MATH $466 / 566$
SPRING 2024

LAST TIME

- Functors
- bifunctors a multifunctors
- natural transformations
- compositions of naturaltransformations

TOPICS:
I. ISOMORPHISM OF CATEGORIES (\$2.4.1)
II. EquIVALENCE OF CATEGORIES (882.4.2-2.4.3)
III. MORITA EQUivalence

LECTURE \#9

CHELSEA WALTON RICE U.

$$
\begin{aligned}
& \equiv \text { RECALL } \equiv \\
& \begin{array}{c}
\text { ACATEGORY } \zeta \\
\text { CONSISTS OF: } \\
\text { (a) OBJECTS. } \\
\text { (b) MORPHISMS } \\
\text { HOMC(X,Y) } \\
\forall x, Y \in \zeta . \\
\text { (c) id }: x \rightarrow x \\
\forall x \in \zeta . \\
\text { (d) } g f: w \rightarrow Y \\
\forall f: w \rightarrow x \\
g: x \rightarrow Y . \\
\text { SATISFyING } \\
\text { ASSOCIATVITy } \\
\text { (hg)f }=h(g f) \\
\text { uNiTALITy } \\
\text { idxf }=f, g i d x=g
\end{array}
\end{aligned}
$$

$$
\equiv \operatorname{RECALL} \equiv
$$

A CATEGORY C CONSISTS OF:
(a) OBJECTS.
(b) MORPHISMS $H_{0} M_{e}(x, y)$ $\forall x, y \in \zeta$.
(c) id $x: x \rightarrow x$ $\forall x \in \mathcal{C}$.
(d) $g f: w \rightarrow y$ $\forall f: w \rightarrow x$ $g: x \rightarrow y$.

SATISFYING
Associativity

$$
(h g) f=h(g f)
$$

unitality

$$
i d_{x} f=f, g i d x=g
$$



Fd Alg
Pose

$\equiv$ Recall $\equiv$ When are two categories considered the same?

A CATEGORY C CONSISTS OF:
(a) OBJECTS.
(b) MORPHISMS Ho me $_{e}(x, y)$ $\forall x, y \in \zeta$.
(c) id $x: x \rightarrow x$ $\forall x \in \mathcal{C}$.
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Pose

引 RECALL $\equiv$ When are Two categories considered the same?

A CATEGORY C CONSISTS OF:
(a) OBJECTS.
(b) MORPHISMS $\operatorname{Hom}_{c}(x, y)$ $\forall x, y \in \zeta$.
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(d) $g f: w \rightarrow y$ $\forall f: w \rightarrow x$ $g: x \rightarrow y$.
SATISFYING
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i d_{x} f=f, g i d_{x}=g
$$

HOW DO WE MOVE ONE FROM CATEGORY TO ANOTHER?


Pose






When are two categories considered the same?


Now Some answers

I. ISOMORPHISM OF CATEGORIES

C, $A$ categories
6 AND $\theta$
are said to be
ISOMORPHIC
IF F FUNCTOR

$$
F: \zeta \rightarrow \theta \& G: \theta \rightarrow C
$$

such that

$$
G F=I d_{e}
$$

\& $F G=I d \theta$
WRITE $\zeta \cong \theta$
I. ISOMORPHISM OF CATEGORIES Ge, $\theta$ categories
$\zeta$ AND $A$
are said to be
ISOMORPHIC
IF 于FUNCTORS

$$
F: \zeta \rightarrow \theta \notin G: \theta \rightarrow \zeta
$$

such That

$$
G F=I d_{e}
$$

$$
\& F G=I d \theta
$$

WRITE $\zeta \cong \theta$

EXAMPLE: $G=G R O U P$
GET: $G-\operatorname{Mod} \cong \operatorname{Rep}(G)$
I. ISOMORPHISM OF CATEGORIES

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such that

$$
\begin{gathered}
G F=I_{d} \\
\& F G=I d \theta \\
\text { WRITE } \zeta \cong \theta
\end{gathered}
$$

