in \mathscr{C}$
- COMPOSITION OF MORPHISMS $F(hg) = F(h)F(g)$ $\forall g : X \to Y,\ h : Y \to Z \in \mathscr{C}$

$\mathscr{C}$ IS THE DOMAIN OF F

$\mathscr{D}$ IS THE CODOMAIN OF F

F IS AN ENDOFUNCTOR WHEN $\mathscr{C} = \mathscr{D}$.

Ex. $\mathrm{Id}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$

$X \mapsto X$

$f \mapsto f$

IDENTITY FUNCTOR

THE COMPOSITION OF TWO FUNCTORS

$\mathscr{C} \xrightarrow{F} \mathscr{D} \xrightarrow{G} \mathscr{E}$

IS A FUNCTOR.

I. FUNCTORS

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

A (COVARIANT) **FUNCTOR**

$F : \mathcal{C} \longrightarrow \mathcal{D}$

CONSISTS OF

(a) AN OBJECT $F(X) \in \mathcal{D}$
FOR EACH $X \in \mathcal{C}$.

(b) A MORPHISM
$F(g) : F(X) \to F(Y) \in \mathcal{D}$
FOR EACH $g : X \to Y \in \mathcal{C}$.

RESPECTING:

- IDENTITY MORPHISMS
  $F(id_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$
- COMPOSITION OF MORPHISMS
  $F(hg) = F(h)F(g)$
  $\forall g : X \to Y,\ h : Y \to Z \in \mathcal{C}$

I. FUNCTORS

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

TOWARDS ANSWERING THIS QUESTION, WE NEED FUNCTORIAL VERSIONS OF "INJECTIVITY" & "SURJECTIVITY"

A (COVARIANT) FUNCTOR
$F : \mathcal{C} \longrightarrow \mathcal{D}$
CONSISTS OF

(a) AN OBJECT $F(X) \in \mathcal{D}$ FOR EACH $X \in \mathcal{C}$.

(b) A MORPHISM $F(g) : F(X) \to F(Y) \in \mathcal{D}$ FOR EACH $g : X \to Y \in \mathcal{C}$.

RESPECTING:

- IDENTITY MORPHISMS
$$F(id_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$$

- COMPOSITION OF MORPHISMS
$$F(hg) = F(h)F(g)$$
$$\forall g : X \to Y,\ h : Y \to Z \in \mathcal{C}$$

I. FUNCTORS

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

TOWARDS ANSWERING THIS QUESTION, WE NEED FUNCTORIAL VERSIONS OF "INJECTIVITY" & "SURJECTIVITY"

⋮

ON MORPHISMS

A (COVARIANT) FUNCTOR $F : \mathscr{C} \to \mathscr{D}$ CONSISTS OF

(a) AN OBJECT $F(X) \in \mathscr{D}$ FOR EACH $X \in \mathscr{C}$.

(b) A MORPHISM $F(g) : F(X) \to F(Y) \in \mathscr{D}$ FOR EACH $g : X \to Y \in \mathscr{C}$.

RESPECTING:

- IDENTITY MORPHISMS
  $$F(id_X) = id_{F(X)} \quad \forall X \in \mathscr{C}$$
- COMPOSITION OF MORPHISMS
  $$F(hg) = F(h)F(g)$$
  $$\forall g : X \to Y,\ h : Y \to Z \in \mathscr{C}$$

(ASSUMING $\mathscr{C}, \mathscr{D}$ LOCALLY SMALL)

CONSIDER THE FUNCTION:

$$F_{X,Y} : Hom_{\mathscr{C}}(X,Y) \to Hom_{\mathscr{D}}(F(X), F(Y))$$
$$g \longmapsto F(g)$$

I. FUNCTORS

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

TOWARDS ANSWERING THIS QUESTION, WE NEED FUNCTORIAL VERSIONS OF "INJECTIVITY" & "SURJECTIVITY"

⋮

ON MORPHISMS

A (COVARIANT) **FUNCTOR** $F : \mathcal{C} \longrightarrow \mathcal{D}$

CONSISTS OF

(a) AN OBJECT $F(X) \in \mathcal{D}$ FOR EACH $X \in \mathcal{C}$.

(b) A MORPHISM $F(g) : F(X) \to F(Y) \in \mathcal{D}$ FOR EACH $g : X \to Y \in \mathcal{C}$.

RESPECTING:

- IDENTITY MORPHISMS
$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$$

- COMPOSITION OF MORPHISMS
$$F(hg) = F(h)F(g)$$
$$\forall g : X \to Y,\ h : Y \to Z \in \mathcal{C}$$

(ASSUMING $\mathcal{C}, \mathcal{D}$ LOCALLY SMALL)

CONSIDER THE FUNCTION:

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$
$$g \longmapsto F(g)$$

F IS **FAITHFUL** IF $F_{X,Y}$ IS INJECTIVE $\forall X, Y \in \mathcal{C}$

F IS **FULL** IF $F_{X,Y}$ IS SURJECTIVE $\forall X, Y \in \mathcal{C}$

F IS **FULLY FAITHFUL** IF $F_{X,Y}$ IS BIJECTIVE $\forall X, Y \in \mathcal{C}$

I. FUNCTORS

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

TOWARDS ANSWERING THIS QUESTION, WE NEED FUNCTORIAL VERSIONS OF "INJECTIVITY" & "SURJECTIVITY"

⋮

ON OBJECTS

A (COVARIANT) FUNCTOR $F: \mathcal{C} \longrightarrow \mathcal{D}$ CONSISTS OF

(a) AN OBJECT $F(X) \in \mathcal{D}$ FOR EACH $X \in \mathcal{C}$.

(b) A MORPHISM $F(g): F(X) \to F(Y) \in \mathcal{D}$ FOR EACH $g: X \to Y \in \mathcal{C}$.

RESPECTING:

- IDENTITY MORPHISMS

$$F(id_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$$

- COMPOSITION OF MORPHISMS

$$F(hg) = F(h)F(g)$$
$$\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$$

(ASSUMING $\mathcal{C}, \mathcal{D}$ LOCALLY SMALL) CONSIDER THE FUNCTION:

$$F_{X,Y}: Hom_{\mathcal{C}}(X,Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$$
$$g \longmapsto F(g)$$

F IS FAITHFUL IF $F_{X,Y}$ IS INJECTIVE $\forall X,Y \in \mathcal{C}$

F IS FULL IF $F_{X,Y}$ IS SURJECTIVE $\forall X,Y \in \mathcal{C}$

F IS FULLY FAITHFUL IF $F_{X,Y}$ IS BIJECTIVE $\forall X,Y \in \mathcal{C}$

I. FUNCTORS

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

TOWARDS ANSWERING THIS QUESTION, WE NEED FUNCTORIAL VERSIONS OF "INJECTIVITY" & "SURJECTIVITY"

⋮

ON OBJECTS

A (COVARIANT) **FUNCTOR** $F : \mathcal{C} \longrightarrow \mathcal{D}$

CONSISTS OF

(a) AN OBJECT $F(X) \in \mathcal{D}$ FOR EACH $X \in \mathcal{C}$.

(b) A MORPHISM $F(g) : F(X) \to F(Y) \in \mathcal{D}$ FOR EACH $g : X \to Y \in \mathcal{C}$.

RESPECTING:

- IDENTITY MORPHISMS
$$F(id_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$$
- COMPOSITION OF MORPHISMS
$$F(hg) = F(h)F(g)$$
$$\forall g : X \to Y,\ h : Y \to Z \in \mathcal{C}$$

(ASSUMING $\mathcal{C}, \mathcal{D}$ LOCALLY SMALL)
CONSIDER THE FUNCTION:

$$F_{X,Y} : Hom_{\mathcal{C}}(X,Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$$
$$g \longmapsto F(g)$$

F IS FAITHFUL IF $F_{X,Y}$ IS INJECTIVE $\forall X, Y \in \mathcal{C}$

F IS FULL IF $F_{X,Y}$ IS SURJECTIVE $\forall X, Y \in \mathcal{C}$

F IS FULLY FAITHFUL IF $F_{X,Y}$ IS BIJECTIVE $\forall X, Y \in \mathcal{C}$

F IS AN **EMBEDDING** IF F IS FAITHFUL & F IS INJ. ON OBJS.

F IS **ESSENTIALLY SURJECTIVE** IF $\forall Y \in \mathcal{D}\ \exists X \in \mathcal{C}\ \ni\ Y \cong F(X)$.

I. FUNCTORS

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

TOWARDS ANSWERING THIS QUESTION, WE NEED FUNCTORIAL VERSIONS OF "INJECTIVITY" & "SURJECTIVITY"

WE'LL ANSWER THE QUESTION IN THE NEXT LECTURE WITH THESE NOTIONS

A (COVARIANT) **FUNCTOR** $F: \mathcal{C} \longrightarrow \mathcal{D}$ CONSISTS OF

(a) AN OBJECT $F(X) \in \mathcal{D}$ FOR EACH $X \in \mathcal{C}$.

(b) A MORPHISM $F(g): F(X) \to F(Y) \in \mathcal{D}$ FOR EACH $g: X \to Y \in \mathcal{C}$.

RESPECTING:

- IDENTITY MORPHISMS $F(id_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$
- COMPOSITION OF MORPHISMS $F(hg) = F(h)F(g)$ $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$

CONSIDER THE FUNCTION:

$$F_{X,Y}: Hom_{\mathcal{C}}(X,Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$$
$$g \longmapsto F(g)$$

F IS **FAITHFUL** IF $F_{X,Y}$ IS INJECTIVE $\forall X, Y \in \mathcal{C}$

F IS **FULL** IF $F_{X,Y}$ IS SURJECTIVE $\forall X, Y \in \mathcal{C}$

F IS **FULLY FAITHFUL** IF $F_{X,Y}$ IS BIJECTIVE $\forall X, Y \in \mathcal{C}$

F IS AN **EMBEDDING** IF F IS FAITHFUL & F IS INJ. ON OBJS.

F IS **ESSENTIALLY SURJECTIVE** IF $\forall Y \in \mathcal{D}\ \exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$.

I. FUNCTORS

A FUNCTOR $F : \mathcal{C} \longrightarrow \mathcal{D}$
(RESP., CONTRAVARIANT)
CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

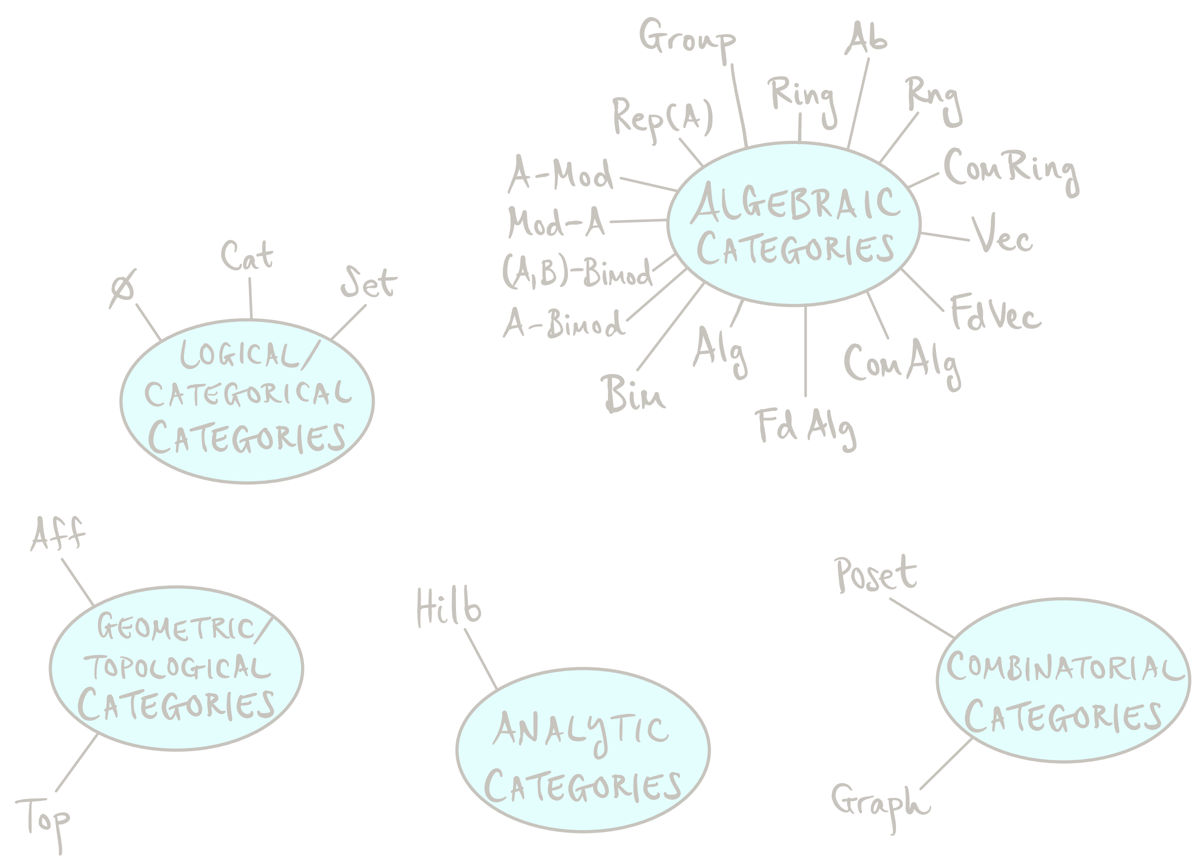
(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
(RESP.,
$F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)
$\forall g : X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$

- $F(hg) = F(h)F(g)$
  $\forall g : X \to Y,\ h : Y \to Z \in \mathcal{C}$
  (RESP.,
  $F(gf) = F(f)F(g)$
  $\forall f : W \to X,\ g : X \to Y \in \mathcal{C}$)

EXAMPLES...



<table>
<tr><td colspan="3">$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)),\ g \mapsto F(g)$</td><td>F EMBEDDING:
F FAITH & INJ ON OBJS</td></tr>
<tr><td>F FAITHFUL:
$F_{X,Y}$ INJ. $\forall X,Y$</td><td>F FULL:
$F_{X,Y}$ SURJ. $\forall X,Y$</td><td>F FULLY FAITHFUL:
$F_{X,Y}$ BIJ. $\forall X,Y$</td><td>F ESS. SURJ:
$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni.\ Y \cong F(X)$</td></tr>
</table>

I. FUNCTORS

$\mathrm{Forg}: \mathcal{C} \to \mathcal{D}$ FORGETFUL FUNCTOR (FORGET STRUCTURE)

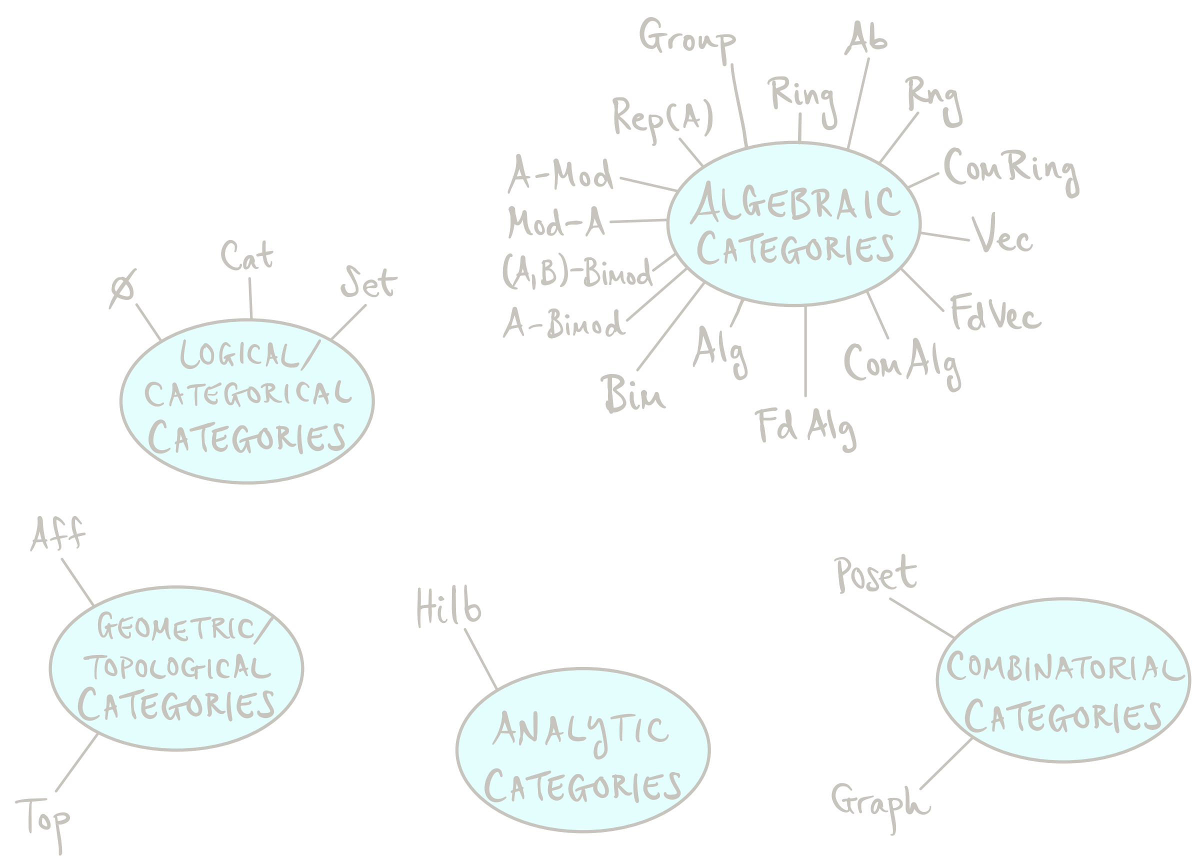
A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D}$ $\forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ (RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$) $\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ $\forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$ $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$ (RESP., $F(gf) = F(f)F(g)$ $\forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)



$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)),\ g \mapsto F(g)$			F EMBEDDING: F FAITH & INJ ON OBJS
F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$	F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$	F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$	F ESS. SURJ: $\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni Y \cong F(X)$

I. FUNCTORS

A FUNCTOR $F : \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT)

CONSISTS OF:

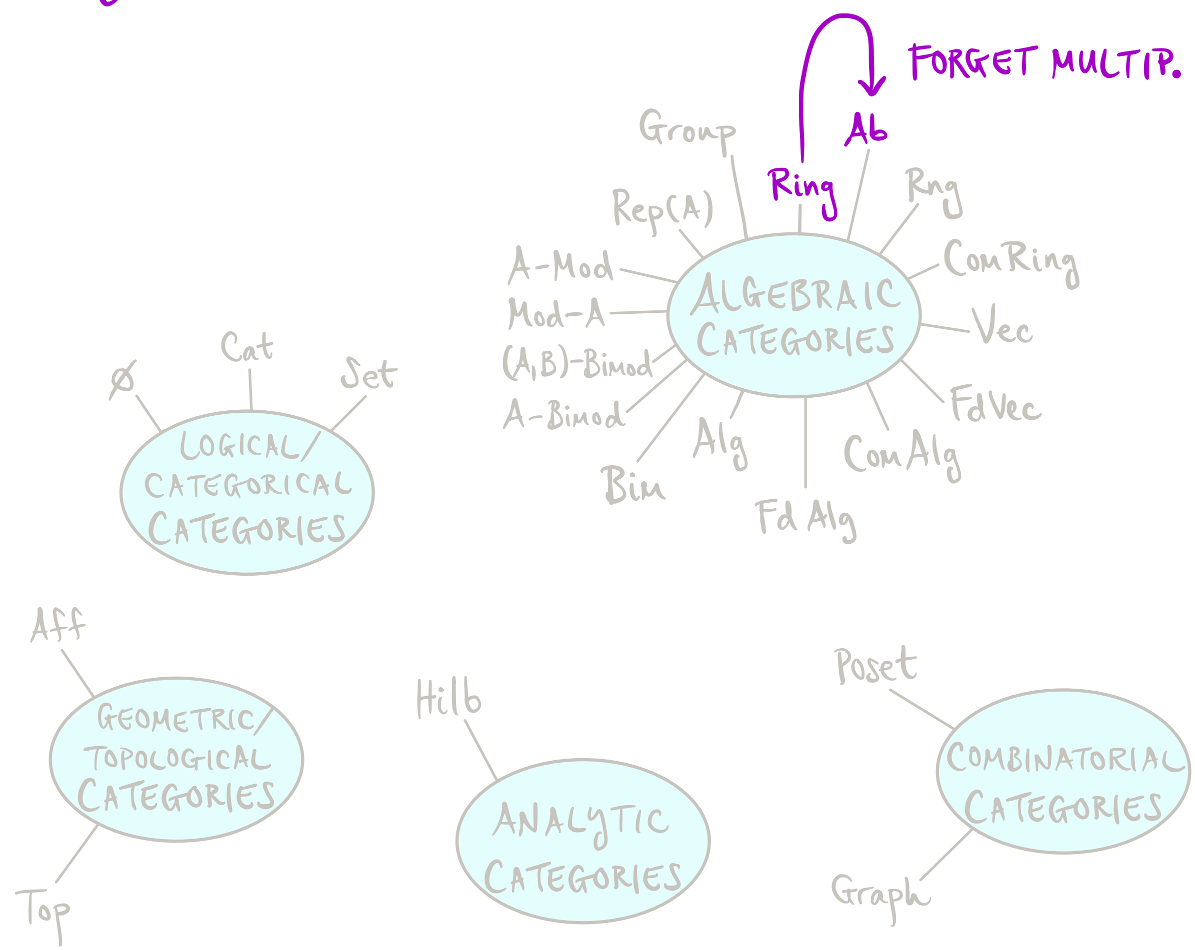
(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ (RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$) $\forall g : X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g) \quad \forall g : X \to Y,\ h : Y \to Z \in \mathcal{C}$ (RESP., $F(gf) = F(f)F(g) \quad \forall f : W \to X,\ g : X \to Y \in \mathcal{C}$)

$\mathrm{Forg} : \mathcal{C} \to \mathcal{D}$ FORGETFUL FUNCTOR (FORGET STRUCTURE)



<table>
<tr><td colspan="3">$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)),\ g \mapsto F(g)$</td><td>$F$ EMBEDDING: F FAITH & INJ ON OBJS</td></tr>
<tr><td>F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$</td><td>F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$</td><td>F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$</td><td>F ESS. SURJ: $\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni Y \cong F(X)$</td></tr>
</table>

I. FUNCTORS

A FUNCTOR $F: \mathscr{C} \to \mathscr{D}$
(RESP., CONTRAVARIANT)
CONSISTS OF:

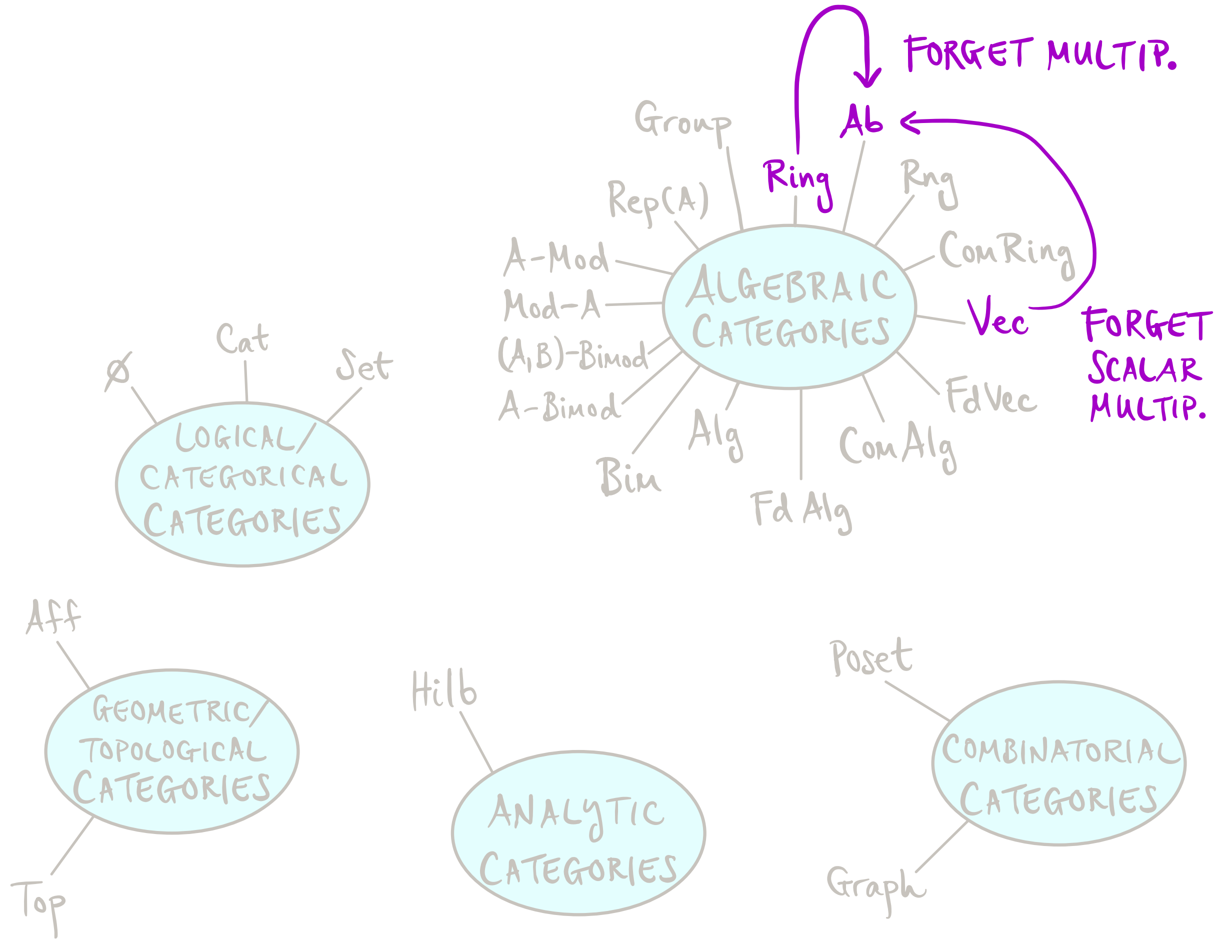
(a) $F(X) \in \mathscr{D} \quad \forall X \in \mathscr{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$
(RESP., $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))$)
$\forall g: X \to Y \in \mathscr{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathscr{C}$
- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathscr{C}$
  (RESP., $F(gf) = F(f)F(g)$
  $\forall f: W \to X,\ g: X \to Y \in \mathscr{C}$)

$\mathrm{Forg}: \mathscr{C} \to \mathscr{D}$ FORGETFUL FUNCTOR (FORGET STRUCTURE)



$F_{X,Y}: \mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y)),\ g \mapsto F(g)$			F EMBEDDING: F FAITH & INJ ON OBJS
F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$	F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$	F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$	F ESS. SURJ: $\forall Y \in \mathscr{D}, \exists X \in \mathscr{C}$ s.t. $Y \cong F(X)$

I. FUNCTORS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

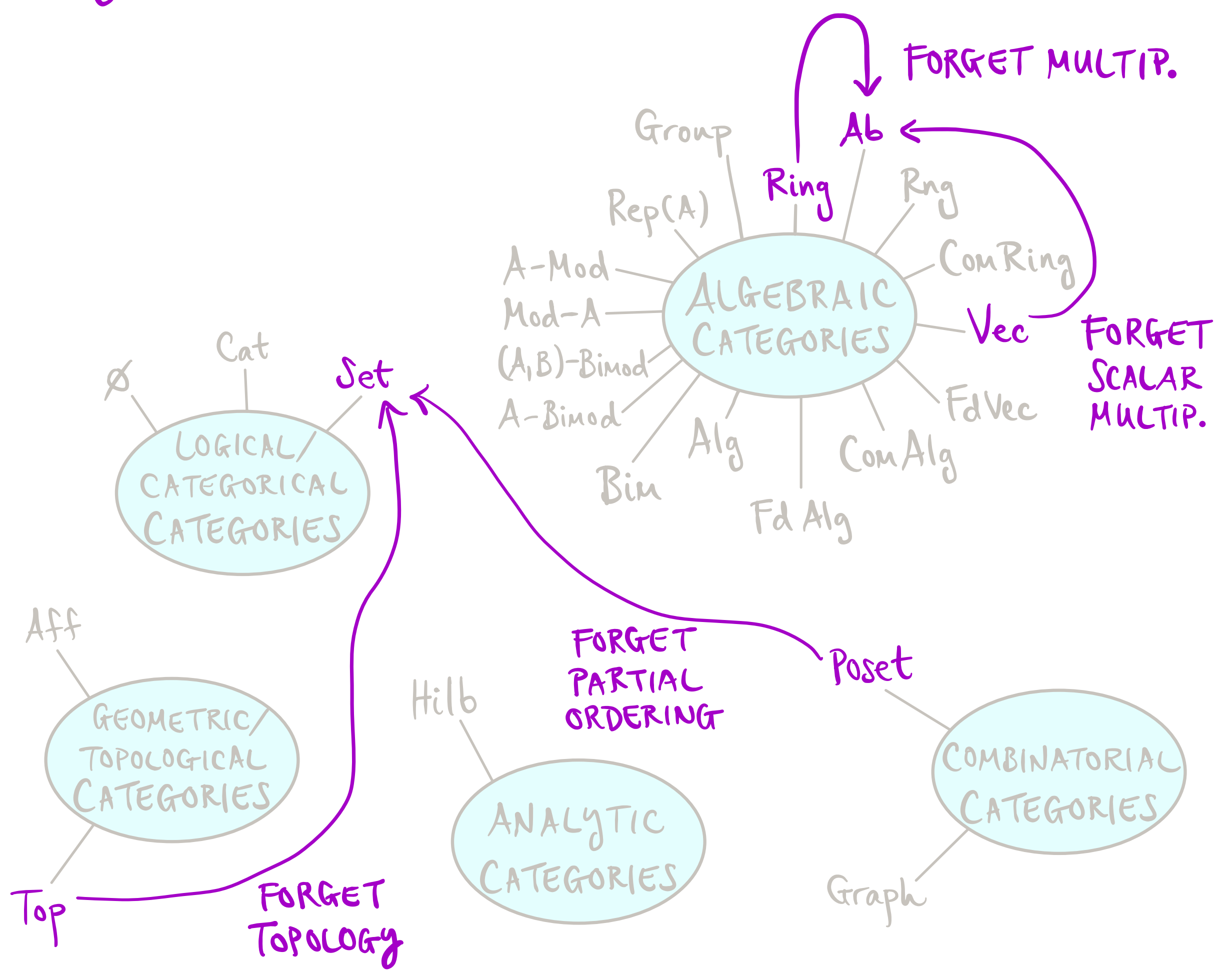
(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ (RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$) $\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$ $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$ (RESP., $F(gf) = F(f)F(g)$ $\forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)

$\mathrm{Forg}: \mathcal{C} \to \mathcal{D}$ FORGETFUL FUNCTOR (FORGET STRUCTURE)



<table>
<tr><td colspan="4">$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)),\ g \mapsto F(g)$</td><td>$F$ EMBEDDING:
F FAITH & INJ ON OBJS</td></tr>
<tr><td>F FAITHFUL:
$F_{X,Y}$ INJ. $\forall X,Y$</td><td>F FULL:
$F_{X,Y}$ SURJ. $\forall X,Y$</td><td>F FULLY FAITHFUL:
$F_{X,Y}$ BIJ. $\forall X,Y$</td><td></td><td>F ESS. SURJ:
$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni Y \cong F(X)$</td></tr>
</table>

I. FUNCTORS

A FUNCTOR $F: \mathscr{C} \to \mathscr{D}$ (RESP., CONTRAVARIANT)

CONSISTS OF:

(a) $F(X) \in \mathscr{D} \quad \forall X \in \mathscr{C}$.

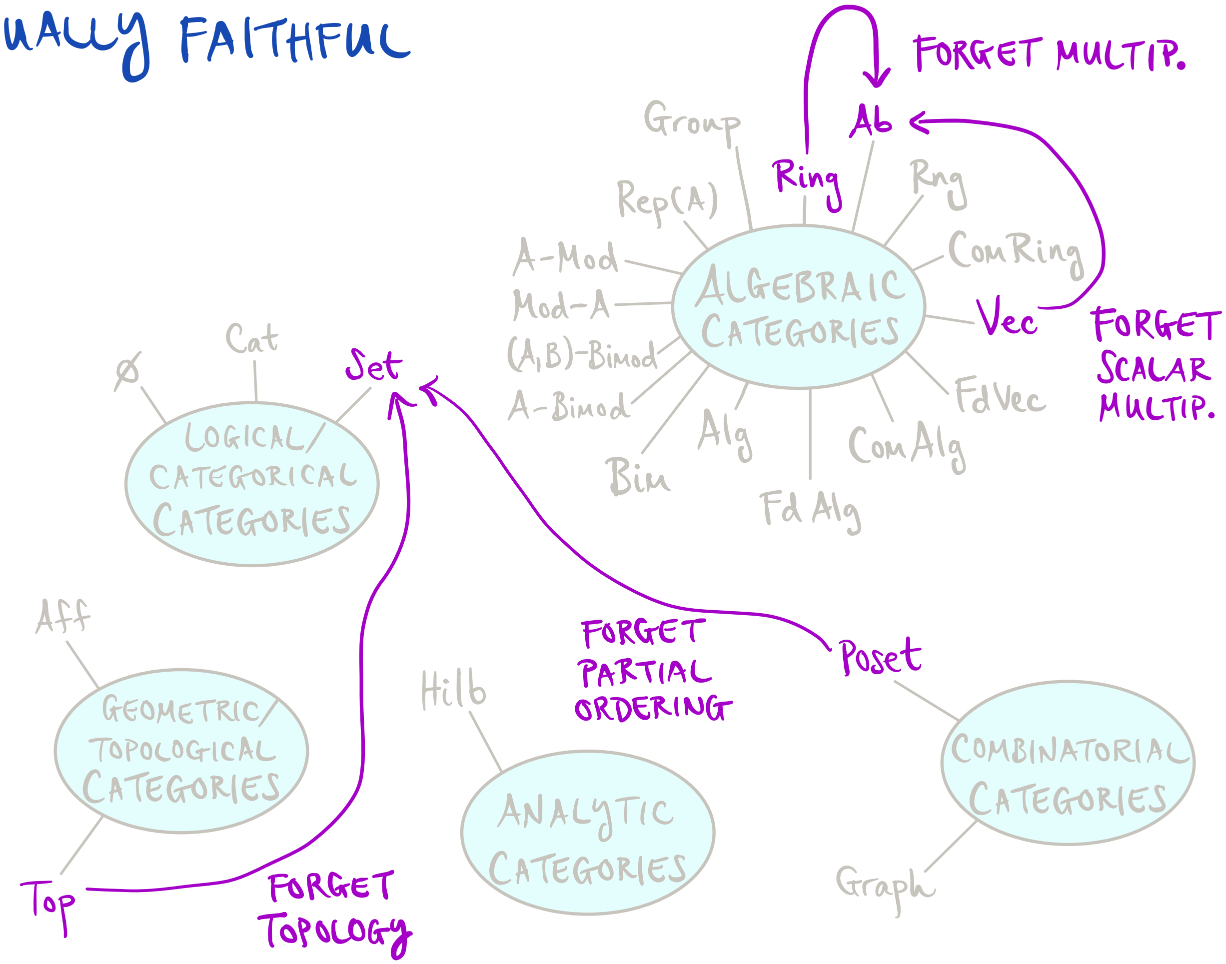
(b) $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$
(RESP., $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))$)
$\forall g: X \to Y \in \mathscr{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathscr{C}$
- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathscr{C}$
  (RESP., $F(gf) = F(f)F(g)$
  $\forall f: W \to X,\ g: X \to Y \in \mathscr{C}$)

$\mathrm{Forg}: \mathscr{C} \to \mathscr{D}$ FORGETFUL FUNCTOR (FORGET STRUCTURE)

USUALLY FAITHFUL



$F_{X,Y}: \mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y)),\ g \mapsto F(g)$			F EMBEDDING: F FAITH & INJ ON OBJS
F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$	F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$	F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$	F ESS. SURJ: $\forall Y \in \mathscr{D}, \exists X \in \mathscr{C}$ s.t. $Y \cong F(X)$

I. FUNCTORS

A FUNCTOR $F: \mathscr{C} \to \mathscr{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathscr{D} \quad \forall X \in \mathscr{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$ (RESP., $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))$) $\forall g: X \to Y \in \mathscr{C}$.

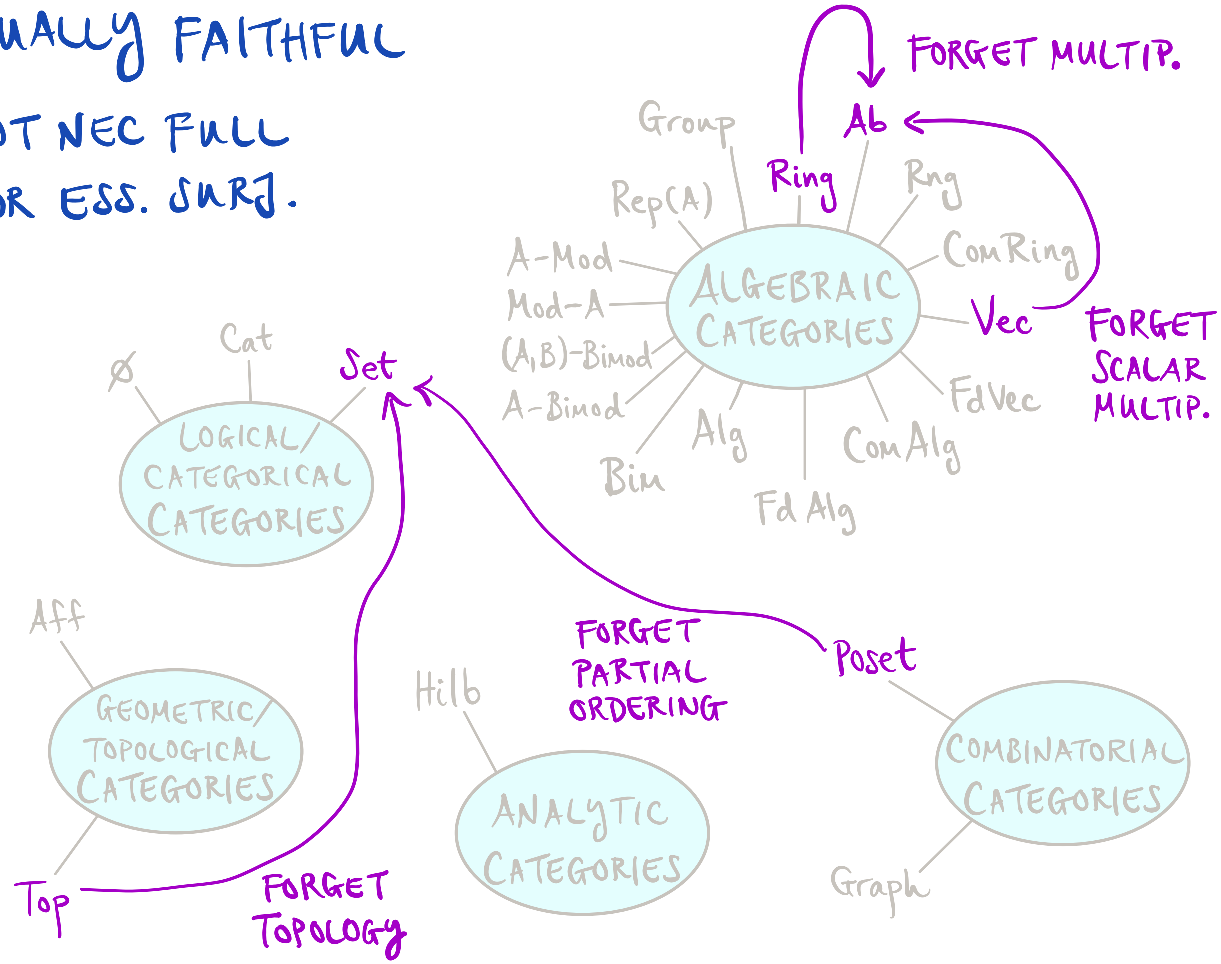
RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathscr{C}$
- $F(hg) = F(h)F(g) \quad \forall g: X \to Y,\ h: Y \to Z \in \mathscr{C}$ (RESP., $F(gf) = F(f)F(g) \quad \forall f: W \to X,\ g: X \to Y \in \mathscr{C}$)

$\mathrm{Forg}: \mathscr{C} \to \mathscr{D}$ FORGETFUL FUNCTOR (FORGET STRUCTURE)

USUALLY FAITHFUL

NOT NEC FULL NOR ESS. SURJ.



$F_{X,Y}: \mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y)),\ g \mapsto F(g)$			F EMBEDDING: F FAITH & INJ ON OBJS
F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$	F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$	F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$	F ESS. SURJ: $\forall Y \in \mathscr{D}, \exists X \in \mathscr{C}$ s.t. $Y \cong F(X)$

I. FUNCTORS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D}$ $\forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ (RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$) $\forall g: X \to Y \in \mathcal{C}$.

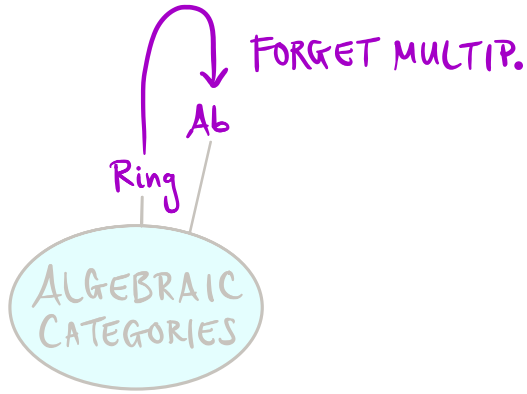
RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ $\forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$ $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$ (RESP., $F(gf) = F(f)F(g)$ $\forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)

$\mathrm{Forg}: \mathcal{C} \to \mathcal{D}$ FORGETFUL FUNCTOR (FORGET STRUCTURE)

USUALLY FAITHFUL

NOT NEC FULL NOR ESS. SURJ.



$\mathrm{Forg}: \mathrm{Ring} \longrightarrow \mathrm{Ab}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)),\ g \mapsto F(g)$			F EMBEDDING: F FAITH & INJ ON OBJS
F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$	F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$	F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$	F ESS. SURJ: $\forall Y \in \mathcal{D}, \exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$

I. FUNCTORS

A FUNCTOR $F: \mathscr{C} \to \mathscr{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathscr{D} \quad \forall X \in \mathscr{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$ (RESP., $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))$) $\forall g: X \to Y \in \mathscr{C}$.

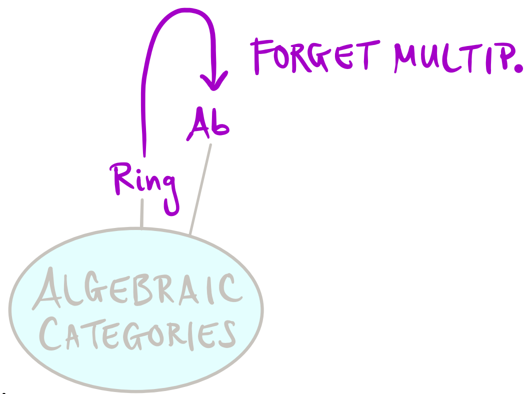
RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathscr{C}$
- $F(hg) = F(h)F(g) \quad \forall g: X \to Y,\ h: Y \to Z \in \mathscr{C}$ (RESP., $F(gf) = F(f)F(g) \quad \forall f: W \to X,\ g: X \to Y \in \mathscr{C}$)

$\mathrm{Forg}: \mathscr{C} \to \mathscr{D}$ FORGETFUL FUNCTOR (FORGET STRUCTURE)

USUALLY FAITHFUL

NOT NEC FULL NOR ESS. SURJ.



$\mathrm{Forg}: \mathrm{Ring} \longrightarrow \mathrm{Ab}$

CANNOT BE UPGRADED TO RING HOMOM. $\rightsquigarrow$ $\left[\begin{array}{l} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z} \\ d + n\mathbb{Z} \mapsto 0 \end{array}\right]$

$F_{X,Y}: \mathrm{Hom}_{\mathscr{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y)),\ g \mapsto F(g)$			F EMBEDDING: F FAITH & INJ ON OBJS
F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$	F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$	F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$	F ESS. SURJ: $\forall Y \in \mathscr{D}, \exists X \in \mathscr{C} \ni Y \cong F(X)$

I. FUNCTORS

A FUNCTOR $F : \mathscr{C} \to \mathscr{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathscr{D} \quad \forall X \in \mathscr{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$
(RESP., $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))$)
$\forall g : X \to Y \in \mathscr{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathscr{C}$
- $F(hg) = F(h)F(g)$ $\forall g : X \to Y,\ h : Y \to Z \in \mathscr{C}$
(RESP., $F(gf) = F(f)F(g)$ $\forall f : W \to X,\ g : X \to Y \in \mathscr{C}$)

$\mathrm{Forg} : \mathscr{C} \to \mathscr{D}$ FORGETFUL FUNCTOR (FORGET STRUCTURE)

USUALLY FAITHFUL

NOT NEC FULL

NOR ESS. SURJ.

$\mathrm{Ring} \xrightarrow{\text{FORGET MULTIP.}} \mathrm{Ab}$

ALGEBRAIC CATEGORIES

$\mathrm{Forg} : \mathrm{Ring} \longrightarrow \mathrm{Ab}$

CANNOT BE UPGRADED TO RING HOMOM. $\rightsquigarrow$ $\left[\begin{array}{l} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z} \\ d + n\mathbb{Z} \mapsto 0 \end{array}\right]$

CANNOT BE UPGRADED TO RING $\rightsquigarrow$ $\mathbb{Q}/\mathbb{Z}$

$F_{X,Y} : \mathrm{Hom}_{\mathscr{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y)),\ g \mapsto F(g)$			F EMBEDDING: F FAITH & INJ ON OBJS
F FAITHFUL: $F_{X,Y}$ INJ. $\forall X, Y$	F FULL: $F_{X,Y}$ SURJ. $\forall X, Y$	F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X, Y$	F ESS. SURJ: $\forall Y \in \mathscr{D}, \exists X \in \mathscr{C}$ s.t. $Y \cong F(X)$

I. FUNCTORS

$\mathrm{Forg}: \mathcal{C} \to \mathcal{D}$ FORGETFUL FUNCTOR (FORGET STRUCTURE)

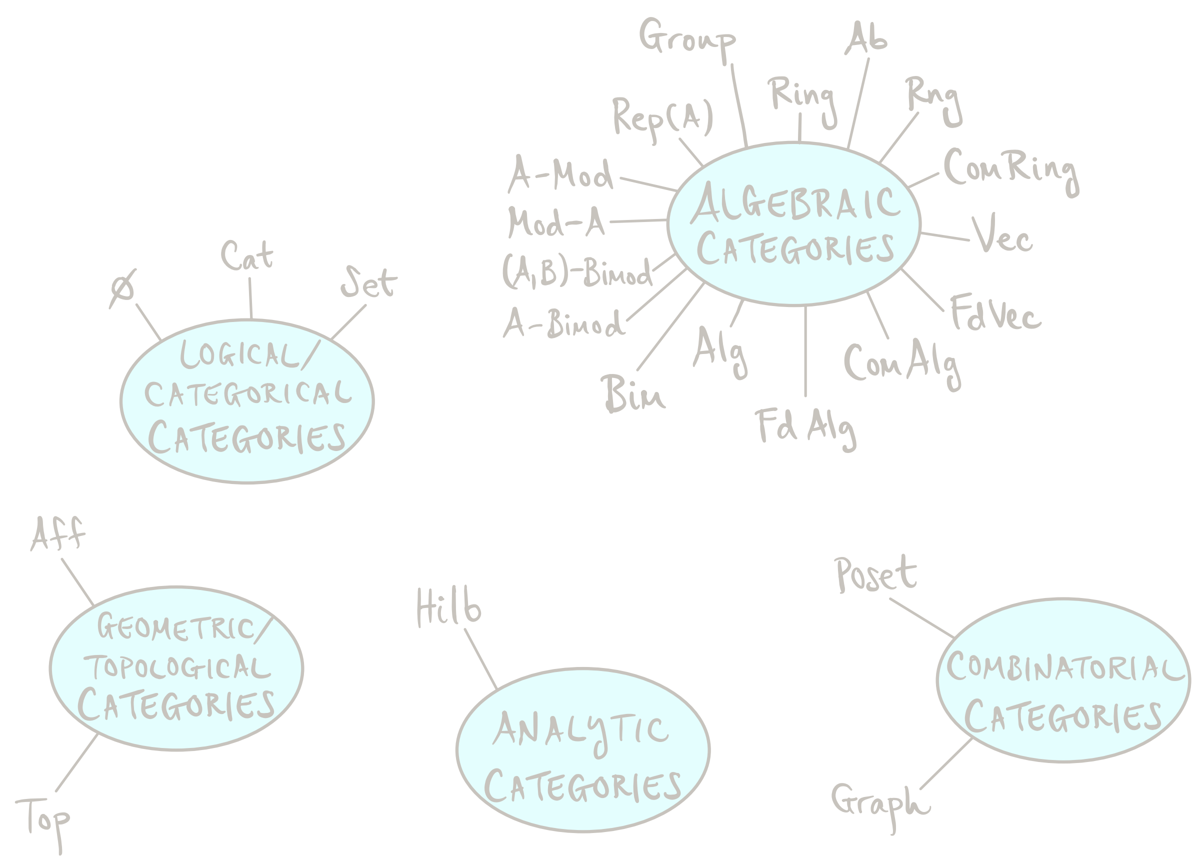
A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D}\ \ \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
(RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)
$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}\ \ \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$
  (RESP., $F(gf) = F(f)F(g)$
  $\forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)



$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)),\ g \mapsto F(g)$			F EMBEDDING: F FAITH & INJ ON OBJS
F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$	F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$	F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$	F ESS. SURJ: $\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni. Y \cong F(X)$

I. FUNCTORS

Inc $: \mathcal{C} \to \mathcal{D}$ **inclusion** (IMPOSING A CERTAIN PROPERTY OF OBJS/HOMS IN $\mathcal{D}$ ON OBJS/HOMS IN $\mathcal{C}$)

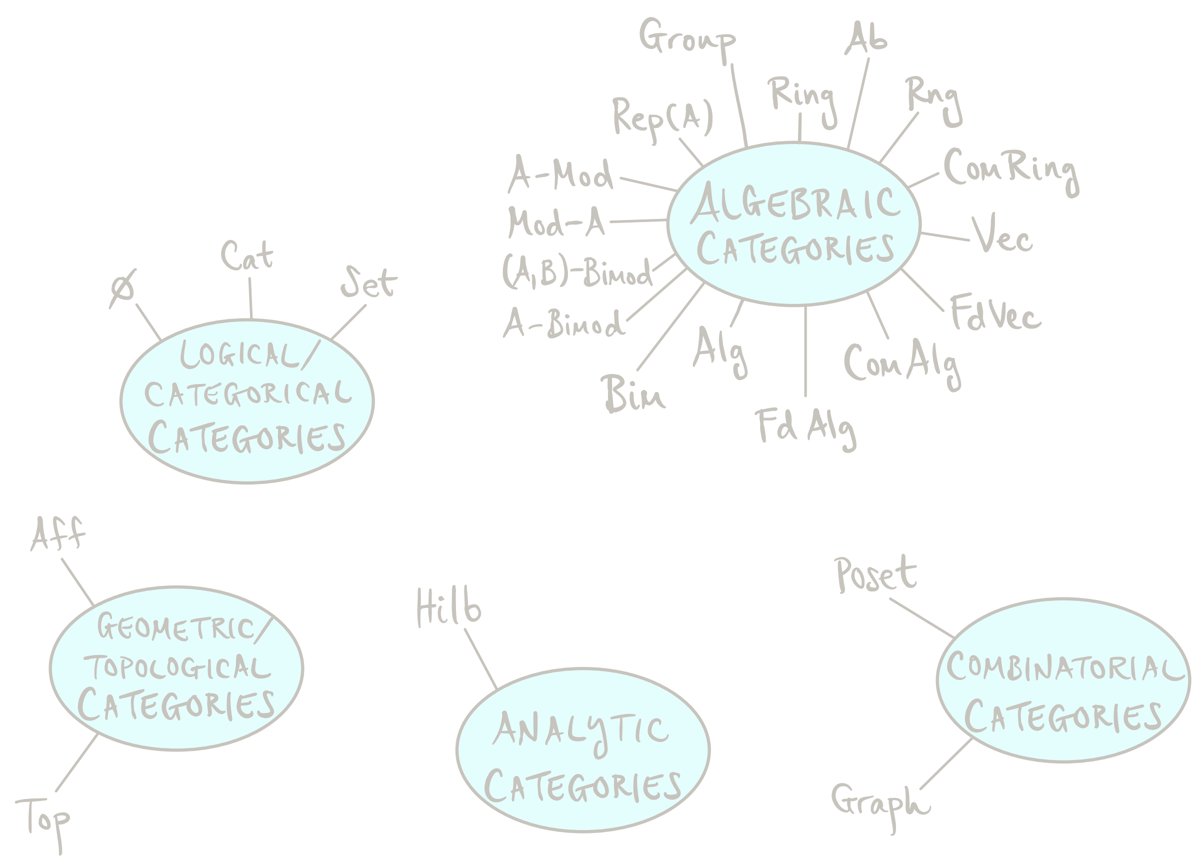
A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ (RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$) $\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g) \quad \forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$ (RESP., $F(gf) = F(f)F(g) \quad \forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)