EXAMPLE: $G=G R O U P$
GET: $G-\operatorname{Mod} \cong \operatorname{Rep}(G)$

$$
\begin{aligned}
& F: G-\operatorname{Mod} \longrightarrow \operatorname{Rep}(G) \\
& \equiv \text { ACTON }=\text { Group tonom. }= \\
& \left(V_{n}, D: G \times V \rightarrow V\right) \mapsto\left(V, P_{V}: G \rightarrow G L(V)\right) \\
& \text { Ven } \\
& (g h) D v=g D(h D v) . \\
& g \mapsto\left[\begin{array}{l}
V \rightarrow V \\
v \mapsto g v v
\end{array}\right] \\
& p(g h)(v)=g h) D v^{\approx}=g D(h D v) \\
& p(g) \rho(h)(v)=g D(\rho(h)(r)) \\
& F^{\prime}: \operatorname{Rep}(G) \longrightarrow G-\operatorname{Mod} \\
& \left(V_{n}, p: G \rightarrow G L(V)\right) \mapsto\left(V, D_{v}: G \times V \rightarrow V\right) \\
& (g, v) \mapsto \rho g(v)
\end{aligned}
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EXAMPLE: $G=G R O U P$
GET: $G-\operatorname{Mod} \cong \operatorname{Rep}(G)$

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F: G-M \text { od } \longrightarrow \operatorname{Rep}(G)
$$

ven

$$
(g h) \nabla v=g D(h \nabla v)
$$

$$
g \mapsto\left[\begin{array}{l}
V \rightarrow V \\
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\end{array}\right]
$$

$$
\begin{aligned}
& p(g h)(v)=g h) D v \\
& \equiv g D(h D v) \\
& p(g) p(h)(v)=g D(p(h)(v))
\end{aligned}
$$

$$
\begin{aligned}
& F^{\prime}: \operatorname{Rep}(G) \longrightarrow G-M_{01} \\
& \left(V_{n}, p: G \rightarrow G L(V)\right) \mapsto\left(V, D_{v}: G \times V \rightarrow V\right) \\
& \begin{array}{l}
\text { Voc } \\
(g, V) \mapsto \rho(\sigma)
\end{array} \\
& \left.C H E C K F^{\prime} F=I d_{G-\operatorname{Mod}} \& F F^{\prime}=I d_{\operatorname{Rep}(G)}\right)
\end{aligned}
$$

I. ISOMORPHISM OF CATEGORIES
$\zeta$ AND $\theta$
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\end{aligned}
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g \mapsto\left[\begin{array}{l}
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& p(g h)(v)=g h) D v \\
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\begin{aligned}
& F^{\prime}: \operatorname{Rep}(G) \longrightarrow G-\operatorname{Mod} \\
& \left(V_{m}, p: G \rightarrow G L(V)\right) \mapsto\left(V, D_{v}: G \times V \rightarrow V\right) \\
& \text { vc } \\
& (g, \sigma) \mapsto \rho g(\sigma) \\
& \text { CHECK } F^{\prime} F=I d_{G-\operatorname{Mod}} \& F F^{\prime}=I d_{\operatorname{Rep}(G)}
\end{aligned}
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F: \zeta \rightarrow \theta \not G: \theta \rightarrow \zeta
$$

such that

$$
G F=I d_{\varphi}
$$

$$
\& F G=I d \theta
$$

WRITE $\zeta \cong \theta$

EXAMPLE: $G=G R O U P$
UPGRADE OF EYER. 1.13
GET: $G-\operatorname{Mod} \cong \operatorname{Rep}(G)$

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\begin{aligned}
F: G-M o d & \longrightarrow \operatorname{Rep}(G) \\
=\text { Action } \equiv & =\operatorname{Group} \text { Hon } о \text { M. }=
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p(g h)(v) & =g h) D v \\
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\end{aligned}
$$

$$
F^{\prime}: \operatorname{Rep}(G) \longrightarrow G-\operatorname{Mod}
$$

$$
\begin{aligned}
&\underset{\underset{N}{\text { Ice }}}{(V, p: G} \rightarrow G L(V)) \mapsto\left(V, D_{V}: G \times V\right. \\
&(g, v) \mapsto p g(v)
\end{aligned}
$$

CHECK $F^{\prime} F=I d_{G-\operatorname{Mod}} \& F F^{\prime}=I d_{\operatorname{Rep}(G)}$
I. ISOMORPHISM OF CATEGORIES
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\begin{aligned}
& p(g h)(v)=g h\left(D v^{v} g D(h \nabla v)\right. \\
& p(g) p(h)(v)=g D(p(h)(v))
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$F^{\prime}: \operatorname{Rep}(G) \longrightarrow G-\operatorname{Mod}$
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\begin{aligned}
& \left(V_{n}, p: G \rightarrow G L(V)\right) \mapsto\left(V, D_{v}: G \times V \rightarrow V\right) \\
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I. ISOMORPHISM OF CATEGORIES
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ARE ISOMORPHIC IF $\exists F: \zeta \rightarrow \theta \notin G: \theta \rightarrow \zeta$
.Э.
$G F=I d_{e} \& F G=I d \theta$
WRITE $\zeta \cong \theta$