$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)),\ g \mapsto F(g)$			F EMBEDDING: F FAITH & INJ ON OBJS
F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$	F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$	F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$	F ESS. SURJ: $\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni Y \cong F(X)$

I. FUNCTORS

$\mathrm{Inc}: \mathscr{C} \longrightarrow \mathscr{D}$ INCLUSION (IMPOSING A CERTAIN PROPERTY OF OBJS/HOMS IN $\mathscr{D}$ ON OBJS/HOMS IN $\mathscr{C}$)

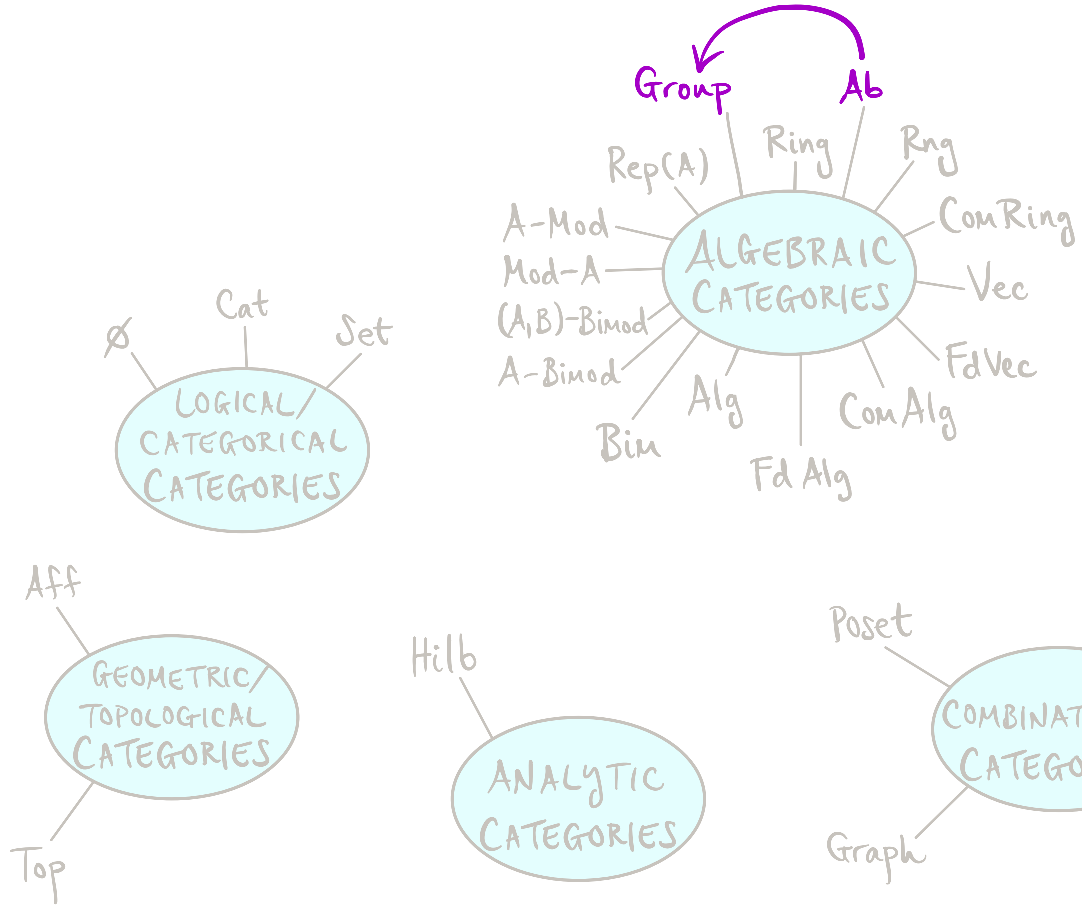
A (RESP., CONTRAVARIANT) FUNCTOR $F: \mathscr{C} \longrightarrow \mathscr{D}$ CONSISTS OF:

(a) $F(X) \in \mathscr{D} \quad \forall X \in \mathscr{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$ (RESP., $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))$) $\forall g: X \to Y \in \mathscr{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathscr{C}$
- $F(hg) = F(h)F(g) \quad \forall g: X \to Y,\ h: Y \to Z \in \mathscr{C}$ (RESP., $F(gf) = F(f)F(g) \quad \forall f: W \to X,\ g: X \to Y \in \mathscr{C}$)



$F_{X,Y}: \mathrm{Hom}_{\mathscr{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y)),\ g \mapsto F(g)$			F EMBEDDING: F FAITH & INJ ON OBJS
F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$	F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$	F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$	F ESS. SURJ: $\forall Y \in \mathscr{D}, \exists X \in \mathscr{C} \ni Y \cong F(X)$

I. FUNCTORS

$\mathrm{Inc}: \mathcal{C} \to \mathcal{D}$ INCLUSION (IMPOSING A CERTAIN PROPERTY OF OBJS/HOMS IN $\mathcal{D}$ ON OBJS/HOMS IN $\mathcal{C}$)

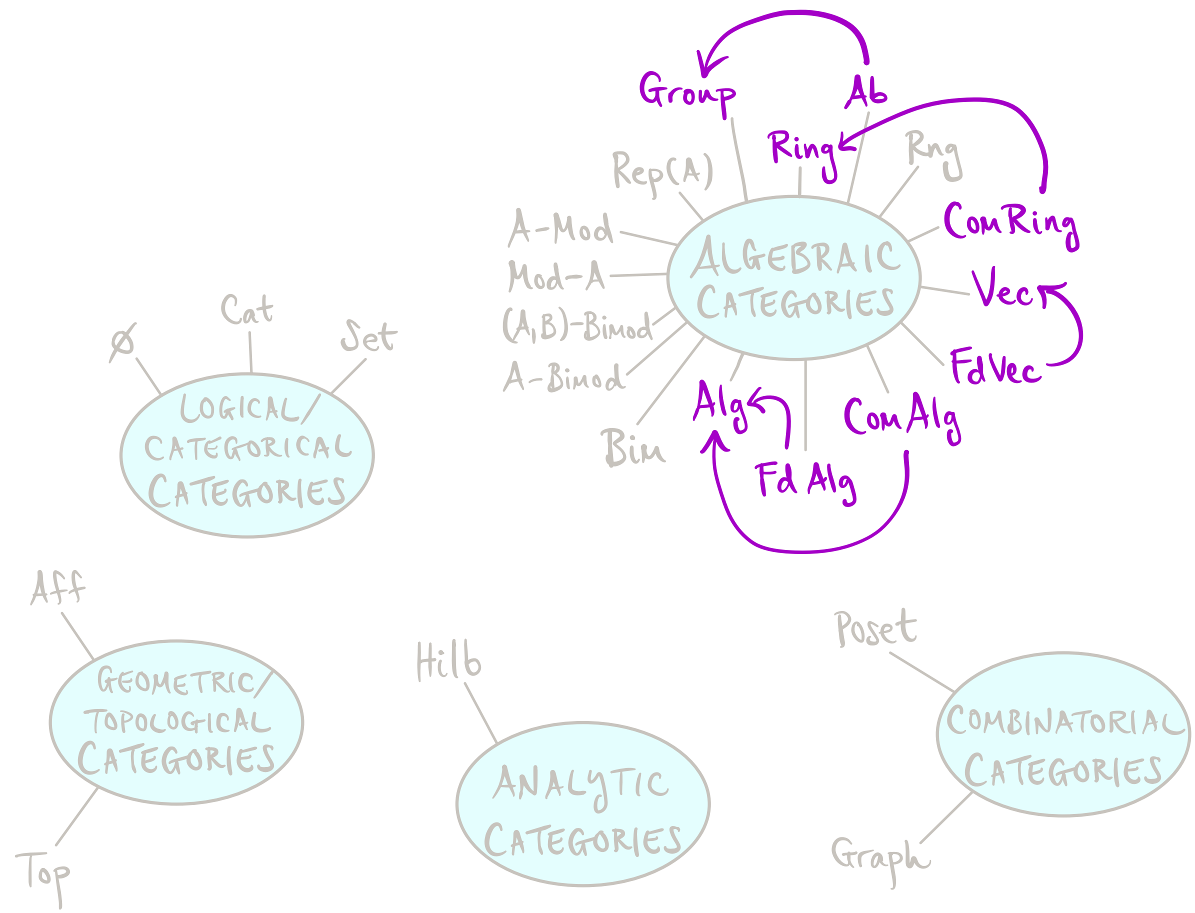
A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ (RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$) $\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g) \quad \forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$ (RESP., $F(gf) = F(f)F(g) \quad \forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)



$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)),\ g \mapsto F(g)$			F EMBEDDING: F FAITH & INJ ON OBJS
F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$	F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$	F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$	F ESS. SURJ: $\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni Y \cong F(X)$

I. FUNCTORS

A FUNCTOR $F: \mathcal{C} \longrightarrow \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ (RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$) $\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g) \quad \forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$ (RESP., $F(gf) = F(f)F(g) \quad \forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)

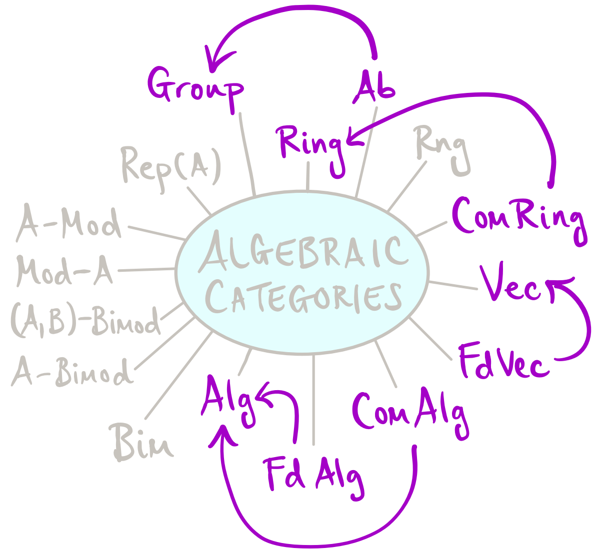
$\mathrm{Inc}: \mathcal{C} \longrightarrow \mathcal{D}$ INCLUSION (IMPOSING A CERTAIN PROPERTY OF OBJS/HOMS IN $\mathcal{D}$ ON OBJS/HOMS IN $\mathcal{C}$)

ALWAYS FAITHFUL

IF FULL, THEN

$\mathcal{C}$ = FULL SUBCAT OF $\mathcal{D}$:

$$\mathrm{Hom}_{\mathcal{D}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y) \quad \forall X, Y \in \mathcal{C}$$



$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)),\ g \mapsto F(g)$			F EMBEDDING: F FAITH & INJ ON OBJS
F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$	F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$	F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$	F ESS. SURJ: $\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni Y \cong F(X)$

I. FUNCTORS

A FUNCTOR $F: \mathcal{C} \longrightarrow \mathcal{D}$ (RESP., CONTRAVARIANT)

CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$

(RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)

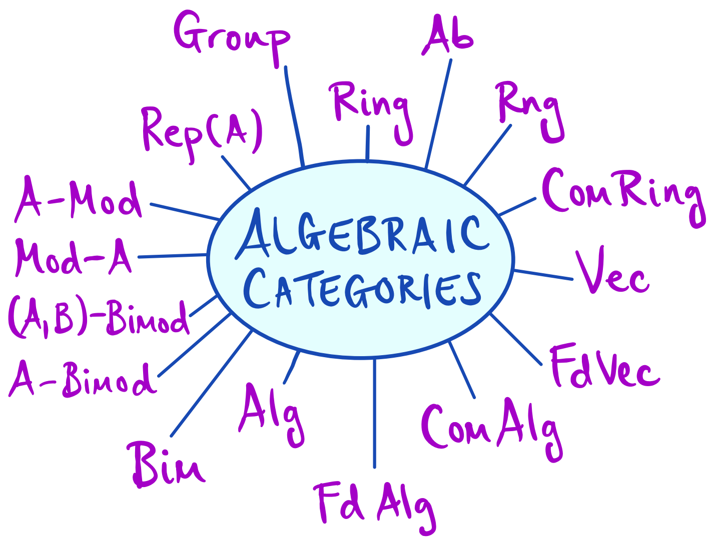
$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$

- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$
  (RESP., $F(gf) = F(f)F(g)$ $\forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)

ALGEBRAIC FUNCTORS...



I. FUNCTORS

ALGEBRAIC FUNCTORS...

A FUNCTOR $F: \mathcal{C} \longrightarrow \mathcal{D}$
(RESP., CONTRAVARIANT)

CONSISTS OF:

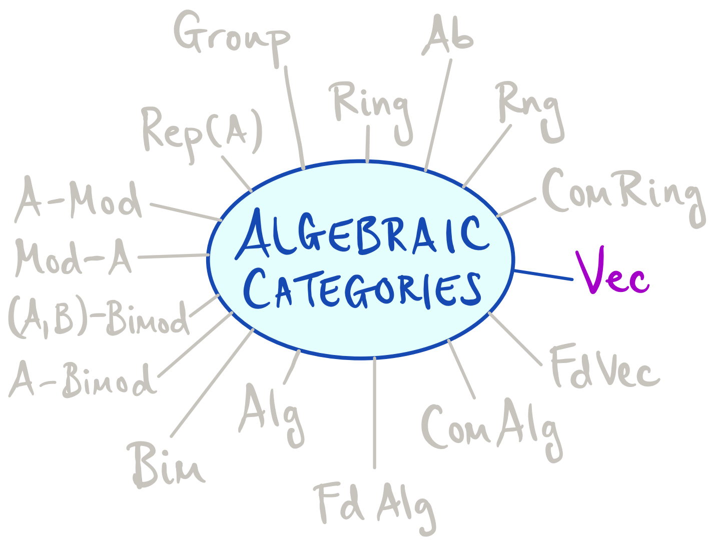
(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
$\left(\text{RESP., } F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))\right)$
$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$

- $F(hg) = F(h)F(g)$
$\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$
$\left(\text{RESP., } F(gf) = F(f)F(g) \quad \forall f: W \to X,\ g: X \to Y \in \mathcal{C}\right)$



I. FUNCTORS

ALGEBRAIC FUNCTORS...

/ $\Bbbk$ FIELD

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT)

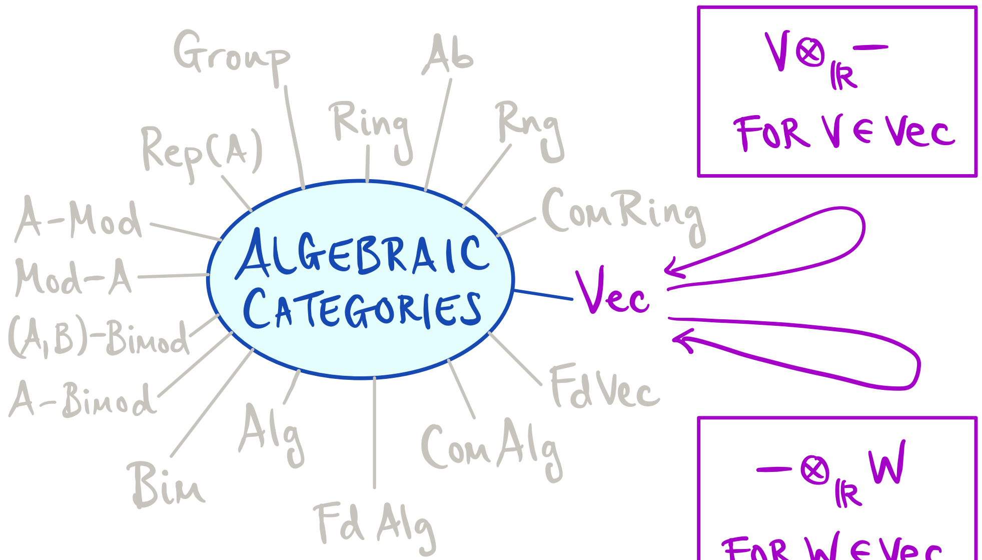
CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
(RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)
$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$
  (RESP., $F(gf) = F(f)F(g)$
  $\forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)



I. FUNCTORS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$
(RESP., CONTRAVARIANT)
CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
$\left(\text{RESP., } F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))\right)$
$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$

- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$
  $\left(\text{RESP., } F(gf) = F(f)F(g) \quad \forall f: W \to X,\ g: X \to Y \in \mathcal{C}\right)$

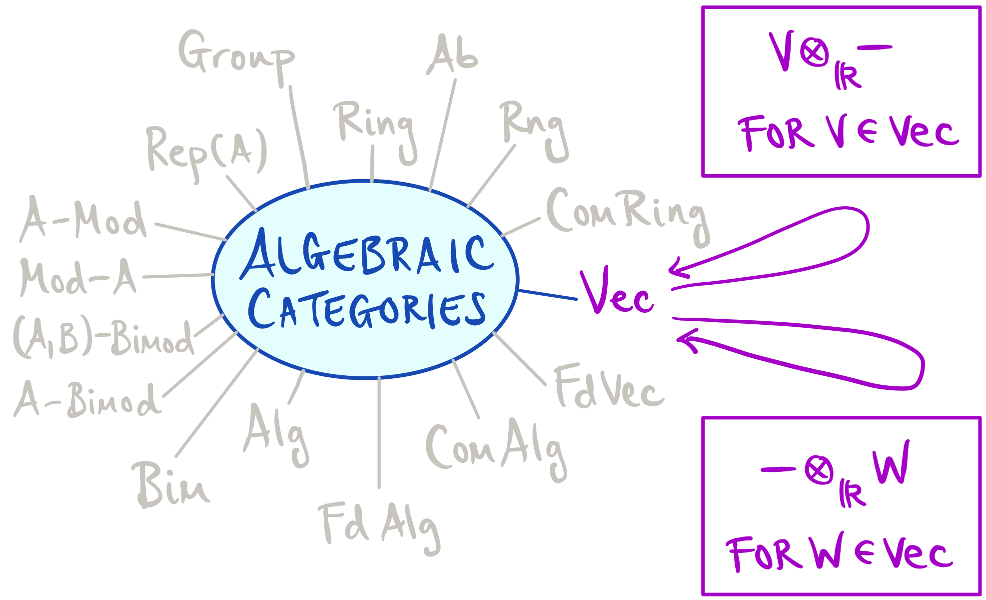
ALGEBRAIC FUNCTORS...

/ $\Bbbk$ FIELD

$V \otimes_{\Bbbk} - : \mathrm{Vec} \longrightarrow \mathrm{Vec}$ FOR FIXED $V \in \mathrm{Vec}$

$$W \longmapsto V \otimes_{\Bbbk} W$$

$$\left[W \xrightarrow{g} W'\right] \longmapsto \left[V \otimes_{\Bbbk} W \xrightarrow{\mathrm{id}_V \otimes_{\Bbbk} g} V \otimes_{\Bbbk} W'\right]$$



I. FUNCTORS

ALGEBRAIC FUNCTORS...

/ $\Bbbk$ FIELD

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
(RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)
$\forall g: X \to Y \in \mathcal{C}$.

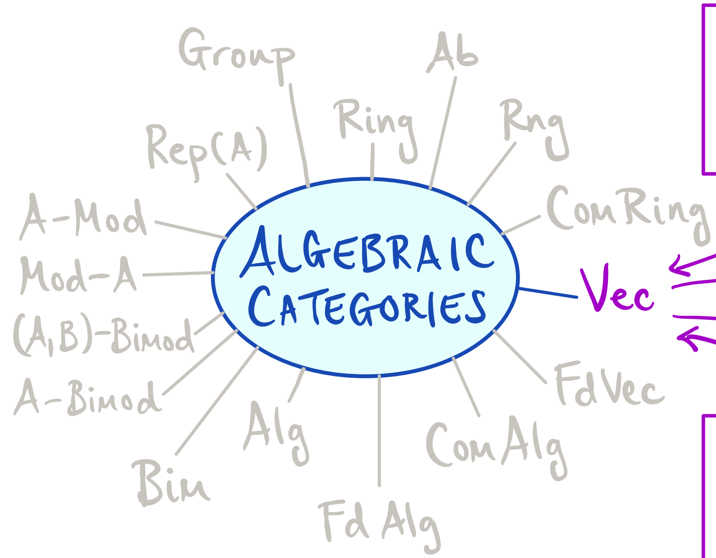
RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$
  (RESP., $F(gf) = F(f)F(g)$
  $\forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)

$V \otimes_{\Bbbk} - : \mathrm{Vec} \longrightarrow \mathrm{Vec}$ FOR FIXED $V \in \mathrm{Vec}$

$$W \longmapsto V \otimes_{\Bbbk} W$$

$$[W \xrightarrow{g} W'] \mapsto [V \otimes_{\Bbbk} W \xrightarrow{\mathrm{id}_V \otimes_{\Bbbk} g} V \otimes_{\Bbbk} W']$$



$V \otimes_{\Bbbk} -$ FOR $V \in \mathrm{Vec}$

$- \otimes_{\Bbbk} W$ FOR $W \in \mathrm{Vec}$

$- \otimes_{\Bbbk} W : \mathrm{Vec} \longrightarrow \mathrm{Vec}$ FOR FIXED $W \in \mathrm{Vec}$

$$V \longmapsto V \otimes_{\Bbbk} W$$

$$[V \xrightarrow{g} V'] \mapsto [V \otimes_{\Bbbk} W \xrightarrow{g \otimes_{\Bbbk} \mathrm{id}_W} V' \otimes_{\Bbbk} W]$$

I. FUNCTORS

A FUNCTOR $F:\mathcal{C}\to\mathcal{D}$
(RESP., CONTRAVARIANT)
CONSISTS OF:

(a) $F(X)\in\mathcal{D}\ \ \forall X\in\mathcal{C}$.

(b) $F(g)\in \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$\left(\text{RESP., } F(g)\in \mathrm{Hom}_{\mathcal{D}}(F(Y),F(X))\right)$
$\forall g:X\to Y\in\mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X)=\mathrm{id}_{F(X)}\ \ \forall X\in\mathcal{C}$
- $F(hg)=F(h)F(g)$
  $\forall g:X\to Y,\ h:Y\to Z\in\mathcal{C}$
  $\left(\text{RESP., } F(gf)=F(f)F(g)\ \ \forall f:W\to X,\ g:X\to Y\in\mathcal{C}\right)$

ALGEBRAIC FUNCTORS...

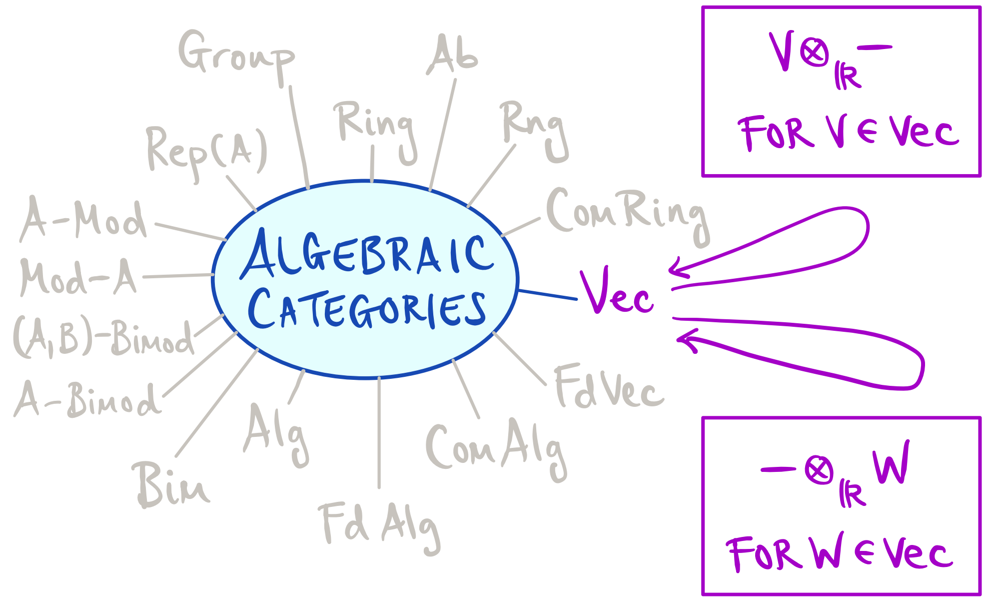
/ $\Bbbk$ FIELD

$V\otimes_{\Bbbk} - : \mathsf{Vec}\longrightarrow\mathsf{Vec}$ FOR FIXED $V\in\mathsf{Vec}$

$W\longmapsto V\otimes_{\Bbbk} W$

$[W\xrightarrow{g}W'] \mapsto [V\otimes_{\Bbbk} W\xrightarrow{\mathrm{id}_V\otimes_{\Bbbk} g} V\otimes_{\Bbbk} W']$

THESE ARE COVARIANT



$-\otimes_{\Bbbk} W : \mathsf{Vec}\longrightarrow\mathsf{Vec}$ FOR FIXED $W\in\mathsf{Vec}$

$V\longmapsto V\otimes_{\Bbbk} W$

$[V\xrightarrow{g}V'] \mapsto [V\otimes_{\Bbbk} W\xrightarrow{g\otimes_{\Bbbk}\mathrm{id}_W} V'\otimes_{\Bbbk} W]$

I. FUNCTORS

ALGEBRAIC FUNCTORS...

A FUNCTOR $F: \mathscr{C} \to \mathscr{D}$
(RESP., CONTRAVARIANT)

CONSISTS OF:

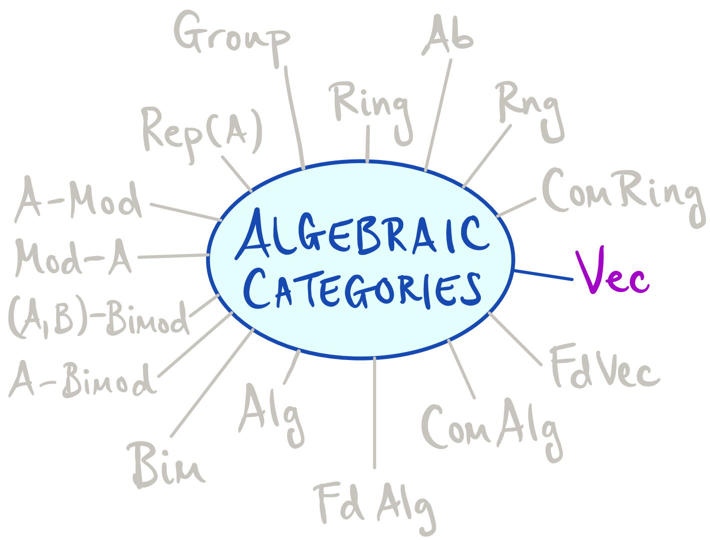
(a) $F(X) \in \mathscr{D} \quad \forall X \in \mathscr{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$
$\left(\text{RESP., } F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))\right)$
$\forall g: X \to Y \in \mathscr{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathscr{C}$

- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathscr{C}$
  $\left(\text{RESP., } F(gf) = F(f)F(g) \quad \forall f: W \to X,\ g: X \to Y \in \mathscr{C}\right)$



I. FUNCTORS

ALGEBRAIC FUNCTORS...

/ $\Bbbk$ FIELD

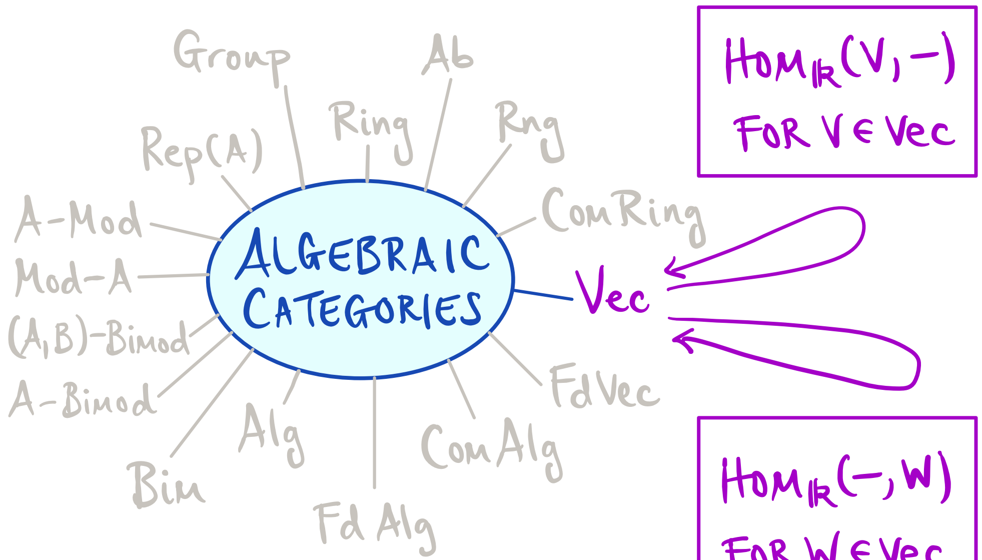
A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D}$ $\forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ (RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$) $\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ $\forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$ $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$ (RESP., $F(gf) = F(f)F(g)$ $\forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)



I. FUNCTORS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D}$ $\forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ $\left(\text{RESP., } F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))\right)$ $\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ $\forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$ $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$ $\left(\text{RESP., } F(gf) = F(f)F(g) \ \forall f: W \to X,\ g: X \to Y \in \mathcal{C}\right)$

ALGEBRAIC FUNCTORS...

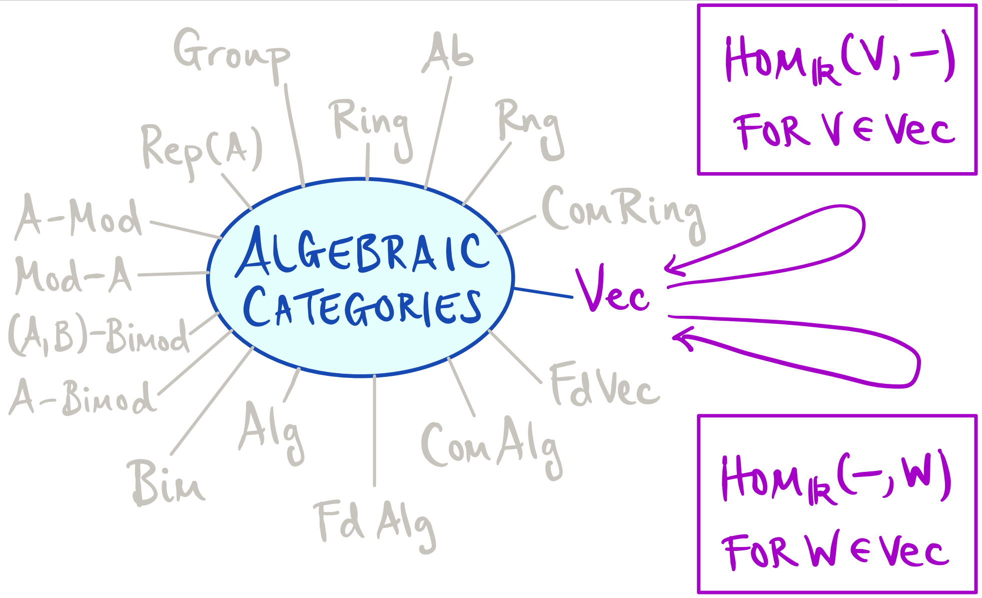
/ $\Bbbk$ FIELD

$\mathrm{Hom}_{\Bbbk}(V,-): \mathsf{Vec} \longrightarrow \mathsf{Vec}$ FOR FIXED $V \in \mathsf{Vec}$

$$W \longmapsto \mathrm{Hom}_{\Bbbk}(V,W)$$

$$[W \xrightarrow{g} W'] \longmapsto [\mathrm{Hom}_{\Bbbk}(V,W) \to \mathrm{Hom}_{\Bbbk}(V,W')]$$

$$f \longmapsto gf$$



I. FUNCTORS

A FUNCTOR $F : \mathscr{C} \to \mathscr{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathscr{D} \quad \forall X \in \mathscr{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$ (RESP., $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))$) $\forall g : X \to Y \in \mathscr{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathscr{C}$
- $F(hg) = F(h)F(g) \quad \forall g : X \to Y,\ h : Y \to Z \in \mathscr{C}$ (RESP., $F(gf) = F(f)F(g) \quad \forall f : W \to X,\ g : X \to Y \in \mathscr{C}$)

ALGEBRAIC FUNCTORS...

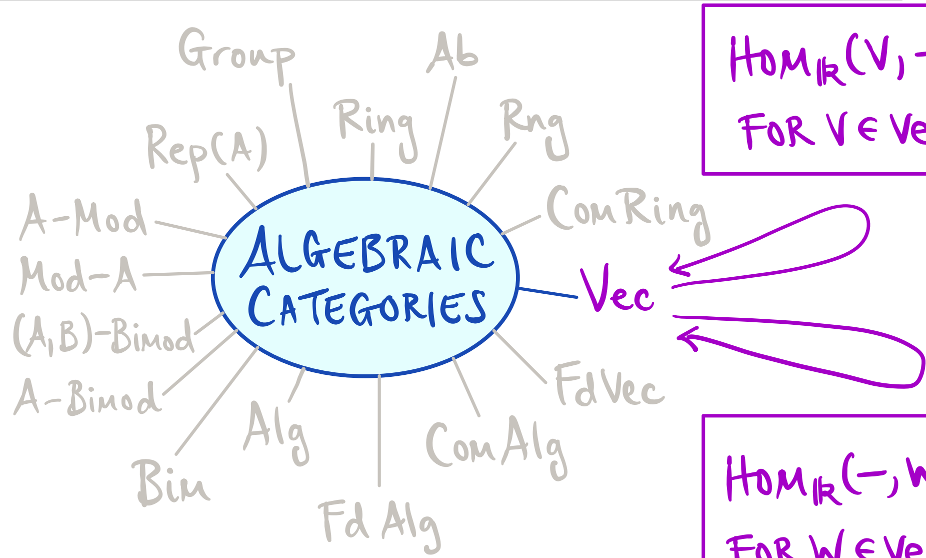
/ $\Bbbk$ FIELD

$\mathrm{Hom}_{\Bbbk}(V, -) : \mathsf{Vec} \longrightarrow \mathsf{Vec}$ FOR FIXED $V \in \mathsf{Vec}$

$$W \longmapsto \mathrm{Hom}_{\Bbbk}(V, W)$$

$$[W \xrightarrow{g} W'] \longmapsto [\mathrm{Hom}_{\Bbbk}(V, W) \longrightarrow \mathrm{Hom}_{\Bbbk}(V, W')]$$

$$f \longmapsto gf$$



$\mathrm{Hom}_{\Bbbk}(V, -)$ FOR $V \in \mathsf{Vec}$

$\mathrm{Hom}_{\Bbbk}(-, W)$ FOR $W \in \mathsf{Vec}$

$\mathrm{Hom}_{\Bbbk}(-, W) : \mathsf{Vec} \longrightarrow \mathsf{Vec}$ FOR FIXED $W \in \mathsf{Vec}$

$$V \longmapsto \mathrm{Hom}_{\Bbbk}(V, W)$$

$$[V \xrightarrow{g} V'] \longmapsto [\mathrm{Hom}_{\Bbbk}(V', W) \longrightarrow \mathrm{Hom}_{\Bbbk}(V, W)]$$

$$f \longmapsto fg$$

I. FUNCTORS

ALGEBRAIC FUNCTORS... / $\Bbbk$ FIELD

A FUNCTOR $F : \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D}$ $\forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ (RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$) $\forall g : X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ $\forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$ $\forall g : X \to Y,\ h : Y \to Z \in \mathcal{C}$ (RESP., $F(gf) = F(f)F(g)$ $\forall f : W \to X,\ g : X \to Y \in \mathcal{C}$)

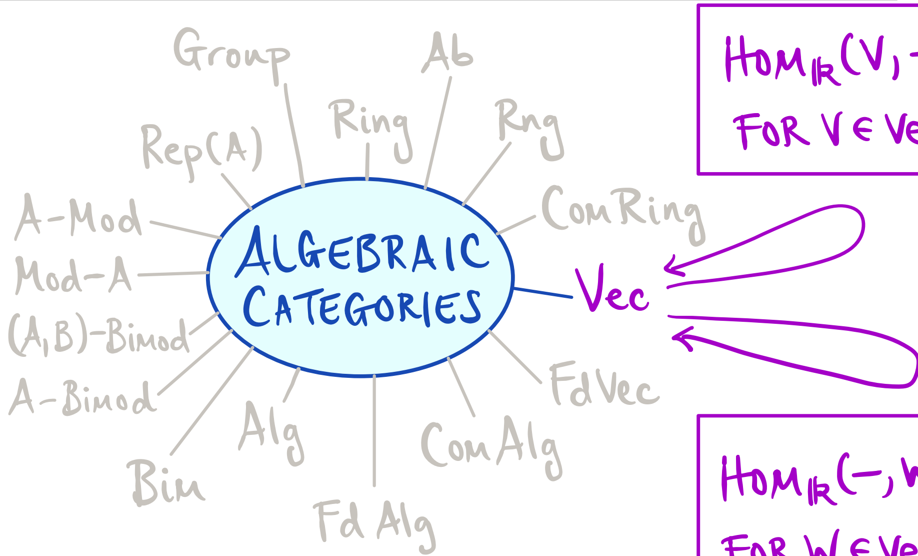
$\mathrm{Hom}_{\Bbbk}(V,-) : \mathsf{Vec} \longrightarrow \mathsf{Vec}$ FOR FIXED $V \in \mathsf{Vec}$

$$W \longmapsto \mathrm{Hom}_{\Bbbk}(V,W)$$

$$[W \xrightarrow{g} W'] \longmapsto [\mathrm{Hom}_{\Bbbk}(V,W) \longrightarrow \mathrm{Hom}_{\Bbbk}(V,W')]$$

$$f \longmapsto gf$$

COVARIANT



$\mathrm{Hom}_{\Bbbk}(V,-)$ FOR $V \in \mathsf{Vec}$

$\mathrm{Hom}_{\Bbbk}(-,W)$ FOR $W \in \mathsf{Vec}$

CONTRA-VARIANT

$\mathrm{Hom}_{\Bbbk}(-,W) : \mathsf{Vec} \longrightarrow \mathsf{Vec}$ FOR FIXED $W \in \mathsf{Vec}$

$$V \longmapsto \mathrm{Hom}_{\Bbbk}(V,W)$$

$$[V \xrightarrow{g} V'] \longmapsto [\mathrm{Hom}_{\Bbbk}(V',W) \longrightarrow \mathrm{Hom}_{\Bbbk}(V,W)]$$

$$f \longmapsto fg$$

I. FUNCTORS

ALGEBRAIC FUNCTORS...

A FUNCTOR $F:\mathcal{C}\to\mathcal{D}$ (RESP., CONTRAVARIANT)

CONSISTS OF:

(a) $F(X)\in\mathcal{D}\ \ \forall X\in\mathcal{C}$.

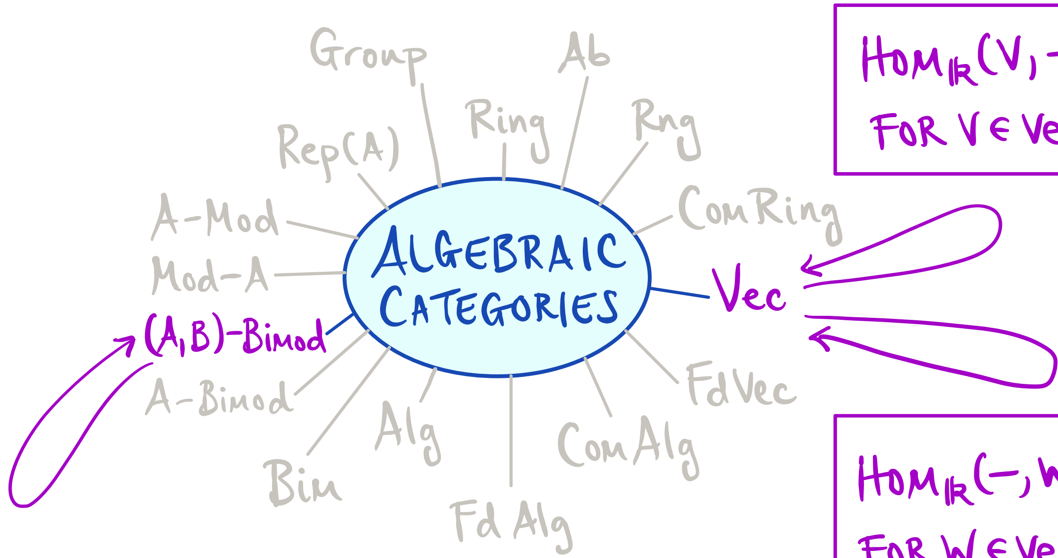
(b) $F(g)\in \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$

(RESP., $F(g)\in \mathrm{Hom}_{\mathcal{D}}(F(Y),F(X))$)

$\forall g:X\to Y\in\mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X)=\mathrm{id}_{F(X)}\ \ \forall X\in\mathcal{C}$
- $F(hg)=F(h)F(g)$
  $\forall g:X\to Y,\ h:Y\to Z\in\mathcal{C}$
  (RESP., $F(gf)=F(f)F(g)$
  $\forall f:W\to X,\ g:X\to Y\in\mathcal{C}$)



$V\otimes_{\mathbb{R}}-$

FOR $V\in$ Vec

$\mathrm{Hom}_{\mathbb{R}}(V,-)$

FOR $V\in$ Vec

$\mathrm{Hom}_{\mathbb{R}}(-,W)$

FOR $W\in$ Vec

$-\otimes_{\mathbb{R}}W$

FOR $W\in$ Vec

I. FUNCTORS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D}$ $\forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ (RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$) $\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

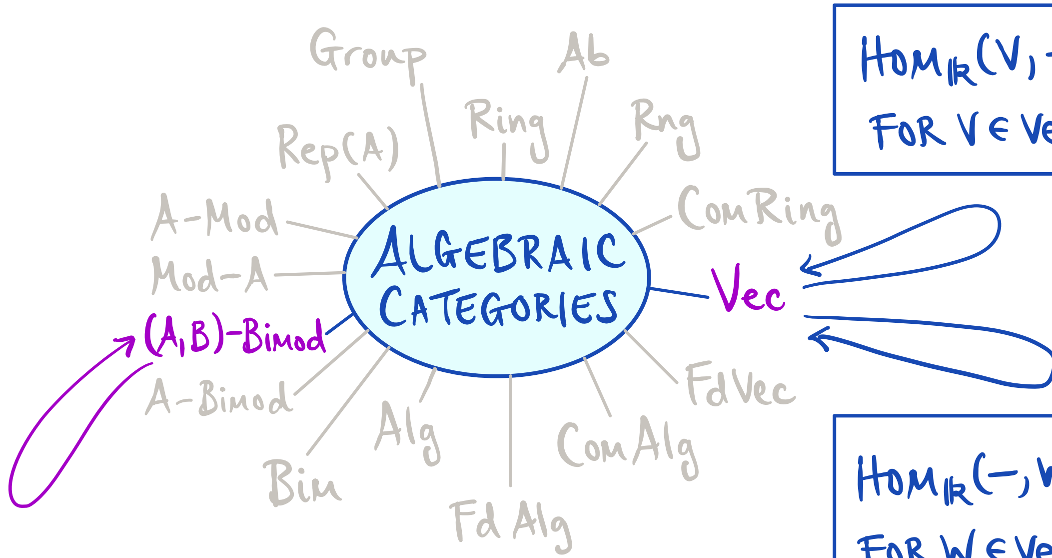
- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ $\forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$ $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$ (RESP., $F(gf) = F(f)F(g)$ $\forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)

ALGEBRAIC FUNCTORS...

FOR $V = {}_{B_1}V_A$,
$V \otimes_A - : (A, B_2)\text{-Bimod} \longrightarrow (B_1, B_2)\text{-Bimod}$

$V \otimes_{\mathbb{R}} -$ FOR $V \in \mathrm{Vec}$

$\mathrm{Hom}_{\mathbb{R}}(V, -)$ FOR $V \in \mathrm{Vec}$



$\mathrm{Hom}_{\mathbb{R}}(-, W)$ FOR $W \in \mathrm{Vec}$

$- \otimes_{\mathbb{R}} W$ FOR $W \in \mathrm{Vec}$

FOR $W = {}_A W_{B_2}$,
$- \otimes_A W : (B_1, A)\text{-Bimod} \longrightarrow (B_1, B_2)\text{-Bimod}$

I. FUNCTORS

ALGEBRAIC FUNCTORS...

A FUNCTOR $F : \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
(RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)
$\forall g : X \to Y \in \mathcal{C}$.

RESPECTING:

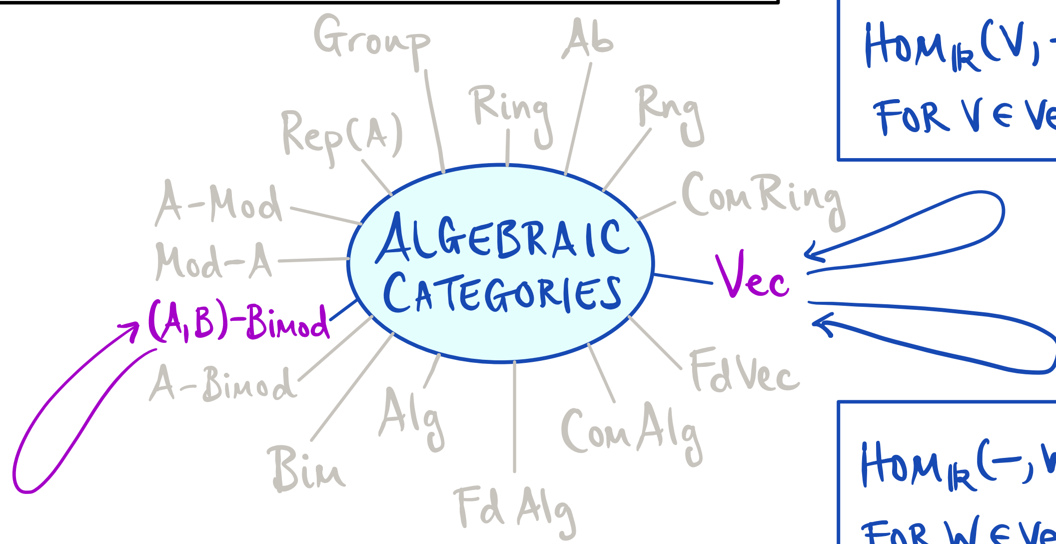
- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$ $\forall g : X \to Y,\ h : Y \to Z \in \mathcal{C}$
  (RESP., $F(gf) = F(f)F(g)$ $\forall f : W \to X,\ g : X \to Y \in \mathcal{C}$)

FOR $V = {}_{B_1}V_A$,
$V \otimes_A - : (A, B_2)\text{-Bimod} \longrightarrow (B_1, B_2)\text{-Bimod}$

FOR $V = {}_A V_{B_1}$,
$\mathrm{Hom}_{A\text{-Mod}}(V, -) : (A, B_2)\text{-Bimod} \longrightarrow (B_1, B_2)\text{-Bimod}$

$V \otimes_{\mathbb{k}} -$ FOR $V \in \mathrm{Vec}$

$\mathrm{Hom}_{\mathbb{k}}(V, -)$ FOR $V \in \mathrm{Vec}$



$\mathrm{Hom}_{\mathbb{k}}(-, W)$ FOR $W \in \mathrm{Vec}$

FOR $W = {}_A W_{B_2}$,
$\mathrm{Hom}_{A\text{-Mod}}(-, W) : (A, B_1)\text{-Bimod} \longrightarrow (B_1, B_2)\text{-Bimod}$

$- \otimes_{\mathbb{k}} W$ FOR $W \in \mathrm{Vec}$

FOR $W = {}_A W_{B_2}$,
$- \otimes_A W : (B_1, A)\text{-Bimod} \longrightarrow (B_1, B_2)\text{-Bimod}$

I. FUNCTORS

A FUNCTOR $F : \mathcal{C} \to \mathcal{D}$
(RESP., CONTRAVARIANT)

CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
(RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)
$\forall g : X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$
  $\forall g : X \to Y,\ h : Y \to Z \in \mathcal{C}$
  (RESP., $F(gf) = F(f)F(g)$ $\forall f : W \to X,\ g : X \to Y \in \mathcal{C}$)

ALGEBRAIC FUNCTORS...