CONSIDER FdVec
/iR FIELD
TAKE $\&=$ FULL SUBCATEGORY OF $F V_{\text {Pec }}{ }_{R}$ ON OBJECTS $\left\{\mathbb{k}^{\oplus n} y_{n \in N}\right.$

Perhaps Favec \& are the "same" as EVERY F.D. VECTOR SPACE $1 S \cong \mathbb{R}^{\oplus n}$ FOR $\begin{aligned} & \text { SOME } \\ & n \in N \\ & \text {. }\end{aligned}$
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& F: \text { FdVec } \longrightarrow \delta \nexists G: \downarrow \longrightarrow F a V e c \\
& V \mapsto \mathbb{k}^{\oplus d i m_{k} v} \quad \mathbb{k}^{\oplus n} \mapsto \mathbb{k}^{\oplus n}
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$$
\begin{aligned}
& F: \text { FdVec } \longrightarrow \delta \\
& V \mapsto \mathbb{k}^{\oplus \operatorname{din}_{k_{k}} V} \neq G: \& \longrightarrow F_{d} V \text { ec } \\
& \quad \|^{\oplus n} \mapsto \mathbb{R}^{\oplus n}
\end{aligned}
$$

HERE, $F G\left(\mathbb{k}^{\oplus n}\right)=F\left(\mathbb{R}^{\oplus n}\right)=\mathbb{k}^{\oplus n}$.
BUT $G F(V)=G\left(\mathbb{R}^{\left.\oplus{ }^{\oplus i m_{k} V}\right)}\right)=\mathbb{k}^{\left(\text {dim }_{k} V\right.}$
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$\therefore$ FdVec $\geqslant \&$
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WILL WEAKEN NOTION OF "SAMENESS"...
I. ISOMORPHISM OF CATEGORIES
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SKELETON OF 6
三 Full subcategory Skel(ce) of Y ON ISOCLASSES OF Obj (E)

CONSIDER FdVec
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TAKE $\&=$ FULL SUBCATEGORY OF $F d V e c_{\mathbb{R}}$ ON OBJECTS $\left\{\mathbb{k}^{\oplus n}\right\}_{n \in N}$

Perhaps FaVec \& are the "same" as EVERY FWD. VECTOR SPACE $1 S \cong \mathbb{R}^{\oplus n}$ FOR $\begin{gathered}\text { SOME } \\ n \in N \\ \text {. }\end{gathered}$
Try:

$$
\begin{aligned}
& F: \text { FdVec } \longrightarrow \& \quad \neq G: \downarrow \longrightarrow \text { FaVe } \\
& V \mapsto \mathbb{k}^{\oplus \operatorname{dim}_{\mathbb{k}} v} \quad \mathbb{k}^{\oplus n} \mapsto \mathbb{k}^{\oplus n}
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HERE, $F G\left(\mathbb{k}^{\oplus n}\right)=F\left(\mathbb{R}^{\oplus n}\right)=\mathbb{k}^{\oplus n}$.
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WILl WEAKEN NOTION OF "SAMENESS"...
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CONSIDER FdVec
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TAKE $\&=$ FULL SUBCATEGORY OF $F V_{\text {Pec }}^{\mathbb{R}}$ ON OBJECTS $\left\{\mathbb{k}^{\oplus n}\right\}_{n \in N}$