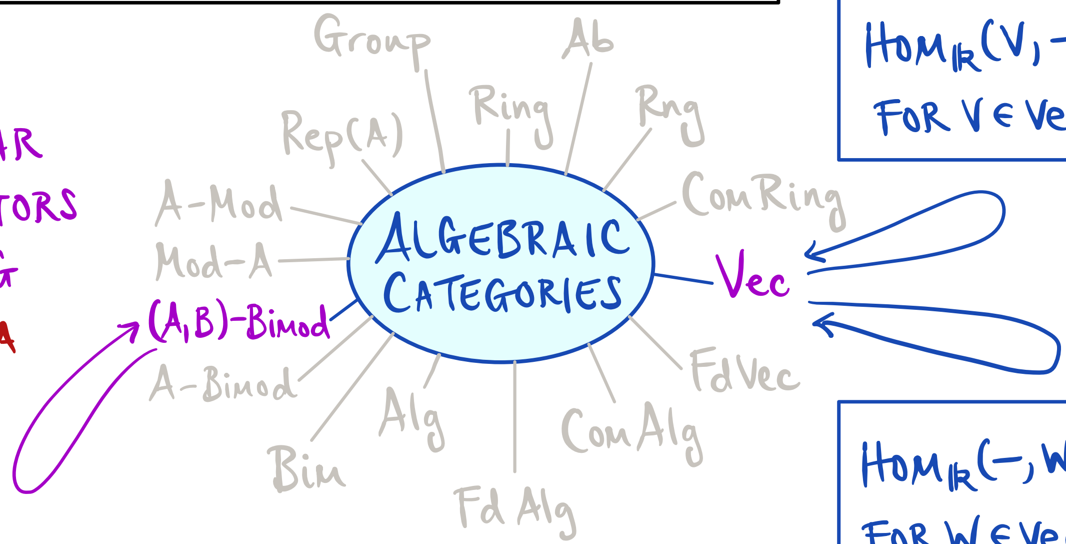
FOR $V = {}_{B_1}V_A$,
$V \otimes_A - : (A, B_2)\text{-Bimod} \longrightarrow (B_1, B_2)\text{-Bimod}$

FOR $V = {}_{A}V_{B_1}$,
$\mathrm{Hom}_{A\text{-Mod}}(V, -) : (A, B_2)\text{-Bimod} \longrightarrow (B_1, B_2)\text{-Bimod}$

$V \otimes_{\mathbb{R}} -$
FOR $V \in \mathrm{Vec}$

$\mathrm{Hom}_{\mathbb{R}}(V, -)$
FOR $V \in \mathrm{Vec}$

HAVE SIMILAR FUNCTORS USING Mod-A



$\mathrm{Hom}_{\mathbb{R}}(-, W)$
FOR $W \in \mathrm{Vec}$

FOR $W = {}_{A}W_{B_2}$,
$\mathrm{Hom}_{A\text{-Mod}}(-, W) : (A, B_1)\text{-Bimod} \longrightarrow (B_1, B_2)\text{-Bimod}$

$- \otimes_{\mathbb{R}} W$
FOR $W \in \mathrm{Vec}$

FOR $W = {}_{A}W_{B_2}$,
$- \otimes_A W : (B_1, A)\text{-Bimod} \longrightarrow (B_1, B_2)\text{-Bimod}$

I. FUNCTORS ... BETWEEN FIELDS

A FUNCTOR $F: \mathscr{C} \to \mathscr{D}$ (RESP., CONTRAVARIANT)

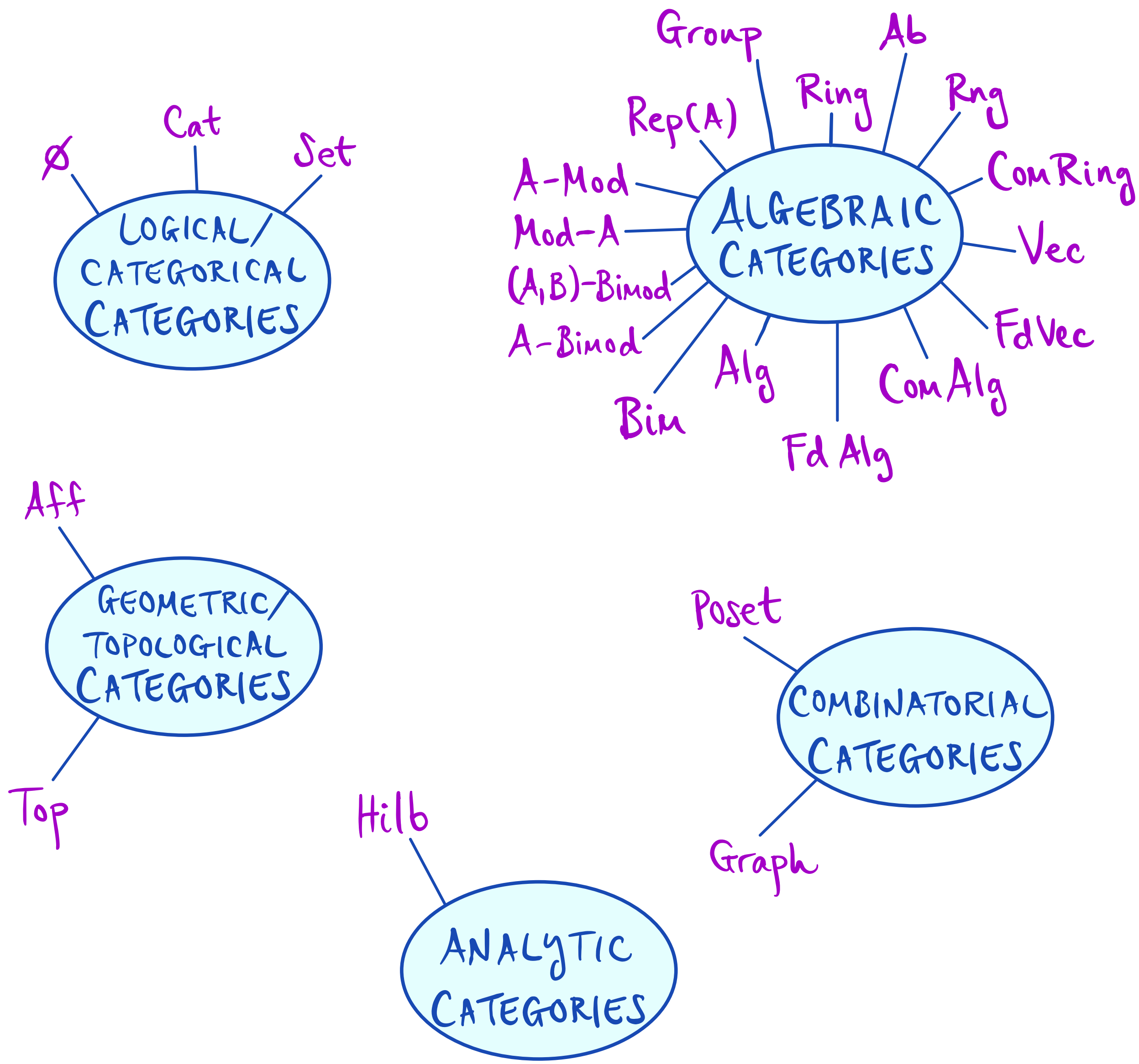
CONSISTS OF:

(a) $F(X) \in \mathscr{D} \quad \forall X \in \mathscr{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$
(RESP., $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))$)
$\forall g: X \to Y \in \mathscr{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathscr{C}$
- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathscr{C}$
  (RESP., $F(gf) = F(f)F(g)$
  $\forall f: W \to X,\ g: X \to Y \in \mathscr{C}$)



I. FUNCTORS ... BETWEEN FIELDS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT)

CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

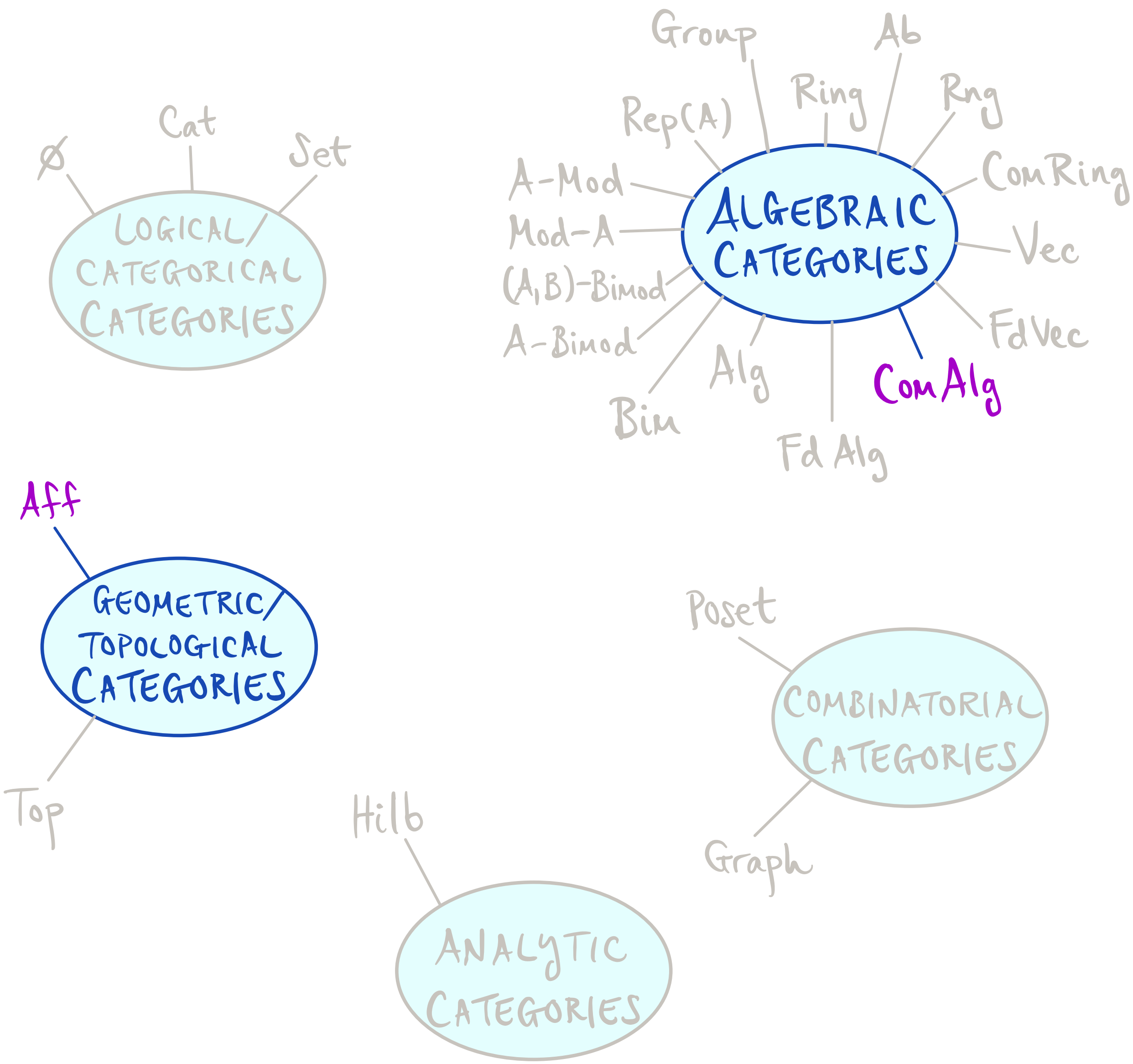
(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$

(RESP.,
$F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)

$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$
  (RESP.,
  $F(gf) = F(f)F(g)$
  $\forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)



I. FUNCTORS ... BETWEEN FIELDS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D}$ $\forall X \in \mathcal{C}$.

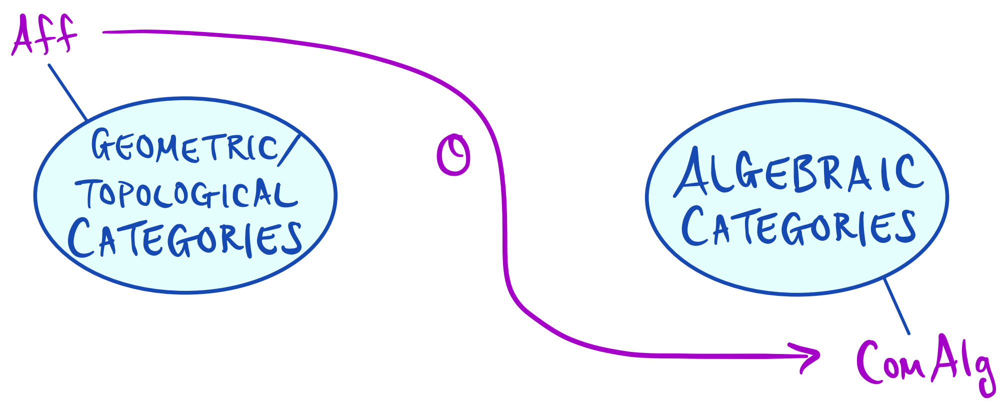
(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
(RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)
$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ $\forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$
  (RESP., $F(gf) = F(f)F(g)$ $\forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)

$$\mathcal{O}: \mathsf{Aff} \longrightarrow \mathsf{ComAlg}$$
$$X \longmapsto \mathcal{O}(X)$$

AFFINE VARIETY — COORDINATE ALGEBRA OF X



I. FUNCTORS ... BETWEEN FIELDS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ $\left(\text{RESP.,}\ F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))\right)$ $\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

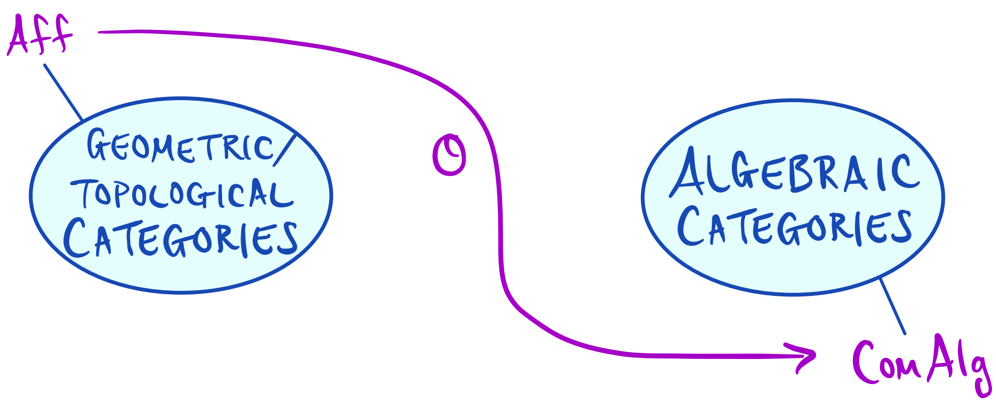
- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$ $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$ $\left(\text{RESP.,}\ F(gf) = F(f)F(g)\ \forall f: W \to X,\ g: X \to Y \in \mathcal{C}\right)$

$$\mathcal{O}: \mathsf{Aff}_{\mathbb{C}} \longrightarrow \mathsf{ComAlg}_{\mathbb{C}}$$

$$X \longmapsto \mathcal{O}(X) = \frac{\mathbb{C}[x_1, \dots, x_n]}{(\text{IDEAL OF POLY'LS DEFINING } X)}$$

X: AFFINE VARIETY — SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1, \dots, x_n]$ EQUAL TO 0

$\mathcal{O}(X)$: COORDINATE ALGEBRA OF X



I. FUNCTORS ... BETWEEN FIELDS

A FUNCTOR $F : \mathscr{C} \to \mathscr{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathscr{D}$ $\forall X \in \mathscr{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$
$\left(\text{RESP., } F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))\right)$
$\forall g : X \to Y \in \mathscr{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ $\forall X \in \mathscr{C}$
- $F(hg) = F(h)F(g)$ $\forall g : X \to Y,\ h : Y \to Z \in \mathscr{C}$
  $\left(\text{RESP., } F(gf) = F(f)F(g) \ \forall f : W \to X,\ g : X \to Y \in \mathscr{C}\right)$

$\mathcal{O} : \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

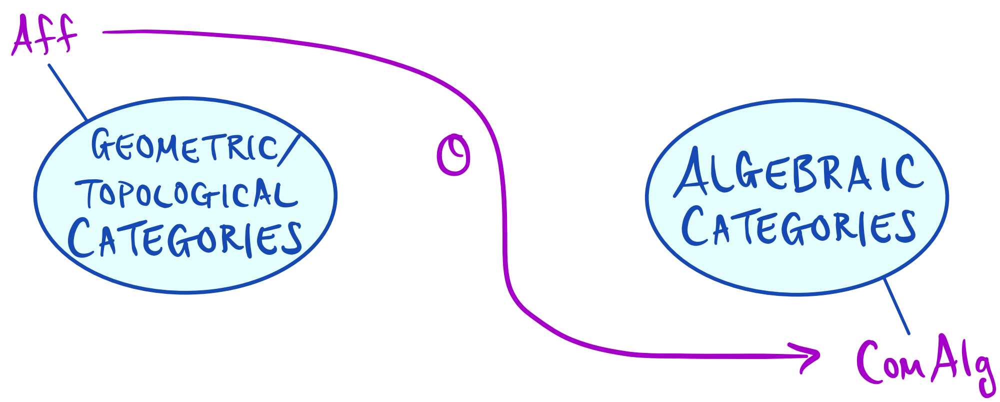
$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1, \dots, x_n]}{(\text{IDEAL OF POLY'LS DEFINING } X)}$

X: AFFINE VARIETY — SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1, \dots, x_n]$ EQUAL TO 0

$\mathcal{O}(X)$: COORDINATE ALGEBRA OF X

Ex. $n = 2$

$$\mathcal{O}(\mathbb{C}^2) = \mathbb{C}[x, y]$$



I. FUNCTORS ... BETWEEN FIELDS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
(RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)
$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

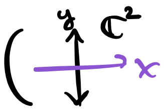
- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$
  (RESP., $F(gf) = F(f)F(g)$
  $\forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1, \dots, x_n]$ EQUAL TO 0

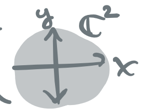
$$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$$

$$\underset{\text{AFFINE VARIETY}}{X} \longmapsto \underset{\text{COORDINATE ALGEBRA OF } X}{\mathcal{O}(X)} = \frac{\mathbb{C}[x_1, \dots, x_n]}{(\text{IDEAL OF POLY'LS DEFINING } X)}$$

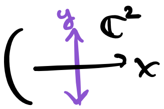
Ex. $n = 2$

$\mathcal{O}($ 

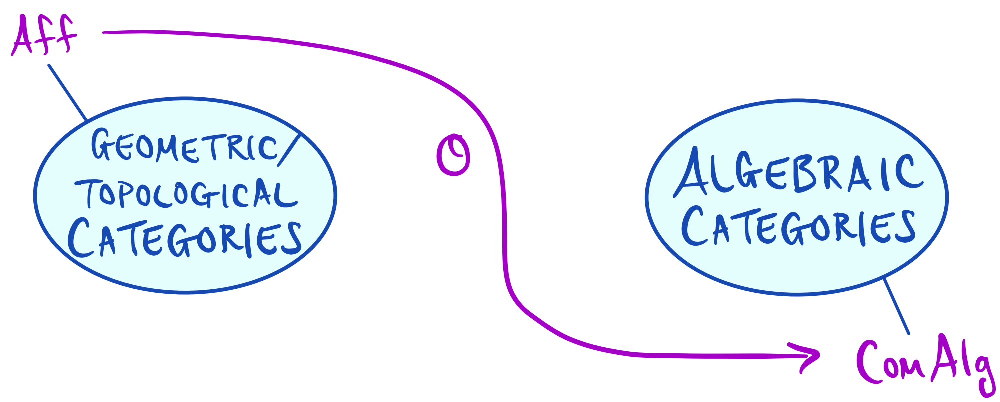
$) = \dfrac{\mathbb{C}[x,y]}{(y)} \cong \mathbb{C}[x]$

$\mathcal{O}($ 

$) = \mathbb{C}[x,y]$

$\mathcal{O}($ 

$) = \dfrac{\mathbb{C}[x,y]}{(x)} \cong \mathbb{C}[y]$



I. FUNCTORS ... BETWEEN FIELDS

A FUNCTOR $F : \mathscr{C} \to \mathscr{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathscr{D} \quad \forall X \in \mathscr{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$
$\left(\text{RESP., } F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))\right)$
$\forall g : X \to Y \in \mathscr{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathscr{C}$
- $F(hg) = F(h)F(g)$
  $\forall g : X \to Y,\ h : Y \to Z \in \mathscr{C}$
  $\left(\text{RESP., } F(gf) = F(f)F(g) \quad \forall f : W \to X,\ g : X \to Y \in \mathscr{C}\right)$

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1, \dots, x_n]$ EQUAL TO 0

$\mathcal{O} : \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1, \dots, x_n]}{\left(\begin{array}{c}\text{IDEAL OF}\\ \text{POLY'LS}\\ \text{DEFINING } X\end{array}\right)}$

AFFINE VARIETY — COORDINATE ALGEBRA OF X

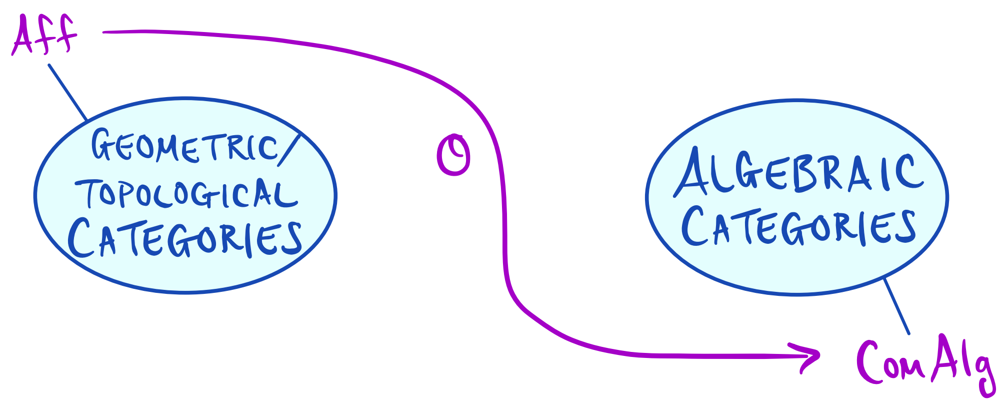
Ex. $n = 2$

$$\mathcal{O}\left(\ {}^{y}\ \mathbb{C}^2\ {}_{x}\right) = \frac{\mathbb{C}[x,y]}{(y)} \cong \mathbb{C}[x]$$

$$\mathcal{O}\left(\ {}^{y}\ \mathbb{C}^2\ {}_{x}\right) = \mathbb{C}[x,y]$$

$$\mathcal{O}\left(\ {}^{y}\ \mathbb{C}^2\ {}_{x}\right) = \frac{\mathbb{C}[x,y]}{(x,y)} \cong \mathbb{C}$$

$$\mathcal{O}\left(\ {}^{y}\ \mathbb{C}^2\ {}_{x}\right) = \frac{\mathbb{C}[x,y]}{(x)} \cong \mathbb{C}[y]$$



I. FUNCTORS ... BETWEEN FIELDS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
$\left(\text{RESP., } F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))\right)$
$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$

- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$
  $\left(\text{RESP., } F(gf) = F(f)F(g) \quad \forall f: W \to X,\ g: X \to Y \in \mathcal{C}\right)$

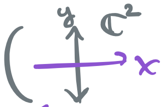
SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1, \dots, x_n]$ EQUAL TO 0

$$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$$
$$X \longmapsto \mathcal{O}(X) = \frac{\mathbb{C}[x_1, \dots, x_n]}{\left(\text{IDEAL OF POLY'LS DEFINING } X\right)}$$

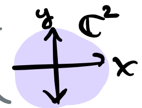
X: AFFINE VARIETY

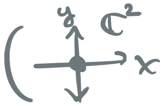
$\mathcal{O}(X)$: COORDINATE ALGEBRA OF X

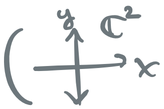
Ex. $n=2$

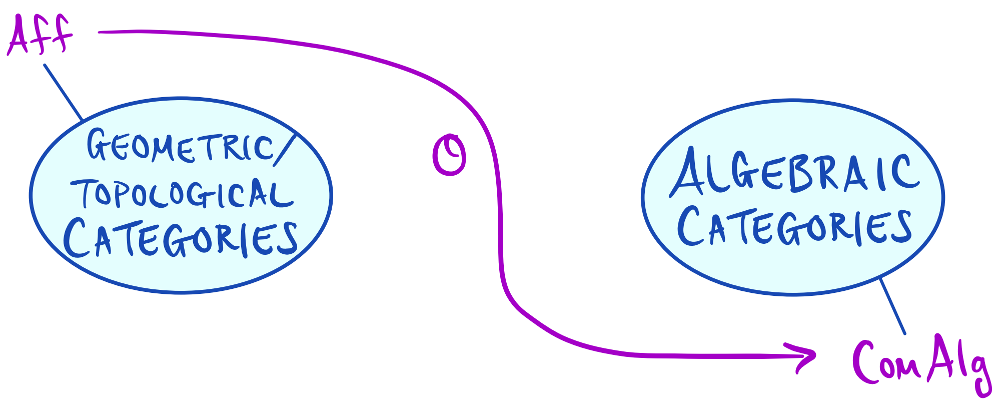
$\mathcal{O}($  $) = \dfrac{\mathbb{C}[x,y]}{(y)} \cong \mathbb{C}[x]$

INCLUSION

$\mathcal{O}($  $) = \mathbb{C}[x,y]$

$\mathcal{O}($  $) = \dfrac{\mathbb{C}[x,y]}{(x,y)} \cong \mathbb{C}$

$\mathcal{O}($  $) = \dfrac{\mathbb{C}[x,y]}{(x)} \cong \mathbb{C}[y]$



I. FUNCTORS ... BETWEEN FIELDS

A FUNCTOR $F: \mathscr{C} \to \mathscr{D}$ (RESP., CONTRAVARIANT)

CONSISTS OF:

(a) $F(X) \in \mathscr{D} \quad \forall X \in \mathscr{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$
$\left(\text{RESP., } F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))\right)$
$\forall g: X \to Y \in \mathscr{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathscr{C}$
- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathscr{C}$
  $\left(\text{RESP., } F(gf) = F(f)F(g) \quad \forall f: W \to X,\ g: X \to Y \in \mathscr{C}\right)$

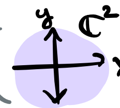
$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1, \dots, x_n]}{(\text{IDEAL OF POLY'LS DEFINING } X)}$

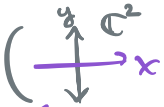
AFFINE VARIETY: SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1, \dots, x_n]$ EQUAL TO 0

$\mathcal{O}(X)$: COORDINATE ALGEBRA OF X

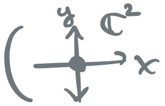
Ex. $n=2$

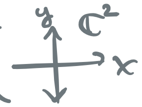
$\mathcal{O}($  $) = \mathbb{C}[x,y]$

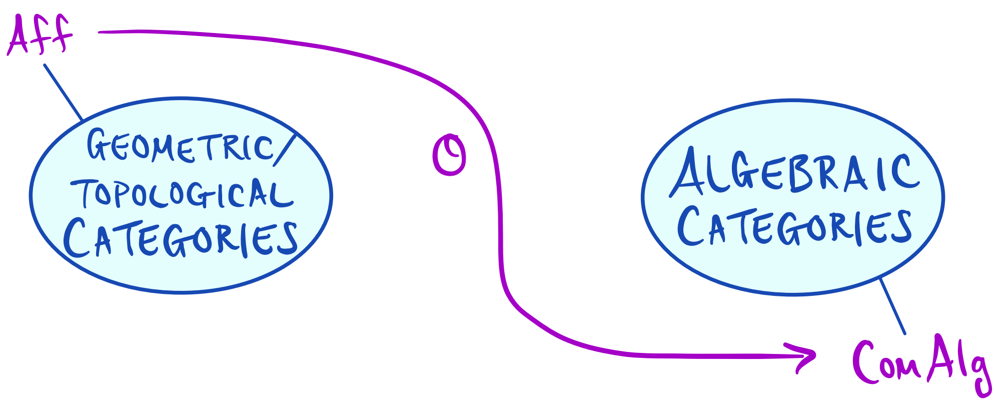
INCLUSION

$\mathcal{O}($  $) = \dfrac{\mathbb{C}[x,y]}{(y)} \cong \mathbb{C}[x]$

PROJECTION

$\mathcal{O}($  $) = \dfrac{\mathbb{C}[x,y]}{(x,y)} \cong \mathbb{C}$

$\mathcal{O}($  $) = \dfrac{\mathbb{C}[x,y]}{(x)} \cong \mathbb{C}[y]$



I. FUNCTORS ... BETWEEN FIELDS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ (RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)
$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(id_X) = id_{F(X)} \quad \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$ $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$ (RESP., $F(gf) = F(f)F(g)$ $\forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)