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& V \mapsto \mathbb{k}^{\oplus \alpha_{1 i} m_{k} V} \quad \mathbb{k}^{\oplus n} \mapsto \mathbb{k}^{\oplus n}
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HERE, $F G\left(k^{\oplus n}\right)=F\left(\mathbb{k}^{\oplus n}\right)=\mathbb{k}^{\oplus n}$.
BUT $G F(V)=G\left(\mathbb{R}^{\oplus d i m_{\mathbb{R}} V}\right)=\mathbb{R}^{\text {dim } M_{R} V} \neq V$
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WILL WEAKEN NOTION OF "SAMENESS"...
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CONSIDER FdVec
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TAKE $\&=$ FULL SUBCATEGORY OF $F d V e c_{\mathbb{R}}$ SKel(FdVec) ON OBJECTS $\mathbb{R}^{\oplus n} y_{n \in \mathbb{N}}$ perhaps Favec \& are the "same" as EVERY F.D. VECTOR SPACE is $\cong \mathbb{R}^{\text {Bn }}$ for dome $\underset{n \in N .}{ }$
Try:
$F:$ FdVec $\longrightarrow \& \quad \neq G: \downarrow \longrightarrow$ FaVe

$$
V \mapsto \mathbb{k}^{\oplus \operatorname{dim}_{\mathbb{k}} V} \quad \mathbb{k}^{\oplus n} \mapsto \mathbb{k}^{\oplus n}
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SKELETON OF 6
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EXERt. 2.33

$$
\operatorname{Skel}(\varphi) \cong \zeta \Leftrightarrow \operatorname{Skel}(\varphi)=\zeta
$$

CONSIDER FdVec
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TAKE $\&=$ FULL SUBCATEGORY OF $F d V e c_{\mathbb{R}}$ SKel(FdVec) ON OBJECTS $\left\{\mathbb{k}^{\oplus n} y_{n \in \mathbb{N}}\right.$ perhaps Favec \& are the "same" as EVERY F.D. VECTOR SPACE is $\cong \mathbb{R}^{\text {Bn }}$ for dome $\underset{n \in N .}{ }$
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$F:$ FdVec $\longrightarrow \delta \neq G: \downarrow \longrightarrow$ FaVe

$$
V \mapsto \mathbb{k}^{\oplus d i m_{k} v} \quad \mathbb{k}^{\oplus n} \mapsto \mathbb{k}^{\oplus n}
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HERE, $F G\left(\mathbb{k}^{\oplus n}\right)=F\left(\mathbb{k}^{\oplus n}\right)=\mathbb{k}^{\left({ }^{\oplus n}\right.}$.
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WIL WEAKEN NOTION OF "SAMENESS"... EQuIv.
II. equivalence of categories

$$
\varphi \text { AND } \theta
$$

ARE ISOMORPHIC IF

$$
\begin{aligned}
& \exists F: \zeta \rightarrow \theta \notin G: \theta \rightarrow \zeta \\
& G F=I d_{\varphi} \& F G=I d \theta
\end{aligned}
$$

WRITE $\zeta \cong \theta$
SKELETON OF $\zeta$
三 FULL SUBCATEGORY Skel( $\mathfrak{C}$ ) of $\zeta$ ONISOCLASSES OF Obj (e)

EYER. 2.33

$$
|\operatorname{SKel}(\varphi) \cong \zeta \Leftrightarrow \operatorname{Skel}(\varphi)=\zeta|
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CONSIDER FdVec
TAKE $\notin=$ FULL SUBCATEGORY OF FdVec $\mathbb{R}$ Skel(FdVec) ON OBJECTS $\left\{\mathbb{k}^{\oplus n}\right\}_{n \in \mathbb{N}}$ $F:$ FdVec $\longrightarrow \& \quad \& G: \searrow \longrightarrow F d V e c$ $V \mapsto \mathbb{k}^{\oplus d i m_{k} V} \quad \mathbb{k}^{\oplus n} \mapsto \mathbb{k}^{\oplus n}$

HERE, $F G\left(\mathbb{R}^{\oplus n}\right)=F\left(\mathbb{R}^{\oplus n}\right)=\mathbb{R}^{\oplus n}$.
BUT $G F(V)=G\left(\mathbb{R}^{\oplus \text { dim } M_{k} V}\right)=\mathbb{R}^{\text {dim } M_{k} V} \neq V$
$\therefore$ FdVec 丰 \&
WIL WEAKEN NOTION OF "SAMENESS"... EQuIV.
II. equivalence of categories

$$
\varphi \text { AND } \theta
$$

are equivalent if

. $\rightarrow$.
$G F \cong I d_{e} \& F G \cong I d \theta$
WRITE $\zeta \simeq \theta$
SKELETON OF $\zeta$
三 FULL SUBCATEGORY Skel( $\mathfrak{C}$ ) of $\zeta$ ONISOCLASSES OF Obj (e)

EYER. 2.33
$\operatorname{Skel}(\zeta) \cong \zeta \Leftrightarrow \operatorname{Skel}(\varphi)=\zeta$

CONSIDER FdVec
TAKE $\&=$ FUCL SUBCATEGORY OF FdVec $\mathbb{R}$ Skel(FdVec) ON OBJECTS $\left\{\mathbb{k}^{\oplus n}\right\}_{n \in \mathbb{N}}$ $F:$ FdVec $\longrightarrow \& \& G: \& \longrightarrow F d V e c$ $V \mapsto \mathbb{k}^{\oplus d i m_{k} V} \quad \mathbb{k}^{\oplus n} \mapsto \mathbb{k}^{\oplus n}$
HERE, $F G\left(\mathbb{R}^{\oplus n}\right)=F\left(\mathbb{R}^{\oplus n}\right)=\mathbb{R}^{\oplus n}$.
BUT $G F(V)=G\left(\mathbb{R}^{\oplus \text { dim } M_{k} V}\right)=\mathbb{R}^{\text {dim } m_{k} V} \neq V$
$\therefore$ FdVec 丰 \&
WIL WEAKEN NOTION OF "SAMENESS"... EQuIV.
II. EqUIVALENCE OF CATEGORIES

Ce, $\theta$ categories

6 AND $\theta$
are equivalent if $\mathcal{F}: \zeta \rightarrow \theta \notin G: \theta \rightarrow \zeta$
..
$G F \cong I d_{e} \& F G \cong I d \theta$
WRITE $\zeta \simeq \theta$
SKELETON OF $\zeta$
三 FULL SUBCATEGORY Skew( $\cdot$ ) of $\zeta$ ONISOCLASSES OF Obj (e)

EYER. 2.33
$\operatorname{Skel}(\zeta) \cong \zeta \Leftrightarrow \operatorname{Skel}(\zeta)=\zeta$

J natural
ISOMORPHISMS:

$$
\begin{gathered}
G \equiv \text { "QUASI-INVERSE" } \\
\text { OFF }
\end{gathered}
$$



CONSIDER FdVec
TAKE $\underset{\|}{\|}=$ FUCL SUBCATEGORY OF FdVec $\mathbb{R}^{2}$
Skel(FdVec) ON OBJECTS $\left\{\mathbb{k}^{\oplus n} y_{n \in \mathbb{N}}\right.$ $F:$ FdVec $\longrightarrow \& \quad \& G: \searrow \longrightarrow F d V e c$

$$
V \mapsto \mathbb{k}^{\oplus d_{i} m_{\mathbb{k}} v} \quad \mathbb{k}^{\oplus n} \mapsto \mathbb{k}^{\oplus n}
$$

HERE, $F G\left(\mathbb{R}^{\oplus n}\right)=F\left(\mathbb{R}^{\oplus n}\right)=\mathbb{R}^{\oplus n}$.
BUT $G F(V)=G\left(\mathbb{R}^{\oplus \text { dim } M_{k} V}\right)=\mathbb{R}^{\text {dim } m_{k} V} \neq V$
$\therefore$ FdVec 丰 $\delta$
WIL WEAKEN NOTION OF "SAMENESS"... EQuIV.
II. equivalence of categories

Ce, $\theta$ categories


SKELETON OF $\zeta$
三 FULL SUBCATEGORY Skel(c) of $\zeta$ ONISOCLASSES OF Obj (e)

EYER. 2.33
$\operatorname{Skel}(\zeta) \cong \zeta \Leftrightarrow \operatorname{Skel}(\zeta)=\zeta$

J natural
ISOMORPHISMS:


CONSIDER FdVec
/ IR FIELD
TAKE $\notin=$ FUCL SUBCATEGORY OF FdVec ${ }_{k}$
Skel(FdVec) ON OBJECTS $\left\{\mathbb{k}^{\oplus n} y_{n \in \mathbb{N}}\right.$
$F:$ FdVec $\longrightarrow \& \quad \& G: \searrow \longrightarrow F d V e c$

$$
V \mapsto \mathbb{k}^{\oplus d_{i} m_{\mathbb{k}} v} \quad \mathbb{k}^{\oplus n} \mapsto \mathbb{k}^{\oplus n}
$$