$\equiv$ CONTRAVARIANT $\equiv$

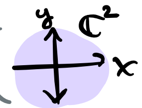
$$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$$

$$X \longmapsto \mathcal{O}(X) = \frac{\mathbb{C}[x_1, \dots, x_n]}{(\text{IDEAL OF POLY'LS DEFINING } X)}$$

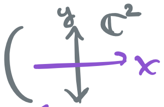
AFFINE VARIETY: SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1, \dots, x_n]$ EQUAL TO 0

$\mathcal{O}(X)$: COORDINATE ALGEBRA OF X

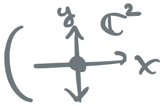
Ex. $n=2$

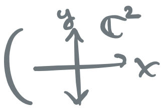
$\mathcal{O}($  $) = \mathbb{C}[x,y]$

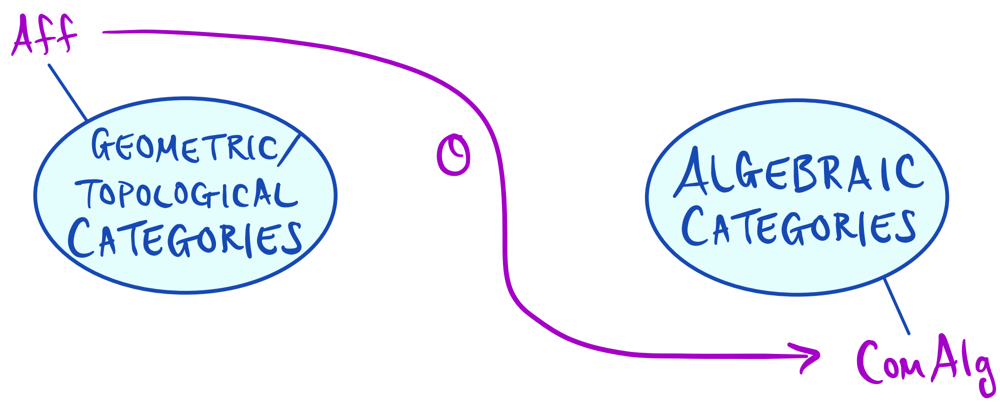
INCLUSION

$\mathcal{O}($  $) = \dfrac{\mathbb{C}[x,y]}{(y)} \cong \mathbb{C}[x]$

PROJECTION

$\mathcal{O}($  $) = \dfrac{\mathbb{C}[x,y]}{(x,y)} \cong \mathbb{C}$

$\mathcal{O}($  $) = \dfrac{\mathbb{C}[x,y]}{(x)} \cong \mathbb{C}[y]$



I. FUNCTORS ... BETWEEN FIELDS

A FUNCTOR $F : \mathscr{C} \to \mathscr{D}$ (RESP., CONTRAVARIANT)

CONSISTS OF:

(a) $F(X) \in \mathscr{D} \quad \forall X \in \mathscr{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$

(RESP.,
$F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))$)

$\forall g : X \to Y \in \mathscr{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathscr{C}$

- $F(hg) = F(h)F(g)$
  $\forall g : X \to Y,\ h : Y \to Z \in \mathscr{C}$
  (RESP.,
  $F(gf) = F(f)F(g)$
  $\forall f : W \to X,\ g : X \to Y \in \mathscr{C}$)

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1, \dots, x_n]$ EQUAL TO 0

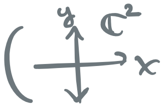
$\mathcal{O} : \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

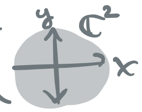
$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1, \dots, x_n]}{\left(\text{IDEAL OF POLY'LS DEFINING } X\right)}$

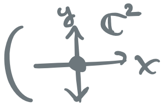
AFFINE VARIETY

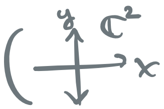
COORDINATE ALGEBRA OF X

Ex. $n=2$

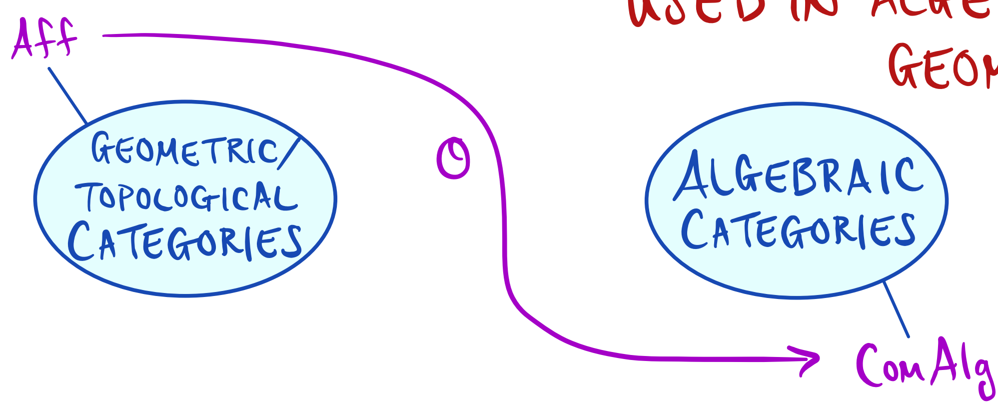
$\mathcal{O}\Big($  $\Big) = \dfrac{\mathbb{C}[x,y]}{(y)} \cong \mathbb{C}[x]$

$\mathcal{O}\Big($  $\Big) = \mathbb{C}[x,y]$

$\mathcal{O}\Big($  $\Big) = \dfrac{\mathbb{C}[x,y]}{(x,y)} \cong \mathbb{C}$

$\mathcal{O}\Big($  $\Big) = \dfrac{\mathbb{C}[x,y]}{(x)} \cong \mathbb{C}[y]$

USED IN ALGEBRAIC GEOMETRY



I. FUNCTORS ... BETWEEN FIELDS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$
(RESP., CONTRAVARIANT)

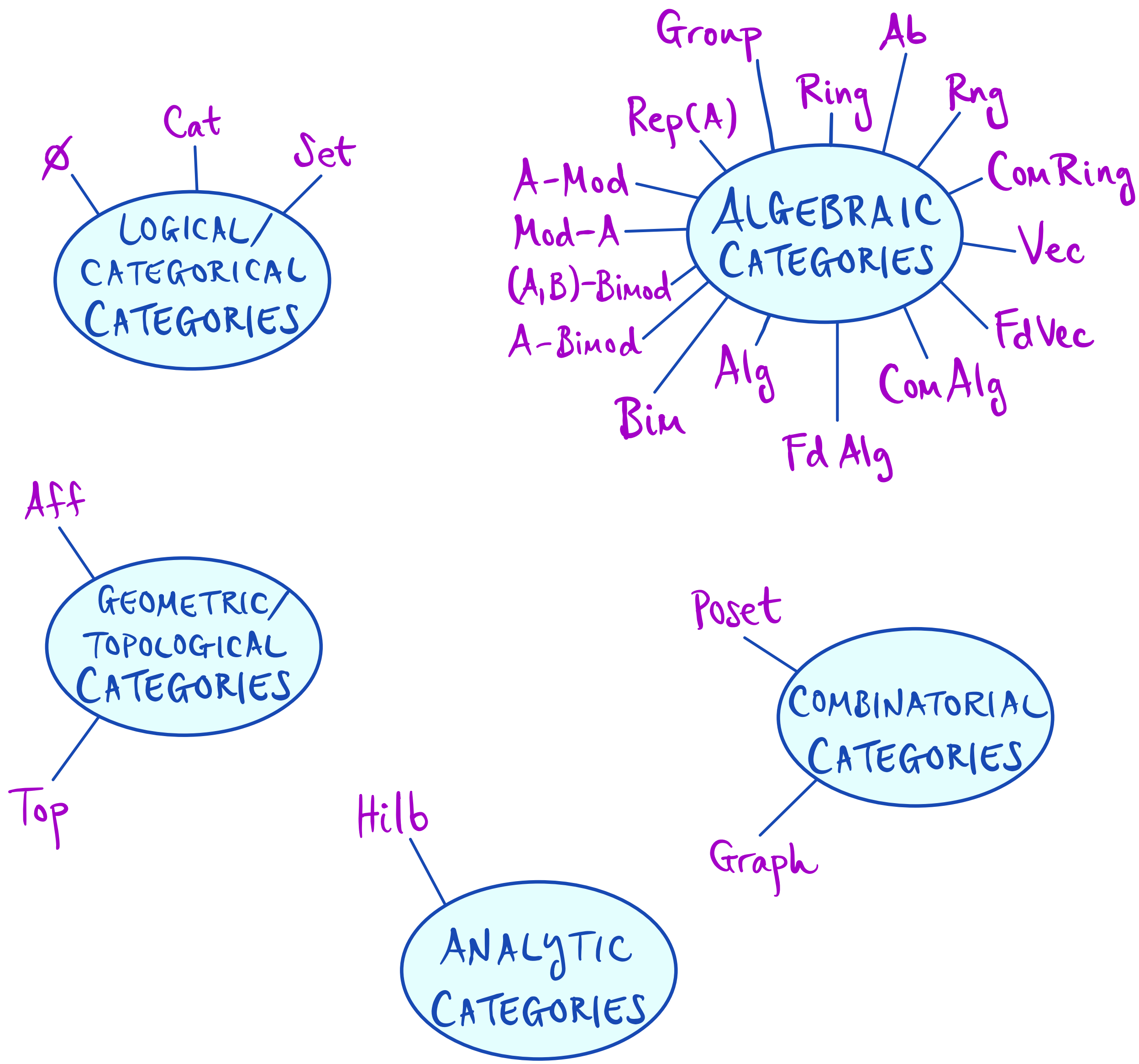
CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
$\left(\text{RESP.,}\ F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))\right)$
$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$
  $\left(\text{RESP.,}\ F(gf) = F(f)F(g) \quad \forall f: W \to X,\ g: X \to Y \in \mathcal{C}\right)$



I. FUNCTORS ... BETWEEN FIELDS

A FUNCTOR $F: \mathcal{C} \longrightarrow \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
$\left(\text{RESP., } F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))\right)$
$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$
  $\left(\text{RESP., } F(gf) = F(f)F(g) \quad \forall f: W \to X,\ g: X \to Y \in \mathcal{C}\right)$

USED IN ALGEBRAIC TOPOLOGY

GEOMETRIC/TOPOLOGICAL CATEGORIES

ALGEBRAIC CATEGORIES

$\mathrm{Top}_* \xrightarrow{\pi_1} \mathrm{Group}$

Top_*: TOPOLOGICAL SPACES WITH BASE POINT

π_1: FUNDAMENTAL GROUP

I. FUNCTORS

... PRESERVING PROPERTIES

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT)

CONSISTS OF:

(a) $F(X) \in \mathcal{D}$ $\forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$

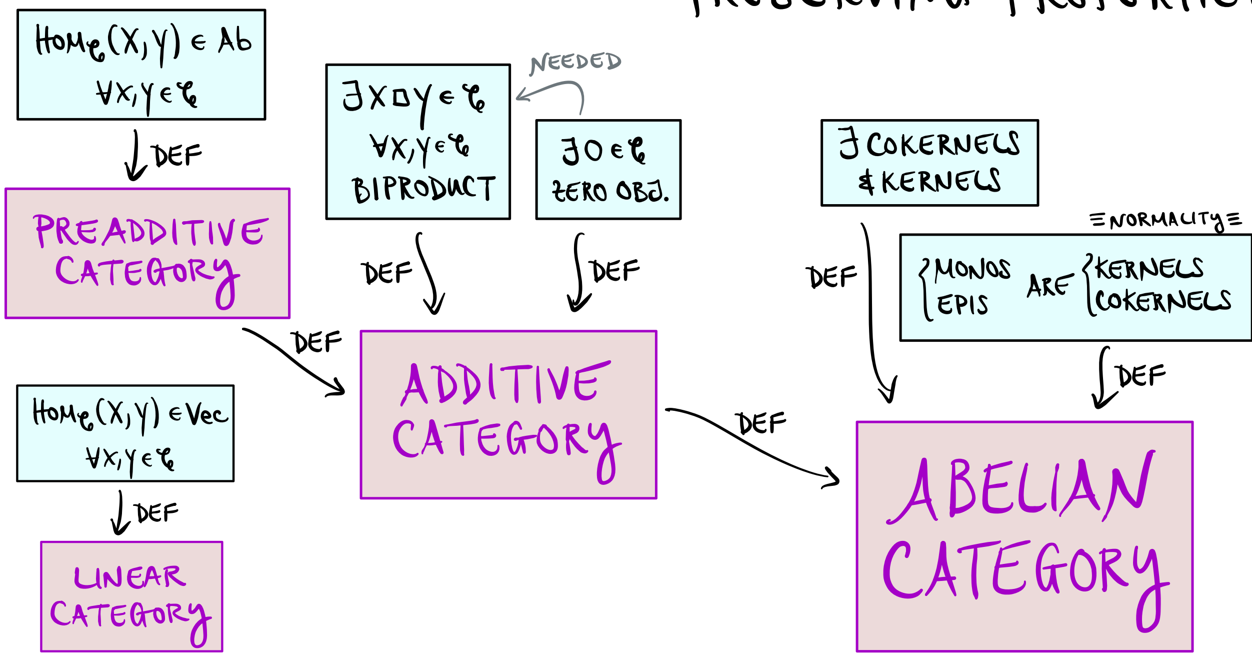
$\left(\text{RESP., } F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))\right)$

$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ $\forall X \in \mathcal{C}$

- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$
  $\left(\text{RESP., } F(gf) = F(f)F(g) \quad \forall f: W \to X,\ g: X \to Y \in \mathcal{C}\right)$



I. FUNCTORS ... PRESERVING PROPERTIES

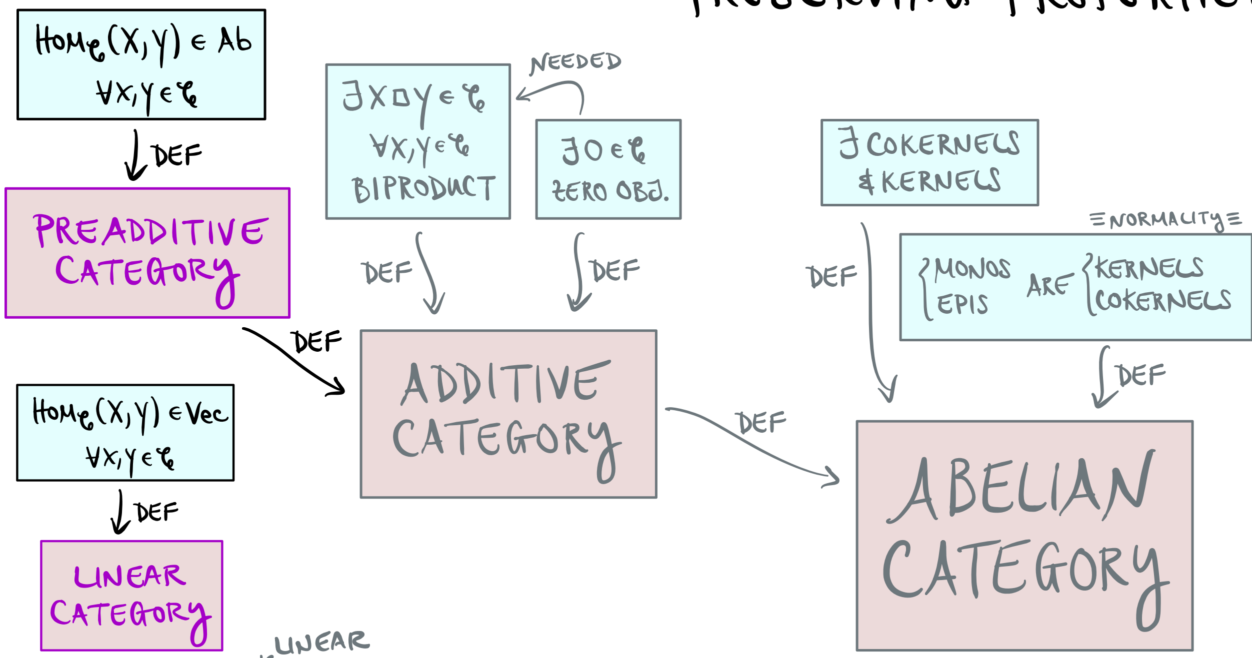
A FUNCTOR $F:\mathcal{C}\to\mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X)\in\mathcal{D}$ $\forall X\in\mathcal{C}$.

(b) $F(g)\in \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ (RESP., $F(g)\in \mathrm{Hom}_{\mathcal{D}}(F(Y),F(X))$) $\forall g:X\to Y\in\mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X)=\mathrm{id}_{F(X)}$ $\forall X\in\mathcal{C}$
- $F(hg)=F(h)F(g)$ $\forall g:X\to Y,\ h:Y\to Z\in\mathcal{C}$ (RESP., $F(gf)=F(f)F(g)$ $\forall f:W\to X,\ g:X\to Y\in\mathcal{C}$)



LINEAR

- $F:\mathcal{C}\to\mathcal{D}$ IS LINEAR IF $F_{X,Y}\in \mathrm{Vec}$ $\forall X,Y\in\mathcal{C}$.
- $F:\mathcal{C}\to\mathcal{D}$ IS ADDITIVE IF $F_{X,Y}\in \mathrm{Group}$ $\forall X,Y\in\mathcal{C}$.

PREADDITIVE

$$\boxed{F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y)\longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)),\quad g\mapsto F(g)}$$

I. FUNCTORS ... PRESERVING PROPERTIES

A FUNCTOR $F:\mathcal{C}\to\mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X)\in\mathcal{D}\quad\forall X\in\mathcal{C}$.

(b) $F(g)\in \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ (RESP., $F(g)\in \mathrm{Hom}_{\mathcal{D}}(F(Y),F(X))$) $\forall g:X\to Y\in\mathcal{C}$.

RESPECTING:

- $F(id_X)=id_{F(X)}\quad\forall X\in\mathcal{C}$
- $F(hg)=F(h)F(g)$ $\forall g:X\to Y,\ h:Y\to Z\in\mathcal{C}$ (RESP., $F(gf)=F(f)F(g)$ $\forall f:W\to X,\ g:X\to Y\in\mathcal{C}$)

$\mathrm{Hom}_{\mathcal{C}}(X,Y)\in Ab\ \forall X,Y\in\mathcal{C}$ $\xrightarrow{\text{DEF}}$ PREADDITIVE CATEGORY

$\exists X\oplus Y\in\mathcal{C}\ \forall X,Y\in\mathcal{C}$ BIPRODUCT

NEEDED

$\exists 0\in\mathcal{C}$ ZERO OBJ.

PREADDITIVE CATEGORY, BIPRODUCT, ZERO OBJ. $\xrightarrow{\text{DEF}}$ ADDITIVE CATEGORY

$\exists$ COKERNELS & KERNELS

≡NORMALITY≡ $\begin{Bmatrix}\text{MONOS}\\ \text{EPIS}\end{Bmatrix}$ ARE $\begin{Bmatrix}\text{KERNELS}\\ \text{COKERNELS}\end{Bmatrix}$

ADDITIVE CATEGORY, $\exists$ COKERNELS & KERNELS, NORMALITY $\xrightarrow{\text{DEF}}$ ABELIAN CATEGORY

$\mathrm{Hom}_{\mathcal{C}}(X,Y)\in Vec\ \forall X,Y\in\mathcal{C}$ $\xrightarrow{\text{DEF}}$ LINEAR CATEGORY

LINEAR

- $F:\mathcal{C}\to\mathcal{D}$ IS **LINEAR** IF $F_{X,Y}\in Vec\quad\forall X,Y\in\mathcal{C}$.
- $F:\mathcal{C}\to\mathcal{D}$ IS **ADDITIVE** IF $F_{X,Y}\in Group\quad\forall X,Y\in\mathcal{C}$.

PREADDITIVE

FACT: IF $\mathcal{C},\mathcal{D}$ ARE ADDITIVE, THEN

$F:\mathcal{C}\to\mathcal{D}$ IS ADDITIVE $\iff$ $F(X\oplus^{\mathcal{C}}Y)\cong F(X)\oplus^{\mathcal{D}}F(Y)\quad\forall X,Y\in\mathcal{C}$.

PF = EXER 2.19

$F_{X,Y}:\mathrm{Hom}_{\mathcal{C}}(X,Y)\longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)),\ g\mapsto F(g)$

I. FUNCTORS

… PRESERVING PROPERTIES

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$
(RESP., CONTRAVARIANT)

CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$

(RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)

$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$
  (RESP., $F(gf) = F(f)F(g)$ $\forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)

$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Ab}\ \ \forall X,Y \in \mathcal{C}$ $\xrightarrow{\text{DEF}}$ PREADDITIVE CATEGORY $\xrightarrow{\text{DEF}}$ ADDITIVE CATEGORY

$\exists X \oplus Y \in \mathcal{C}\ \ \forall X,Y \in \mathcal{C}$ BIPRODUCT $\xrightarrow{\text{DEF}}$ ADDITIVE CATEGORY

NEEDED

$\exists 0 \in \mathcal{C}$ ZERO OBJ. $\xrightarrow{\text{DEF}}$ ADDITIVE CATEGORY

ADDITIVE CATEGORY $\xrightarrow{\text{DEF}}$ ABELIAN CATEGORY

PRESERVED IN A LATER LECTURE

$\exists$ COKERNELS & KERNELS $\xrightarrow{\text{DEF}}$ ABELIAN CATEGORY

≡NORMALITY≡

$\left\{\begin{array}{l}\text{MONOS}\\ \text{EPIS}\end{array}\right.$ ARE $\left\{\begin{array}{l}\text{KERNELS}\\ \text{COKERNELS}\end{array}\right.$ $\xrightarrow{\text{DEF}}$ ABELIAN CATEGORY

$\mathrm{Hom}_{\mathcal{C}}(X,Y) \in \mathrm{Vec}\ \ \forall X,Y \in \mathcal{C}$ $\xrightarrow{\text{DEF}}$ LINEAR CATEGORY

LINEAR

- $F: \mathcal{C} \to \mathcal{D}$ IS **LINEAR** IF $F_{X,Y} \in \mathrm{Vec}\ \ \forall X,Y \in \mathcal{C}$.
- $F: \mathcal{C} \to \mathcal{D}$ IS **ADDITIVE** IF $F_{X,Y} \in \mathrm{Group}\ \ \forall X,Y \in \mathcal{C}$.

PREADDITIVE

FACT: IF $\mathcal{C}, \mathcal{D}$ ARE ADDITIVE, THEN

$F: \mathcal{C} \to \mathcal{D}$ IS ADDITIVE $\iff$ $F(X \oplus^{\mathcal{C}} Y) \cong F(X) \oplus^{\mathcal{D}} F(Y)$
$\forall X,Y \in \mathcal{C}$.