HERE, $F G\left(\mathbb{R}^{\oplus n}\right)=F\left(\mathbb{R}^{\oplus n}\right)=\mathbb{R}^{\oplus n}$.
BUT $G F(V)=G\left(\mathbb{R}^{\oplus \text { dim } m_{k} V}\right)=\mathbb{R}^{\oplus \text { dim } m_{k} V} \neq V$
$\therefore$ FdVec 丰 \&
II. equivalence of categories

Ce, $\theta$ categories


SKELETON OF $\zeta$
三 FULL SUBCATEGORY SEel( $\boldsymbol{C}$ ) of $\zeta$ ONISOCLASSES OF Obj (e)

EYER. 2.33
$\operatorname{Skel}(\zeta) \cong \zeta \Leftrightarrow \operatorname{Skel}(\zeta)=\zeta$

J natural
ISOMORPHISMS:


CONSIDER FdVec
TAKE $A=$ FULL SUBCATEGORY OF FdVec $\mathbb{R}^{2}$
Skel (FdVec) ON OBJECTS $\left\{\mathbb{R}^{\oplus n}\right\}_{n \in \mathbb{N}}$

$$
\begin{aligned}
& F: \text { FdVec } \longrightarrow \phi \quad \& G: \downarrow \longrightarrow F d V e c \\
& V \mapsto \mathbb{k}^{\oplus d i m_{\mathbb{k}} v} \quad \mathbb{k}^{\oplus n} \mapsto \mathbb{k}^{\oplus n}
\end{aligned}
$$

HERE, $F G\left(\mathbb{R}^{\oplus n}\right)=F\left(\mathbb{R}^{\oplus n}\right)=\mathbb{R}^{\oplus n}$.
BUT $G F(V)=G\left(\mathbb{R}^{\oplus \text { dim } M_{\mathbb{R}} V}\right)=\mathbb{R}^{\oplus \operatorname{dim}_{\mathbb{R}} V} \neq V$
$\therefore$ FdVec 丰 \&
II. equivalence of categories

Ce, $\theta$ categories


SKELETON OF $\zeta$
三 FULL SUBCATEGORY SEel( $\boldsymbol{C}$ ) of $\zeta$ ON ISOCLASSES OF Obj (e)

EYER. 2.33

$$
\operatorname{Skel}(\zeta) \cong \zeta \Leftrightarrow \operatorname{Skel}(\zeta)=\zeta
$$

J natural
ISOMORPHISMS:


CONSIDER FAVec
TAKE $\notin=$ FUCL SUBCATEGORY OF FdVec ${ }_{k}$
Skel (FdVec) ON OBJECTS $\left\{\mathbb{R}^{\oplus n}\right\}_{n \in \mathbb{N}}$

$$
\begin{aligned}
& F: \text { FdVec } \longrightarrow \phi \quad \& G: \downarrow \longrightarrow F d V e c \\
& V \mapsto \mathbb{k}^{\oplus d i m_{k} V} \quad \mathbb{k}^{\oplus n} \mapsto \mathbb{k}^{\oplus n}
\end{aligned}
$$

HERE, $F G\left(\mathbb{R}^{\oplus n}\right)=F\left(\mathbb{R}^{\oplus n}\right)=\mathbb{R}^{\oplus n}$.

$$
\begin{aligned}
& \notin G F(V)=G\left(\mathbb{R}^{\oplus d i m_{\mathbb{R}} V}\right)=\mathbb{R}^{\oplus d i M_{\mathfrak{k}} V} \cong V \\
\therefore & F d V e c \simeq \&
\end{aligned}
$$

II. equivalence of categories

Ce, $\theta$ categories


SKELETON OF $\zeta$
三 FULL SUBCATEGORY Skel( $\mathfrak{C}$ ) of $\zeta$ ON ISOCLASSES OF Obj (e)

EXR. 2.33

$$
\operatorname{Skel}(\zeta) \cong \zeta \Leftrightarrow \operatorname{Skel}(\zeta)=\zeta
$$

J natural
ISOMORPHISMS:


CONSIDER FdVec
TAKE $\notin=$ FUCL SUBCATEGORY OF FdVec ${ }_{k}$
Skel(FdVec) ON OBJECTS $\left\{\mathbb{k}^{\oplus n} y_{n \in \mathbb{N}}\right.$

$$
\begin{aligned}
& F: \text { FdVec } \longrightarrow \delta \& G: \& \longrightarrow F d V e c \\
& V \mapsto \mathbb{k}^{\oplus d i m_{k} V} \quad \mathbb{k}^{\oplus n} \mapsto \mathbb{k}^{\oplus n}
\end{aligned}
$$

HERE, $F G\left(\mathbb{R}^{\oplus n}\right)=F\left(\mathbb{R}^{\oplus n}\right)=\mathbb{R}^{\oplus n}$.

$$
\begin{aligned}
& \notin G F(V)=G\left(\mathbb{R}^{\oplus d i m_{12} V}\right)=\mathbb{R}^{\oplus d i M_{12} V} \cong V \\
\therefore & F d V e c \simeq \&
\end{aligned}
$$

II. equivalence of categories

Ce, $\theta$ categories


SKELETON OF $\zeta$
三 FULL SUBCATEGORY Skel( $\mathfrak{C}$ ) of $\zeta$ ONISOCLASSES OF Obj (e)

EXES. 2.33

$$
\operatorname{Skel}(\varphi) \cong \zeta \Leftrightarrow \operatorname{Skel}(\varphi)=\zeta
$$

$\left.S_{\operatorname{kel}(C)}\right) \simeq \zeta$ ALways

$$
\zeta \simeq \theta \Leftrightarrow \operatorname{Skel}(\varphi) \cong \operatorname{Skel}(\theta)
$$

J natural
ISOMORPHISMS:


CONSIDER FdVec
TAKE $\quad A=$ FULL SUBCATEGORY OF FdVec $\mathbb{R}^{2}$
$S \mathrm{Kel}(\mathrm{FdVec})$ ON OBJECTS $\left\{\mathbb{k}^{\oplus n} y_{n \in \mathbb{N}}\right.$

$$
\begin{aligned}
& F: \text { FdVec } \longrightarrow \& \quad \& G: \& \longrightarrow F d V e c \\
& V \mapsto \mathbb{k}^{\oplus d i m_{\mathbb{R}^{2}} V} \quad \mathbb{k}^{\oplus n} \mapsto \mathbb{k}^{\oplus n}
\end{aligned}
$$

HERE, $F G\left(\mathbb{R}^{\oplus n}\right)=F\left(\mathbb{R}^{\oplus n}\right)=\mathbb{R}^{\oplus n}$.

$$
\begin{aligned}
& \notin G F(V)=G\left(\mathbb{R}^{\oplus d i m_{12} V}\right)=\mathbb{R}^{\oplus d i M_{12} V} \cong V \\
\therefore & F d V e c \simeq \&
\end{aligned}
$$

II. EqUIVALENCE OF CATEGORIES

Ce, $A$ categories

natural ISOMORPHISMS:

like Two structures are the same
II. equivalence of categories

Ge, $\theta$ categories

$$
\begin{gathered}
\zeta \text { AND } \theta \\
\text { ARE EQUIVALENT IF } \\
\text { FF: } \zeta \rightarrow \theta \notin G: \theta \rightarrow \zeta \\
\Rightarrow . \\
G F \cong \text { Id } \ddagger G \cong I d \theta \\
\text { WRITE } \zeta \simeq \theta
\end{gathered}
$$

J natural ISOMORPHISMS:


Э MUTUAUY INVERSE STRUCTURE MAPS BETWEEN THEM

like Two structures are the same
II. Equivalence of categories

Ge, $\theta$ categories

$$
\begin{gathered}
\zeta \text { AND } \theta \\
\text { ARE EQuIVALENT IF } \\
\exists F: \zeta \rightarrow \theta \notin G: \theta \rightarrow \zeta \\
\Rightarrow \\
G F \cong I d_{\varphi} \& F G \cong I d \theta \\
\text { WRITE } \zeta \simeq \theta \\
\hline
\end{gathered}
$$

J natural ISOMORPHISMS:


G MUTUAUY INVERSE STRUCTURE MAPS
BETWEEN THEM
I
like Two structures are the same路

F a bidective structure map from one to the other
II. equivalence of categories

Ce, $\theta$ categories
A CHARACTERIZATION
IN TERMS OF ...
$F_{x, y}: H O M_{g}(x, y) \rightarrow H_{0} M_{B}(F(x), F(y))$ F FAITHFUL: $F_{x, y}$ iN. $\forall x, y$
FFucl: $F_{x, y}$ surd. $\forall x, y$
FFucly FAITHFUL: $F_{x, y}$ BId. $\forall x, y$ FESS. SURE:

$$
\forall y \in D, \exists x \in \zeta \rightarrow Y \cong F(x)
$$

$$
\begin{aligned}
& 6 \text { AND } \theta \\
& \text { are equivalent if } \\
& \exists F: \zeta \rightarrow \theta \notin G: \theta \rightarrow \zeta \\
& \text {. } \rightarrow \text {. } \\
& G F \cong I d_{e} \& F G \cong I d \theta \\
& \text { WRITE } \zeta \simeq \theta
\end{aligned}
$$