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)),\ g \mapsto F(g)$

PF = EXER 2.19

II. BIFUNCTORS & MULTIFUNCTORS

A **BIFUNCTOR** IS A FUNCTOR OF THE FORM:

$$F : \underbrace{\mathscr{C} \times \mathscr{C}'}_{\text{PRODUCT CATEGORY}} \longrightarrow \mathscr{D}.$$

II. BIFUNCTORS & MULTIFUNCTORS

A **BIFUNCTOR** IS A FUNCTOR OF THE FORM:

$$F : \underbrace{\mathscr{C} \times \mathscr{C}'}_{\text{PRODUCT CATEGORY}} \longrightarrow \mathscr{D}.$$

HERE, WE GET FUNCTORS

$F(-, X') : \mathscr{C} \longrightarrow \mathscr{D}$

$X \longmapsto F(X, X')$

$g \longmapsto F(g, \mathrm{id}_{X'})$

FOR A FIXED OBJECT $X' \in \mathscr{C}'$

$F(X, -) : \mathscr{C}' \longrightarrow \mathscr{D}$

$X' \longmapsto F(X, X')$

$g \longmapsto F(\mathrm{id}_X, g)$

FOR A FIXED OBJECT $X \in \mathscr{C}$

II. BIFUNCTORS & MULTIFUNCTORS

A **BIFUNCTOR** IS A FUNCTOR OF THE FORM:

$$F : \underbrace{\mathcal{C} \times \mathcal{C}'}_{\text{PRODUCT CATEGORY}} \longrightarrow \mathcal{D}.$$

HERE, WE GET FUNCTORS

$$F(-, X') : \mathcal{C} \longrightarrow \mathcal{D} \qquad\qquad F(X, -) : \mathcal{C}' \longrightarrow \mathcal{D}$$

$$X \longmapsto F(X, X') \qquad\qquad X' \longmapsto F(X, X')$$

$$g \longmapsto F(g, \mathrm{id}_{X'}) \qquad\qquad g \longmapsto F(\mathrm{id}_X, g)$$

FOR A FIXED OBJECT $X' \in \mathcal{C}$ — FOR A FIXED OBJECT $X \in \mathcal{C}$

EXAMPLES:

$- \otimes_{\Bbbk} - : \mathrm{Vec} \times \mathrm{Vec} \longrightarrow \mathrm{Vec}$ $(V, W) \longmapsto V \otimes_{\Bbbk} W$	$- \otimes_A - : (B_1, A)\text{-Bimod} \times (A, B_2)\text{-Bimod} \longrightarrow (B_1, B_2)\text{-Bimod}$ $(V, W) \longmapsto V \otimes_A W$

II. BIFUNCTORS & MULTIFUNCTORS

A **BIFUNCTOR** IS A FUNCTOR OF THE FORM:

$$F : \underbrace{\mathcal{C} \times \mathcal{C}'}_{\text{PRODUCT CATEGORY}} \longrightarrow \mathcal{D}.$$

HERE, WE GET FUNCTORS

$$F(-, X') : \mathcal{C} \longrightarrow \mathcal{D} \qquad X \longmapsto F(X, X') \qquad g \longmapsto F(g, \mathrm{id}_{X'})$$

FOR A FIXED OBJECT $X' \in \mathcal{C}'$

$$F(X, -) : \mathcal{C}' \longrightarrow \mathcal{D} \qquad X' \longmapsto F(X, X') \qquad g \longmapsto F(\mathrm{id}_X, g)$$

FOR A FIXED OBJECT $X \in \mathcal{C}$

EXAMPLES:

$$- \otimes_{\Bbbk} - : \mathrm{Vec} \times \mathrm{Vec} \longrightarrow \mathrm{Vec} \qquad (V, W) \longmapsto V \otimes_{\Bbbk} W$$

$$- \otimes_A - : (B_1, A)\text{-Bimod} \times (A, B_2)\text{-Bimod} \longrightarrow (B_1, B_2)\text{-Bimod} \qquad (V, W) \longmapsto V \otimes_A W$$

$$\mathrm{Hom}_{A\text{-Mod}}(-, -) : (A, B_1)\text{-Bimod} \times (A, B_2)\text{-Bimod} \longrightarrow (B_1, B_2)\text{-Bimod} \qquad (V, W) \longmapsto \mathrm{Hom}_{A\text{-Mod}}(V, W)$$

II. BIFUNCTORS & MULTIFUNCTORS

A **BIFUNCTOR** IS A FUNCTOR OF THE FORM:

$$F : \underbrace{\mathcal{C} \times \mathcal{C}'}_{\text{PRODUCT CATEGORY}} \longrightarrow \mathcal{D}.$$

HERE, WE GET FUNCTORS

$$F(-, X') : \mathcal{C} \longrightarrow \mathcal{D} \qquad X \longmapsto F(X, X') \qquad g \longmapsto F(g, \mathrm{id}_{X'})$$

FOR A FIXED OBJECT $X' \in \mathcal{C}'$

$$F(X, -) : \mathcal{C}' \longrightarrow \mathcal{D} \qquad X' \longmapsto F(X, X') \qquad g \longmapsto F(\mathrm{id}_X, g)$$

FOR A FIXED OBJECT $X \in \mathcal{C}$

EXAMPLES:

$$- \otimes_{\Bbbk} - : \mathrm{Vec} \times \mathrm{Vec} \longrightarrow \mathrm{Vec}, \qquad (V, W) \longmapsto V \otimes_{\Bbbk} W$$

$$- \otimes_A - : (B_1, A)\text{-Bimod} \times (A, B_2)\text{-Bimod} \longrightarrow (B_1, B_2)\text{-Bimod}, \qquad (V, W) \longmapsto V \otimes_A W$$

$$\mathrm{Hom}_{A\text{-Mod}}(-, -) : ((A, B_1)\text{-Bimod})^{op} \times (A, B_2)\text{-Bimod} \longrightarrow (B_1, B_2)\text{-Bimod}, \qquad (V, W) \longmapsto \mathrm{Hom}_{A\text{-mod}}(V, W)$$

$\mathrm{Hom}_{A\text{-mod}}(-, W)$ CONTRAVARIANT

II. BIFUNCTORS & MULTIFUNCTORS

A **BIFUNCTOR** IS A FUNCTOR OF THE FORM:

$$F : \underbrace{\mathcal{C} \times \mathcal{C}'}_{\text{PRODUCT CATEGORY}} \longrightarrow \mathcal{D}.$$

LIKEWISE, A MULTIFUNCTOR IS A FUNCTOR OF THE FORM

$$F : \mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_n \longrightarrow \mathcal{D}$$

HERE, $F(X_1, \ldots, X_{i-1}, -, X_{i+1}, \ldots, X_n) : \mathcal{C}_i \longrightarrow \mathcal{D}$ $\quad \forall i = 1, \ldots, n$

IS A FUNCTOR FOR FIXED $X_j \in \mathcal{C}_j$ $(j \neq i)$

EXAMPLES:

$- \otimes_{\Bbbk} - : \mathrm{Vec} \times \mathrm{Vec} \longrightarrow \mathrm{Vec}$ $(V, W) \longmapsto V \otimes_{\Bbbk} W$	$- \otimes_A - : (B_1, A)\text{-Bimod} \times (A, B_2)\text{-Bimod} \longrightarrow (B_1, B_2)\text{-Bimod}$ $(V, W) \longmapsto V \otimes_A W$

$$\mathrm{Hom}_{A\text{-Mod}}(-,-) : \left((A, B_1)\text{-Bimod}\right)^{op} \times (A, B_2)\text{-Bimod} \longrightarrow (B_1, B_2)\text{-Bimod}$$

$$(V, W) \longmapsto \mathrm{Hom}_{A\text{-mod}}(V, W)$$

$\mathrm{Hom}_{A\text{-mod}}(-, W)$
CONTRAVARIANT

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F : \mathscr{C} \to \mathscr{D}$
(RESP., CONTRAVARIANT)

CONSISTS OF:

(a) $F(X) \in \mathscr{D} \quad \forall X \in \mathscr{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$
$\left(\begin{array}{l}\text{RESP.,} \\ F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))\end{array}\right)$
$\forall g : X \to Y \in \mathscr{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathscr{C}$

- $F(hg) = F(h)F(g)$
  $\forall g : X \to Y,\ h : Y \to Z \in \mathscr{C}$
  $\left(\begin{array}{l}\text{RESP.,} \\ F(gf) = F(f)F(g) \\ \forall f : W \to X,\ g : X \to Y \in \mathscr{C}\end{array}\right)$

HOW TO MOVE FROM ONE CATEGORY TO ANOTHER

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F : \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT)

CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$

$\left(\begin{array}{l}\text{RESP.,}\\ F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))\end{array}\right)$

$\forall g : X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$

- $F(hg) = F(h)F(g)$
  $\forall g : X \to Y,\ h : Y \to Z \in \mathcal{C}$

  $\left(\begin{array}{l}\text{RESP.,}\\ F(gf) = F(f)F(g)\\ \forall f : W \to X,\ g : X \to Y \in \mathcal{C}\end{array}\right)$

HOW TO MOVE FROM ONE CATEGORY TO ANOTHER

LATER: $\mathcal{C} \simeq \mathcal{D}$

IF

$\exists$ FUNCTORS $F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

s.t. $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT)

CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$

(RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)

$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)} \quad \forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$
  $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$
  (RESP., $F(gf) = F(f)F(g)$
  $\forall f: W \to X,\ g: X \to Y \in \mathcal{C}$)

HOW TO MOVE FROM ONE CATEGORY TO ANOTHER

LATER: $\mathcal{C} \simeq \mathcal{D}$

IF

$\exists$ FUNCTORS $F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

$\ni$. $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

ISOMORPHISMS (INSTEAD OF EQUALITIES) ALLOW FOR A RICHER THEORY

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT)

CONSISTS OF:

(a) $F(X) \in \mathcal{D}$ $\forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
$\left(\text{RESP., } F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))\right)$
$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

- $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ $\forall X \in \mathcal{C}$
- $F(hg) = F(h)F(g)$ $\forall g: X \to Y,\ h: Y \to Z \in \mathcal{C}$
$\left(\text{RESP., } F(gf) = F(f)F(g) \ \forall f: W \to X,\ g: X \to Y \in \mathcal{C}\right)$

HOW TO MOVE FROM ONE CATEGORY TO ANOTHER

LATER: $\mathcal{C} \simeq \mathcal{D}$ IF $\exists$ FUNCTORS $F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

s.t. $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

ISOMORPHISMS (INSTEAD OF EQUALITIES) ALLOW FOR A RICHER THEORY

NEED A WAY OF MOVING FROM ONE FUNCTOR TO ANOTHER

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D}$ $\forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ $\left(\text{RESP.,} \; F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))\right)$ $\forall g: X \to Y \in \mathcal{C}$.

RESPECTING IDENTITY & COMPOSED MORPHISMS

GIVEN FUNCTORS $F, F': \mathcal{C} \to \mathcal{D}$,

A NATURAL TRANSFORMATION $\phi: F \Rightarrow F'$

NEED A WAY OF MOVING FROM ONE FUNCTOR TO ANOTHER

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F : \mathcal{C} \longrightarrow \mathcal{D}$
(RESP., CONTRAVARIANT)
CONSISTS OF:
(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.
(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
(RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)
$\forall g : X \to Y \in \mathcal{C}$.
RESPECTING IDENTITY & COMPOSED MORPHISMS

GIVEN FUNCTORS $F, F' : \mathcal{C} \longrightarrow \mathcal{D}$,

A NATURAL TRANSFORMATION $\phi : F \Rightarrow F'$

CONSISTS OF MORPHISMS

$$\{\phi_X : F(X) \longrightarrow F'(X) \text{ IN } \mathcal{D}\}_{X \in \mathcal{C}}$$

SUCH THAT $\forall f : X \to Y \in \mathcal{C}$:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\circlearrowleft"] & F(Y) \arrow[d, "\phi_Y"] \\
F'(X) \arrow[r, "F'(f)"'] & F'(Y)
\end{tikzcd}$$

NEED A WAY OF MOVING
FROM ONE FUNCTOR TO ANOTHER

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D}$ $\forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ (RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$) $\forall g: X \to Y \in \mathcal{C}$.

RESPECTING IDENTITY & COMPOSED MORPHISMS

GIVEN FUNCTORS $F, F': \mathcal{C} \to \mathcal{D}$,

A NATURAL TRANSFORMATION $\phi: F \Rightarrow F'$

CONSISTS OF MORPHISMS

$$\{\phi_X : F(X) \longrightarrow F'(X) \text{ in } \mathcal{D}\}_{X \in \mathcal{C}}$$

"COMPONENT OF ϕ AT X"

SUCH THAT $\forall f: X \to Y \in \mathcal{C}$:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\circlearrowleft"] & F(Y) \arrow[d, "\phi_Y"] \\ F'(X) \arrow[r, "F'(f)"'] & F'(Y) \end{tikzcd}$$

"NATURALITY OF ϕ AT f"

NEED A WAY OF MOVING FROM ONE FUNCTOR TO ANOTHER

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F: \mathcal{C} \longrightarrow \mathcal{D}$
(RESP., CONTRAVARIANT)
CONSISTS OF:

(a) $F(X) \in \mathcal{D}$ $\forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$

(RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)

$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING IDENTITY & COMPOSED MORPHISMS

GIVEN FUNCTORS $F, F' : \mathcal{C} \longrightarrow \mathcal{D}$,

A **NATURAL TRANSFORMATION** $\phi : F \Rightarrow F'$

CONSISTS OF MORPHISMS

$$\{\phi_X : F(X) \longrightarrow F'(X) \text{ IN } \mathcal{D}\}_{X \in \mathcal{C}}$$

"COMPONENT OF ϕ AT X"

SUCH THAT $\forall f : X \to Y \in \mathcal{C}$:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y"] \\ F'(X) \arrow[r, "F'(f)"'] & F'(Y) \end{tikzcd}$$

"NATURALITY OF ϕ AT f"

≡ PICTURED AS ≡

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow \phi"] & \mathcal{D} \end{tikzcd}$$

NEED A WAY OF MOVING
FROM ONE FUNCTOR TO ANOTHER

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F : \mathcal{C} \to \mathcal{D}$
(RESP., CONTRAVARIANT)
CONSISTS OF:
(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.
(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
(RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)
$\forall g : X \to Y \in \mathcal{C}$.
RESPECTING IDENTITY & COMPOSED MORPHISMS

GIVEN FUNCTORS $F, F' : \mathcal{C} \to \mathcal{D}$,

A NATURAL ~~TRANSFORMATION~~ ISOMORPHISM $\phi : F \overset{\sim}{\Rightarrow} F'$

CONSISTS OF ~~MORPHISMS~~ ISOS

$$\{\phi_X : F(X) \xrightarrow{\sim} F'(X) \text{ in } \mathcal{D}\}_{X \in \mathcal{C}}$$

"COMPONENT OF ϕ AT X"

SUCH THAT $\forall f : X \to Y \in \mathcal{C}$:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y"] \\
F'(X) \arrow[r, "F'(f)"'] & F'(Y)
\end{tikzcd}$$

"NATURALITY OF ϕ AT f"

= PICTURED AS =

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "F'"'{name=D}] & \mathcal{D}
\arrow[from=U, to=D, Rightarrow, "\sim\ \phi"]
\end{tikzcd}$$

NEED A WAY OF MOVING
FROM ONE FUNCTOR TO ANOTHER

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F: \mathcal{C} \longrightarrow \mathcal{D}$
(RESP., CONTRAVARIANT)
CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
$\left(\text{RESP.,}\ F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))\right)$
$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING IDENTITY & COMPOSED MORPHISMS

A NATURAL TRANSF'N
(RESP., NAT'L ISOMORPHISM)
$\phi: F \overset{\sim}{\Rightarrow} F'$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F'"'] \arrow[r, phantom, "\sim\ \Downarrow \phi"] & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"', "\sim"] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y", "\sim"'] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

$\forall f: X \to Y \in \mathcal{C}$

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D}$ $\forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ (RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$) $\forall g: X \to Y \in \mathcal{C}$.

RESPECTING IDENTITY & COMPOSED MORPHISMS

EXAMPLES

$$\begin{tikzcd} \mathrm{Vec} \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "(\ )^{**}"'] \arrow[r, phantom, "\Downarrow\delta"] & \mathrm{Vec} \end{tikzcd}$$

$$\delta_V : V \longrightarrow V^{**}$$
$$v \longmapsto \left[\begin{array}{l} V^* \longrightarrow \Bbbk \\ f \longmapsto f(v) \end{array}\right]$$

A NATURAL TRANSF'N (RESP., NAT'L ISOMORPHISM) $\phi: F \Rightarrow F'$ ($\phi: F \overset{\sim}{\Rightarrow} F'$)

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\sim\Downarrow\phi"] & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"', "\sim"] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y", "\sim"'] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

$\forall f: X \to Y \in \mathcal{C}$

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F: \mathcal{C} \longrightarrow \mathcal{D}$
(RESP., CONTRAVARIANT)
CONSISTS OF:

(a) $F(X) \in \mathcal{D}$ $\forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$

(RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)

$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING IDENTITY & COMPOSED MORPHISMS

EXAMPLES

$$\begin{tikzcd} \mathrm{FdVec} \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "(\ )^{**}"'] \arrow[r, phantom, "\sim\Downarrow\delta"] & \mathrm{FdVec} \end{tikzcd}$$

$$\delta_V : V \xrightarrow{\sim} V^{**}$$

$$v \longmapsto \begin{bmatrix} V^* \longrightarrow \Bbbk \\ f \longmapsto f(v) \end{bmatrix}$$

A NATURAL TRANSF'N
(RESP., NAT'L ISOMORPHISM)
$\phi : F \xRightarrow{\sim} F'$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\sim\Downarrow\phi"] & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"', "\sim"] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y", "\sim"'] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

$\forall f : X \to Y \in \mathcal{C}$

III. NATURAL TRANSFORMATIONS

A functor $F: \mathcal{C} \to \mathcal{D}$
(resp., contravariant)
consists of:

(a) $F(X) \in \mathcal{D}$ $\forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$

(resp., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)

$\forall g: X \to Y \in \mathcal{C}$.

Respecting identity & composed morphisms

EXAMPLES

DETAILS LEFT TO EXERCISE 2.23

$$\begin{tikzcd} \mathsf{FdVec} \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "(\ )^{**}"'] \arrow[r, phantom, "\sim\Downarrow\delta"] & \mathsf{FdVec} \end{tikzcd}$$

$$\delta_V : V \xrightarrow{\sim} V^{**}$$

$$v \longmapsto \begin{bmatrix} V^* \longrightarrow \Bbbk \\ f \longmapsto f(v) \end{bmatrix}$$

A **NATURAL TRANSF'N**
(RESP., NAT'L ISOMORPHISM)
$\phi : F \xRightarrow{\sim} F'$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\sim\Downarrow\phi"] & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"', "\sim"] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y", "\sim"'] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

$\forall f : X \to Y \in \mathcal{C}$

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F:\mathcal{C}\to\mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X)\in\mathcal{D}\quad\forall X\in\mathcal{C}$.

(b) $F(g)\in \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$ (RESP., $F(g)\in \mathrm{Hom}_{\mathcal{D}}(F(Y),F(X))$)

$\forall g: X\to Y\in\mathcal{C}$.

RESPECTING IDENTITY & COMPOSED MORPHISMS

EXAMPLES

DETAILS LEFT TO EXERCISE 2.23

$$\begin{tikzcd} \mathsf{FdVec} \arrow[rr, bend left, "\mathrm{Id}"] \arrow[rr, bend right, "(\ )^{**}"'] \arrow[rr, phantom, "\sim\Downarrow\delta"] & & \mathsf{FdVec} \end{tikzcd}$$

$$\delta_V : V \xrightarrow{\sim} V^{**}$$

$$v \longmapsto \begin{bmatrix} V^* \longrightarrow \Bbbk \\ f \longmapsto f(v) \end{bmatrix}$$

??

$$\begin{tikzcd} \mathsf{FdVec} \arrow[rr, bend left, "\mathrm{Id}"] \arrow[rr, bend right, "(\ )^{*}"'] \arrow[rr, phantom, "\exists\Downarrow\delta"] & & \mathsf{FdVec} \end{tikzcd}$$

??

A NATURAL TRANSF'N (RESP., NAT'L ISOMORPHISM)

$$\phi : F \overset{\sim}{\Rightarrow} F'$$

$$\begin{tikzcd} \mathcal{C} \arrow[rr, bend left, "F"] \arrow[rr, bend right, "F'"'] \arrow[rr, phantom, "\sim\Downarrow\phi"] & & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"', "\sim"] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y", "\sim"'] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

$\forall f: X\to Y\in\mathcal{C}$

III. Natural Transformations

A functor $F: \mathcal{C} \to \mathcal{D}$ (resp., contravariant) consists of:

(a) $F(X) \in \mathcal{D}$ $\forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$

(resp., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)

$\forall g: X \to Y \in \mathcal{C}$.

respecting identity & composed morphisms

Examples

Details left to Exercise 2.23

$$\begin{tikzcd} \mathsf{FdVec} \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "(\ )^{**}"'] \arrow[r, phantom, "\sim\Downarrow\delta"] & \mathsf{FdVec} \end{tikzcd}$$

$$\delta_V : V \xrightarrow{\sim} V^{**}$$

$$v \longmapsto \begin{bmatrix} V^* \longrightarrow \Bbbk \\ f \longmapsto f(v) \end{bmatrix}$$

Covariant

$$\begin{tikzcd} \mathsf{FdVec} \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "(\ )^{*}"'] \arrow[r, phantom, "\not\sim\Downarrow\delta"] & \mathsf{FdVec} \end{tikzcd}$$

Contravariant

A **natural transf'n** (resp., nat'l isomorphism) $\phi: F \xRightarrow{\sim} F'$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\sim\Downarrow\phi"] & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"', "\sim"] \arrow[dr, phantom, "\circlearrowleft"] & F(Y) \arrow[d, "\sim"', "\phi_Y"] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

$\forall f: X \to Y \in \mathcal{C}$

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F: \mathcal{C} \longrightarrow \mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$

(RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)

$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING IDENTITY & COMPOSED MORPHISMS

EXAMPLES

DETAILS LEFT TO EXERCISE 2.23

$$\begin{tikzcd} \mathrm{FdVec} \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "(\ )^{**}"'] \arrow[r, phantom, "\sim\Downarrow\delta"] & \mathrm{FdVec} \end{tikzcd}$$

$$\delta_V : V \xrightarrow{\sim} V^{**}$$

$$v \longmapsto \left[\begin{array}{l} V^{*} \longrightarrow \Bbbk \\ f \longmapsto f(v) \end{array} \right]$$

COVARIANT

$$\begin{tikzcd} \mathrm{FdVec} \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "(\ )^{*}"'] \arrow[r, phantom, "\not\cong\Downarrow\delta"] & \mathrm{FdVec} \end{tikzcd}$$

CONTRAVARIANT

A NATURAL TRANSF'N (RESP., NAT'L ISOMORPHISM)

$$\phi : F \overset{\sim}{\Rightarrow} F'$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\sim\Downarrow\phi"] & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"', "\sim"] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y", "\sim"'] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

$$\forall f: X \to Y \in \mathcal{C}$$

ASSOCIATIVITY OF $\otimes := \otimes_{\Bbbk}$

$$\begin{tikzcd} \mathrm{Vec} \times \mathrm{Vec} \times \mathrm{Vec} \arrow[r, bend left, "\otimes \circ (\otimes \times \mathrm{Id})"] \arrow[r, bend right, "\otimes \circ (\mathrm{Id} \times \otimes)"'] \arrow[r, phantom, "\sim\Downarrow a"] & \mathrm{Vec} \end{tikzcd}$$

$$a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

$$(u \otimes v) \otimes w \longmapsto u \otimes (v \otimes w)$$

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F:\mathcal{C}\to\mathcal{D}$ (RESP., CONTRAVARIANT) CONSISTS OF:

(a) $F(X)\in\mathcal{D}\quad \forall X\in\mathcal{C}$.

(b) $F(g)\in \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$

$\left(\text{RESP.,}\ F(g)\in \mathrm{Hom}_{\mathcal{D}}(F(Y),F(X))\right)$

$\forall g:X\to Y\in\mathcal{C}$.

RESPECTING IDENTITY & COMPOSED MORPHISMS

EXAMPLES

DETAILS LEFT TO EXERCISE 2.23

$$\begin{tikzcd} \mathrm{FdVec} \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "(\ )^{**}"'] \arrow[r, phantom, "\sim\Downarrow\delta"] & \mathrm{FdVec} \end{tikzcd}$$

$$\delta_V : V \xrightarrow{\sim} V^{**}$$

$$v \longmapsto \begin{bmatrix} V^* \longrightarrow \Bbbk \\ f \longmapsto f(v) \end{bmatrix}$$

COVARIANT

$$\begin{tikzcd} \mathrm{FdVec} \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "(\ )^{*}"'] \arrow[r, phantom, "\not\sim\Downarrow\delta"] & \mathrm{FdVec} \end{tikzcd}$$

CONTRAVARIANT

A NATURAL TRANSF'N (RESP., NAT'L ISOMORPHISM)

$$\phi : F \overset{\sim}{\Rightarrow} F'$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\sim\Downarrow\phi"] & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"', "\sim"] \arrow[dr, phantom, "\circlearrowleft"] & F(Y) \arrow[d, "\phi_Y", "\sim"'] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

$\forall f: X\to Y \in \mathcal{C}$

DETAILS LEFT TO EXERCISE 2.24

ASSOCIATIVITY OF $\otimes := \otimes_{\Bbbk}$

$$\begin{tikzcd} \mathrm{Vec}\times\mathrm{Vec}\times\mathrm{Vec} \arrow[r, bend left, "\otimes\circ(\otimes\times\mathrm{Id})"] \arrow[r, bend right, "\otimes\circ(\mathrm{Id}\times\otimes)"'] \arrow[r, phantom, "\sim\Downarrow a"] & \mathrm{Vec} \end{tikzcd}$$

$$a_{U,V,W} : (U\otimes V)\otimes W \to U\otimes(V\otimes W)$$

$$(u\otimes v)\otimes w \longmapsto u\otimes(v\otimes w)$$

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F: \mathcal{C} \longrightarrow \mathcal{D}$
(RESP., CONTRAVARIANT)
CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
$\left(\text{RESP., } F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))\right)$
$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING IDENTITY & COMPOSED MORPHISMS

THESE FORM A CATEGORY

A NATURAL TRANSF'N
(RESP., NAT'L ISOMORPHISM)
$\phi: F \Rightarrow F'$ (RESP., $\phi: F \xRightarrow{\sim} F'$)

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "F'"'{name=D}] & \mathcal{D} \arrow[Rightarrow, from=U, to=D, "\phi", "\sim"']
\end{tikzcd}$$

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"', "\sim"] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y", "\sim"'] \\
F'(X) \arrow[r, "F'(f)"] & F'(Y)
\end{tikzcd}$$

$\forall f: X \to Y \in \mathcal{C}$

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$
(RESP., CONTRAVARIANT)
CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$

(RESP., $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))$)

$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING IDENTITY & COMPOSED MORPHISMS

THESE FORM A CATEGORY: $\boxed{\mathrm{Fun}(\mathcal{C}, \mathcal{D})}$

OBJECTS = FUNCTORS $\mathcal{C} \to \mathcal{D}$

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, F') := \mathrm{Nat}_{\mathcal{C},\mathcal{D}}(F, F')$$

NATURAL TRANS'NS $F \Rightarrow F'$

A NATURAL TRANSF'N
(RESP., NAT'L ISOMORPHISM)
$\phi: F \overset{\sim}{\Rightarrow} F'$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "F'"'{name=D}] & \mathcal{D}
\arrow[Rightarrow, from=U, to=D, "\phi", "\sim"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"', "\sim"] \arrow[dr, phantom, "\circlearrowleft"] & F(Y) \arrow[d, "\phi_Y", "\sim"'] \\
F'(X) \arrow[r, "F'(f)"] & F'(Y)
\end{tikzcd}$$

$\forall f: X \to Y \in \mathcal{C}$

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F: \mathcal{C} \longrightarrow \mathcal{D}$
(RESP., CONTRAVARIANT)
CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$

$\left(\text{RESP.,} \; F(g) \in \mathrm{Hom}_{\mathcal{D}}(F(Y), F(X))\right)$

$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING IDENTITY & COMPOSED MORPHISMS

THESE FORM A CATEGORY: $\boxed{\mathrm{Fun}(\mathcal{C}, \mathcal{D})}$

OBJECTS = FUNCTORS $\mathcal{C} \longrightarrow \mathcal{D}$

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, F') := \mathrm{Nat}_{\mathcal{C},\mathcal{D}}(F, F')$$

NATURAL TRANS'NS
$F \Rightarrow F'$

$\mathrm{ID}_F : F \Rightarrow F$
IDENTITY NAT'L TRANSF'N
COMPONENTS $(\mathrm{ID}_F)_X := \mathrm{id}_{F(X)}$
$\forall X \in \mathcal{C}$

A NATURAL TRANSF'N
(RESP., NAT'L ISOMORPHISM)
$\phi : F \overset{\sim}{\Rightarrow} F'$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\sim \Downarrow \phi"] & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"', "\sim"] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y", "\sim"'] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

$\forall f: X \to Y \in \mathcal{C}$

III. NATURAL TRANSFORMATIONS

A FUNCTOR $F : \mathscr{C} \to \mathscr{D}$
(RESP., CONTRAVARIANT)
CONSISTS OF:

(a) $F(X) \in \mathscr{D}$ $\forall X \in \mathscr{C}$.

(b) $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$
(RESP., $F(g) \in \mathrm{Hom}_{\mathscr{D}}(F(Y), F(X))$)
$\forall g : X \to Y \in \mathscr{C}$.

RESPECTING IDENTITY & COMPOSED MORPHISMS

THESE FORM A CATEGORY : $\boxed{\mathrm{Fun}(\mathscr{C}, \mathscr{D})}$

OBJECTS = FUNCTORS $\mathscr{C} \to \mathscr{D}$

$$\mathrm{Hom}_{\mathrm{Fun}(\mathscr{C},\mathscr{D})}(F, F') := \mathrm{Nat}_{\mathscr{C},\mathscr{D}}(F, F')$$

NATURAL TRANS'NS
$F \Rightarrow F'$

A NATURAL TRANSF'N
(RESP., NAT'L ISOMORPHISM)
$\phi : F \Rightarrow F'$ (RESP., $\phi : F \overset{\sim}{\Rightarrow} F'$)

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "F'"'{name=D}] & \mathscr{D} \arrow[Rightarrow, from=U, to=D, "\phi", "\sim"'] \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"', "\sim"] \arrow[dr, phantom, "\circlearrowleft"] & F(Y) \arrow[d, "\phi_Y", "\sim"'] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

$\forall f : X \to Y \in \mathscr{C}$

$\mathrm{ID}_F : F \Rightarrow F$
IDENTITY NAT'L TRANSF'N
COMPONENTS $(\mathrm{ID}_F)_X := \mathrm{id}_{F(X)}$ $\forall X \in \mathscr{C}$

$$F \overset{\phi}{\Rightarrow} F' \overset{\phi'}{\Rightarrow} F''$$

"VERTICAL COMPOSITION"
DEFINED NEXT...

IV. Compositions of Natural Transformations

A NATURAL TRANSF'N

$\phi : F \Rightarrow F'$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "F'"'{name=D}] & \mathcal{D} \arrow[Rightarrow, from=U, to=D, "\phi"] \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y"] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

$\forall f : X \to Y \in \mathcal{C}$

IV. Compositions of Natural Transformations

Vertical Composition

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=70, "F"{name=U}] \arrow[r, "F'"{name=M, description}] \arrow[r, bend right=70, "F''"'{name=L}] & \mathcal{D}
\arrow[from=U, to=M, phantom, "\Downarrow \phi"]
\arrow[from=M, to=L, phantom, "\Downarrow \phi'"]
\end{tikzcd}
\; =: \;
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=40, "F"{name=A}] \arrow[r, bend right=40, "F''"'{name=B}] & \mathcal{D}
\arrow[from=A, to=B, phantom, "\Downarrow \phi' \circ^{\mathrm{VER}} \phi"]
\end{tikzcd}
$$

$$(\phi' \circ^{\mathrm{VER}} \phi)_X : F(X) \xrightarrow{\phi_X} F'(X) \xrightarrow{\phi'_X} F''(X)$$

A natural transf'n

$\phi : F \Rightarrow F'$

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=40, "F"{name=A}] \arrow[r, bend right=40, "F'"'{name=B}] & \mathcal{D}
\arrow[from=A, to=B, phantom, "\Downarrow \phi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\circlearrowleft"] & F(Y) \arrow[d, "\phi_Y"] \\
F'(X) \arrow[r, "F'(f)"] & F'(Y)
\end{tikzcd}
$$

$\forall f : X \to Y \in \mathcal{C}$

IV. COMPOSITIONS OF NATURAL TRANSFORMATIONS

VERTICAL COMPOSITION

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=70, "F"] \arrow[r, "F'" description] \arrow[r, bend right=70, "F''"'] \arrow[r, phantom, shift left=3ex, "\Downarrow\phi"] \arrow[r, phantom, shift right=3ex, "\Downarrow\phi'"] & \mathcal{D} \end{tikzcd} \; =: \; \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F''"'] \arrow[r, phantom, "\Downarrow \phi' \circ^{VER} \phi"] & \mathcal{D} \end{tikzcd}$$

$$(\phi' \circ^{VER} \phi)_X : F(X) \xrightarrow{\phi_X} F'(X) \xrightarrow{\phi'_X} F''(X)$$

HORIZONTAL COMPOSITION

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathcal{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\psi"] & \mathcal{E} \end{tikzcd} \; =: \; \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "G'F'"'] \arrow[r, phantom, "\Downarrow \psi \circ^{HOR} \phi"] & \mathcal{E} \end{tikzcd}$$

$$(\psi \circ^{HOR} \phi)_X : GF(X) \xrightarrow{G(\phi_X)} GF'(X) \xrightarrow{\psi_{F'(X)}} G'F'(X)$$

A NATURAL TRANSF'N

$\phi : F \Rightarrow F'$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y"] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

$\forall f : X \to Y \in \mathcal{C}$

IV. Compositions of Natural Transformations

Vertical Composition

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"{name=U}] \arrow[r, "F'"{name=M, description}] \arrow[r, bend right=60, "F''"'{name=L}] & \mathcal{D} \arrow[from=U, to=M, phantom, "\Downarrow \phi"] \arrow[from=M, to=L, phantom, "\Downarrow \phi'"] \end{tikzcd} \quad =: \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left=40, "F"{name=U}] \arrow[r, bend right=40, "F''"'{name=L}] & \mathcal{D} \arrow[from=U, to=L, phantom, "\Downarrow \phi' \circ^{VER} \phi"] \end{tikzcd}$$

$$(\phi' \circ^{VER} \phi)_X : F(X) \xrightarrow{\phi_X} F'(X) \xrightarrow{\phi'_X} F''(X)$$

Horizontal Composition

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"{name=U1}] \arrow[r, bend right=50, "F'"'{name=L1}] & \mathcal{D} \arrow[r, bend left=50, "G"{name=U2}] \arrow[r, bend right=50, "G'"'{name=L2}] & \mathcal{E} \arrow[from=U1, to=L1, phantom, "\Downarrow \phi"] \arrow[from=U2, to=L2, phantom, "\Downarrow \psi"] \end{tikzcd} \quad =: \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left=40, "GF"{name=U}] \arrow[r, bend right=40, "G'F'"'{name=L}] & \mathcal{E} \arrow[from=U, to=L, phantom, "\Downarrow \psi \circ^{HOR} \phi"] \end{tikzcd}$$

$$(\psi \circ^{HOR} \phi)_X : \begin{tikzcd} & GF'(X) \arrow[dr, "\psi_{F'(X)}"] & \\ GF(X) \arrow[ur, "G(\phi_X)"] \arrow[dr, "\psi_{F(X)}"'] & & G'F'(X) \\ & G'F(X) \arrow[ur, "G'(\phi_X)"'] & \end{tikzcd}$$

A natural transf'n

$\phi : F \Rightarrow F'$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=40, "F"{name=U}] \arrow[r, bend right=40, "F'"'{name=L}] & \mathcal{D} \arrow[from=U, to=L, phantom, "\Downarrow \phi"] \end{tikzcd} \qquad \begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y"] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

$\forall f : X \to Y \in \mathcal{C}$

IV. COMPOSITIONS OF NATURAL TRANSFORMATIONS

VERTICAL COMPOSITION

$$\begin{tikzcd}[column sep=huge]
\mathcal{C} \arrow[r, bend left=60, "F"{name=U}] \arrow[r, "F'"{name=M, description}] \arrow[r, bend right=60, "F''"'{name=L}] & \mathcal{D}
\arrow[from=U, to=M, Rightarrow, "\phi"]
\arrow[from=M, to=L, Rightarrow, "\phi'"]
\end{tikzcd}
\quad =: \quad
\begin{tikzcd}[column sep=huge]
\mathcal{C} \arrow[r, bend left=40, "F"{name=U}] \arrow[r, bend right=40, "F''"'{name=L}] & \mathcal{D}
\arrow[from=U, to=L, Rightarrow, "\phi' \circ^{VER} \phi"]
\end{tikzcd}$$

$$(\phi' \circ^{VER} \phi)_X : F(X) \xrightarrow{\phi_X} F'(X) \xrightarrow{\phi'_X} F''(X)$$

HORIZONTAL COMPOSITION

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"{name=F}] \arrow[r, bend right=50, "F'"'{name=F1}] & \mathcal{D} \arrow[r, bend left=50, "G"{name=G}] \arrow[r, bend right=50, "G'"'{name=G1}] & \mathcal{E}
\arrow[from=F, to=F1, Rightarrow, "\phi"]
\arrow[from=G, to=G1, Rightarrow, "\psi"]
\end{tikzcd}
\quad =: \quad
\begin{tikzcd}[column sep=huge]
\mathcal{C} \arrow[r, bend left=40, "GF"{name=U}] \arrow[r, bend right=40, "G'F'"'{name=L}] & \mathcal{E}
\arrow[from=U, to=L, Rightarrow, "\psi \circ^{HOR} \phi"]
\end{tikzcd}$$

$$(\psi \circ^{HOR} \phi)_X : \begin{tikzcd}
 & GF'(X) \arrow[dr, "\psi_{F'(X)}"] & \\
GF(X) \arrow[ur, "G(\phi_X)"] \arrow[dr, "\psi_{F(X)}"'] \arrow[rr, phantom, "\circlearrowright"] & & G'F'(X) \\
 & G'F(X) \arrow[ur, "G'(\phi_X)"'] &
\end{tikzcd}$$

$\circlearrowright$: NATURALITY OF ψ AT ϕ_X

A NATURAL TRANSF'N $\phi : F \Rightarrow F'$

$$\begin{tikzcd}[column sep=huge]
\mathcal{C} \arrow[r, bend left=40, "F"{name=U}] \arrow[r, bend right=40, "F'"'{name=L}] & \mathcal{D}
\arrow[from=U, to=L, Rightarrow, "\phi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y"] \\
F'(X) \arrow[r, "F'(f)"'] & F'(Y)
\end{tikzcd}$$

$$\forall f : X \to Y \in \mathcal{C}$$

IV. Compositions of Natural Transformations

Vertical Composition

$$\begin{tikzcd}[row sep=large]
\mathcal{C} \arrow[r, bend left=60, "F"{name=A}] \arrow[r, "F'"{name=B, description}] \arrow[r, bend right=60, "F''"'{name=C}] & \mathcal{D}
\arrow[from=A, to=B, Rightarrow, shorten=2pt, "\phi"]
\arrow[from=B, to=C, Rightarrow, shorten=2pt, "\phi'"]
\end{tikzcd}
\quad =: \quad
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=40, "F"{name=A}] \arrow[r, bend right=40, "F''"'{name=C}] & \mathcal{D}
\arrow[from=A, to=C, Rightarrow, shorten=3pt, "\phi' \circ^{\mathrm{VER}} \phi"]
\end{tikzcd}$$

$$(\phi' \circ^{\mathrm{VER}} \phi)_X : F(X) \xrightarrow{\phi_X} F'(X) \xrightarrow{\phi'_X} F''(X)$$

Horizontal Composition

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"{name=A}] \arrow[r, bend right=50, "F'"'{name=B}] & \mathcal{D} \arrow[r, bend left=50, "G"{name=C}] \arrow[r, bend right=50, "G'"'{name=D}] & \mathcal{E}
\arrow[from=A, to=B, Rightarrow, shorten=2pt, "\phi"]
\arrow[from=C, to=D, Rightarrow, shorten=2pt, "\psi"]
\end{tikzcd}
\quad =: \quad
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=40, "GF"{name=A}] \arrow[r, bend right=40, "G'F'"'{name=B}] & \mathcal{E}
\arrow[from=A, to=B, Rightarrow, shorten=3pt, "\psi \circ^{\mathrm{HOR}} \phi"]
\end{tikzcd}$$

$$(\psi \circ^{\mathrm{HOR}} \phi)_X : \begin{tikzcd}
 & GF'(X) \arrow[dr, "\psi_{F'(X)}"] & \\
GF(X) \arrow[ur, "G(\phi_X)"] \arrow[dr, "\psi_{F(X)}"'] \arrow[rr, phantom, "\circlearrowright"] & & G'F'(X) \\
 & G'F(X) \arrow[ur, "G'(\phi_X)"'] &
\end{tikzcd}$$

$\circlearrowright$: Naturality of ψ at ϕ_X

A Natural Transf'n

$\phi : F \Rightarrow F'$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=40, "F"{name=A}] \arrow[r, bend right=40, "F'"'{name=B}] & \mathcal{D}
\arrow[from=A, to=B, Rightarrow, shorten=3pt, "\phi"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y"] \\
F'(X) \arrow[r, "F'(f)"] & F'(Y)
\end{tikzcd}$$

$\forall f : X \to Y \in \mathcal{C}$

Interchange Law

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F"{name=A}] \arrow[r, bend right=50, "F''"'{name=B}] & \mathcal{D} \arrow[r, bend left=50, "G"{name=C}] \arrow[r, bend right=50, "G''"'{name=D}] & \mathcal{E}
\arrow[from=A, to=B, Rightarrow, shorten=2pt, "\phi' \circ^{\mathrm{VER}} \phi"]
\arrow[from=C, to=D, Rightarrow, shorten=2pt, "\psi' \circ^{\mathrm{VER}} \psi"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[row sep=large]
\mathcal{C} \arrow[r, bend left=60, "GF"{name=A}] \arrow[r, "G'F'"{name=B, description}] \arrow[r, bend right=60, "G''F''"'{name=C}] & \mathcal{E}
\arrow[from=A, to=B, Rightarrow, shorten=2pt, "\psi \circ^{\mathrm{HOR}} \phi"]
\arrow[from=B, to=C, Rightarrow, shorten=2pt, "\psi' \circ^{\mathrm{HOR}} \phi'"]
\end{tikzcd}$$

IV. COMPOSITIONS OF NATURAL TRANSFORMATIONS

VERTICAL COMPOSITION

$$\begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=70, "F"{name=U}] \arrow[r, "F'"{name=M, description}] \arrow[r, bend right=70, "F''"'{name=L}] & \mathcal{D} \arrow[Rightarrow, from=U, to=M, "\phi"] \arrow[Rightarrow, from=M, to=L, "\phi'"] \end{tikzcd} \;=:\; \begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=40, "F"{name=U}] \arrow[r, bend right=40, "F''"'{name=L}] & \mathcal{D} \arrow[Rightarrow, from=U, to=L, "\phi' \circ^{VER} \phi"] \end{tikzcd}$$

$$(\phi' \circ^{VER} \phi)_X : F(X) \xrightarrow{\phi_X} F'(X) \xrightarrow{\phi'_X} F''(X)$$

HORIZONTAL COMPOSITION

$$\begin{tikzcd}[column sep=large] \mathcal{C} \arrow[r, bend left=50, "F"{name=U1}] \arrow[r, bend right=50, "F'"'{name=L1}] & \mathcal{D} \arrow[r, bend left=50, "G"{name=U2}] \arrow[r, bend right=50, "G'"'{name=L2}] & \mathcal{E} \arrow[Rightarrow, from=U1, to=L1, "\phi"] \arrow[Rightarrow, from=U2, to=L2, "\psi"] \end{tikzcd} \;=:\; \begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=40, "GF"{name=U}] \arrow[r, bend right=40, "G'F'"'{name=L}] & \mathcal{E} \arrow[Rightarrow, from=U, to=L, "\psi \circ^{HOR} \phi"] \end{tikzcd}$$

$$(\psi \circ^{HOR} \phi)_X : \begin{tikzcd} GF(X) \arrow[r, "G(\phi_X)"] \arrow[d, "\psi_{F(X)}"'] \arrow[dr, phantom, "\circlearrowright"] & GF'(X) \arrow[d, "\psi_{F'(X)}"] \\ G'F(X) \arrow[r, "G'(\phi_X)"'] & G'F'(X) \end{tikzcd}$$

$\circlearrowright$: NATURALITY OF ψ AT ϕ_X

CAN ALSO COMBINE FUNCTORS W/ NAT'L TRANSF'NS...

A NATURAL TRANSF'N $\phi : F \Rightarrow F'$

$$\begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=40, "F"{name=U}] \arrow[r, bend right=40, "F'"'{name=L}] & \mathcal{D} \arrow[Rightarrow, from=U, to=L, "\phi"] \end{tikzcd} \qquad \begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y"] \\ F'(X) \arrow[r, "F'(f)"'] & F'(Y) \end{tikzcd}$$

$\forall f : X \to Y \in \mathcal{C}$

INTERCHANGE LAW

$$\begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=50, "F"{name=U1}] \arrow[r, bend right=50, "F''"'{name=L1}] & \mathcal{D} \arrow[r, bend left=50, "G"{name=U2}] \arrow[r, bend right=50, "G''"'{name=L2}] & \mathcal{E} \arrow[Rightarrow, from=U1, to=L1, "\phi' \circ^{VER} \phi"] \arrow[Rightarrow, from=U2, to=L2, "\psi' \circ^{VER} \psi"] \end{tikzcd} \;=\; \begin{tikzcd}[column sep=huge] \mathcal{C} \arrow[r, bend left=70, "GF"{name=U}] \arrow[r, "G'F'"{name=M, description}] \arrow[r, bend right=70, "G''F''"'{name=L}] & \mathcal{E} \arrow[Rightarrow, from=U, to=M, "\psi \circ^{HOR} \phi"] \arrow[Rightarrow, from=M, to=L, "\psi' \circ^{HOR} \phi'"] \end{tikzcd}$$

IV. COMPOSITIONS OF NATURAL TRANSFORMATIONS

CAN ALSO COMBINE FUNCTORS W/ NAT'L TRANSF'NS...

A NATURAL TRANSF'N $\phi : F \Rightarrow F'$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "F'"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\phi"] \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y"] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

$\forall f : X \to Y \in \mathcal{C}$

IV. COMPOSITIONS OF NATURAL TRANSFORMATIONS

$$\begin{tikzcd} \mathcal{B} \arrow[r, "F_0"] & \mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "F'"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, phantom, "\Downarrow \phi"] \end{tikzcd} \quad =: \quad \begin{tikzcd} \mathcal{B} \arrow[r, bend left=50, "FF_0"{name=U}] \arrow[r, bend right=50, "F'F_0"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, phantom, "\Downarrow \phi * F_0"] \end{tikzcd}$$

FOR $W \in \mathcal{B}$:

$$(\phi * F_0)_W := \phi_{F_0(W)} :$$

$$FF_0(W) \longrightarrow F'F_0(W)$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "F'"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, phantom, "\Downarrow \phi"] \arrow[r, "F_1"] & \mathcal{E} \end{tikzcd} \quad =: \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F_1F"{name=U}] \arrow[r, bend right=50, "F_1F'"'{name=D}] & \mathcal{E} \arrow[from=U, to=D, phantom, "\Downarrow F_1 * \phi"] \end{tikzcd}$$

FOR $X \in \mathcal{C}$:

$$(F_1 * \phi)_X := F_1(\phi_X) :$$

$$F_1F(X) \longrightarrow F_1F'(X)$$

CAN ALSO COMBINE

FUNCTORS W/ NAT'L TRANSF'NS...

A NATURAL TRANSF'N

$\phi : F \Rightarrow F'$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "F'"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, phantom, "\Downarrow \phi"] \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y"] \\ F'(X) \arrow[r, "F'(f)"'] & F'(Y) \end{tikzcd}$$

$\forall f : X \to Y \in \mathcal{C}$

IV. COMPOSITIONS OF NATURAL TRANSFORMATIONS

$$\begin{tikzcd} \mathcal{B} \arrow[r, "F_0", "\equiv \\ \mathrm{ID}_{F_0}"'] & \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "F'"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\phi"] \end{tikzcd} \quad =: \quad \begin{tikzcd} \mathcal{B} \arrow[r, bend left, "FF_0"{name=U}] \arrow[r, bend right, "F'F_0"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\phi * F_0"] \end{tikzcd}$$

FOR $W \in \mathcal{B}$:

$$(\phi * F_0)_W := \phi_{F_0(W)} : \; FF_0(W) \longrightarrow F'F_0(W)$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "F'"'{name=D}] & \mathcal{D} \arrow[r, "F_1", "\equiv \\ \mathrm{ID}_{F_1}"'] & \mathcal{E} \arrow[from=U, to=D, Rightarrow, "\phi"] \end{tikzcd} \quad =: \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1F"{name=U}] \arrow[r, bend right, "F_1F'"'{name=D}] & \mathcal{E} \arrow[from=U, to=D, Rightarrow, "F_1 * \phi"] \end{tikzcd}$$

FOR $X \in \mathcal{C}$:

$$(F_1 * \phi)_X := F_1(\phi_X) : \; F_1F(X) \longrightarrow F_1F'(X)$$

CAN ALSO COMBINE FUNCTORS W/ NAT'L TRANSF'NS...

A NATURAL TRANSF'N $\phi : F \Rightarrow F'$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "F'"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\phi"] \end{tikzcd} \qquad \begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y"] \\ F'(X) \arrow[r, "F'(f)"'] & F'(Y) \end{tikzcd}$$

$$\forall f : X \to Y \in \mathcal{C}$$

SPECIAL CASE OF HORIZONTAL COMPOSITION

$$\begin{cases} \phi * F_0 = \phi \circ^{\mathrm{hor}} \mathrm{ID}_{F_0} \\ F_1 * \phi = \mathrm{ID}_{F_1} \circ^{\mathrm{hor}} \phi \end{cases}$$

IV. Compositions of Natural Transformations

$$\begin{tikzcd} \mathcal{B} \arrow[r, "F_0"] & \mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F'"'] \arrow[r, phantom, "\Downarrow \phi"] & \mathcal{D} \end{tikzcd} \quad =: \quad \begin{tikzcd} \mathcal{B} \arrow[r, bend left=50, "FF_0"] \arrow[r, bend right=50, "F'F_0"'] \arrow[r, phantom, "\Downarrow \phi * F_0"] & \mathcal{D} \end{tikzcd}$$

For $W \in \mathcal{B}$:

$(\phi * F_0)_W := \phi_{F_0(W)}$:

$FF_0(W) \longrightarrow F'F_0(W)$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F'"'] \arrow[r, phantom, "\Downarrow \phi"] & \mathcal{D} \arrow[r, "F_1"] & \mathcal{E} \end{tikzcd} \quad =: \quad \begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F_1F"] \arrow[r, bend right=50, "F_1F'"'] \arrow[r, phantom, "\Downarrow F_1 * \phi"] & \mathcal{E} \end{tikzcd}$$

For $X \in \mathcal{C}$:

$(F_1 * \phi)_X := F_1(\phi_X)$:

$F_1F(X) \longrightarrow F_1F'(X)$

Can also combine functors w/ nat'l transf'ns...

... called "Whiskering"

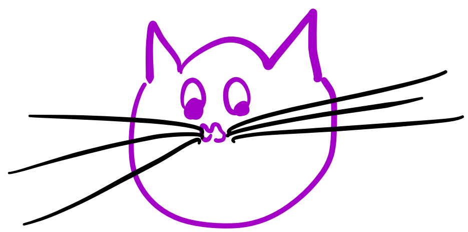
A natural transf'n

$\phi : F \Rightarrow F'$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F'"'] \arrow[r, phantom, "\Downarrow \phi"] & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"'] \arrow[dr, phantom, "\circlearrowleft"] & F(Y) \arrow[d, "\phi_Y"] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

$\forall f : X \to Y \in \mathcal{C}$



MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #8

NEXT TIME:

WILL SAY WHEN CATEGORIES ARE "THE SAME"

TOPICS:

I. FUNCTORS (§§2.3.1– 2.3.2)

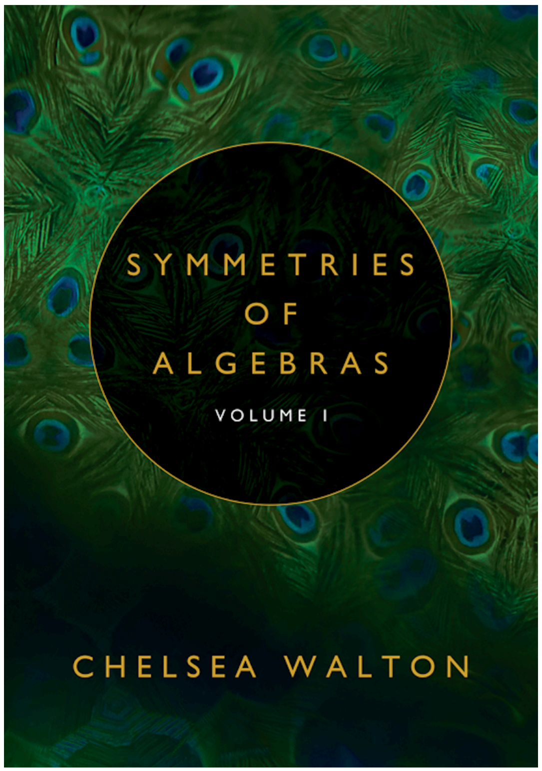
II. BIFUNCTORS & MULTIFUNCTORS (§2.3.3)

III. NATURAL TRANSFORMATIONS (§2.3.4)

IV. COMPOSITIONS OF NATURAL TRANSFORMATIONS (§2.3.5)

Enjoy this lecture?
You'll enjoy the textbook!

C. Walton's "Symmetries of Algebras, Volume 1" (2024)



Available for purchase at :

619 Wreath (at a discount)

https://www.619wreath.com/

Also on Amazon
&
Google Play

Lecture #8 keywords: bifunctor, contravariant, covariant, essentially surjective, faithful, full, fully faithful, functor, horizontal composition, inclusion, multifunctor, natural transformation, vertical composition