J natural ISOMORPHISMS:


G MUTUAUY INVERSE STRUCTURE MAPS BETWEEN THEM

(like Two structures are the same I

Fa bijective structure map FROM ONE TO THE OTHER
II. EqUivalence of categories

$$
\varphi \text { AND } \theta
$$

are equivalent if

$$
\begin{aligned}
& \exists F: \zeta \rightarrow \theta \notin G: \theta \rightarrow \zeta \\
& . \rightarrow .
\end{aligned}
$$

$G F \cong I d_{e} \& F G \cong I d \theta$


WRITE $\zeta \simeq \theta$

$$
\begin{aligned}
& F_{x, y}: H O M_{e}(x, y) \rightarrow H_{0} M_{\theta}(F(x), F(y)) \\
& g \mapsto F(g)
\end{aligned}
$$

F Fattheuc: $F_{x, y}$ iN J. $\forall x, y$
FFucl: $F_{x, y}$ surd. $\forall x, y$
Fancily FAITHFUL: $F_{x, y}$ BIG. $\forall x, y$
FESS. SUR:

$$
\forall y \in B, \exists x \in \zeta \Rightarrow y \cong F(x)
$$

II. equivalence of categories

$$
\zeta \text { AND } \theta
$$

are equivalent if $\exists F: \zeta \rightarrow \theta \notin G: \theta \rightarrow C$
.


WRITE $\zeta \simeq \theta$

$$
F_{x_{1}, y}: H O M O_{Q}(x, y) \rightarrow H_{a} M_{\theta}(F(x), F(y))
$$

$$
g \mapsto F(g)
$$

F Fattheuc: $F_{x, y}$ NJ. $\forall x, y$
FFuCl: $F_{x, y}$ surd. $\forall x, y$
FFucly FAITHFUL: $F_{x, y}$ BIG. $\forall x, y$
FESS. SURF:

$$
\forall y \in D, \exists x \in \zeta \rightarrow y \cong F(x)
$$

THEOREM
$\zeta \simeq \theta \Longleftrightarrow$ FFncly FAITHFUL, EWS. SURJECTIVE FUNCTOR $F: \varphi \rightarrow \theta$
II. equivalence of categories

$$
\begin{gathered}
\zeta \text { AND } \theta \\
\text { ARE EQUIVALENT IF } \\
\exists F: \zeta \rightarrow \theta \& G: \theta \rightarrow \zeta \\
. \rightarrow .
\end{gathered}
$$

$G F \cong I d_{c} \notin F G \cong I d \theta$


WRITE $\zeta \simeq \theta$
$\begin{aligned} F_{x, y}: & H \mu_{0} \mu_{e}(x, y) \\ g & \rightarrow F(g)\end{aligned}$
F FAITHFUL: $F_{x, y}$ iN. $\forall x, y$
F Full: $F_{x, y}$ surd. $\forall x, y$
Fancily FAITHFUL: $F_{x, y}$ BIG. $\forall x, y$
FESS. SUR:

$$
\forall y \in D, \exists x \in \zeta \Rightarrow \quad y \cong F(x)
$$

THEOREM
$\zeta \simeq \theta \Longleftrightarrow$ JFncly FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \zeta \rightarrow \theta$

CLAIM: FIN FULLY FAITHFUL AND BS. SURD.
II. equivalence of categories

$$
\begin{gathered}
\zeta \text { AND } \theta \\
\text { ARE EQUIVALENT IF } \\
\exists F: \zeta \rightarrow \theta \& G: \theta \rightarrow \zeta \\
. \rightarrow .
\end{gathered}
$$

$G F \cong I d_{e} \ddagger F G \cong I d \theta$


WRITE $\zeta \simeq \theta$
$\begin{aligned} F_{x, y}: & H O_{q}(x, y) \\ g & \rightarrow H_{0} \mu_{\theta}(F(x), F(y)) \\ & \rightarrow F(g)\end{aligned}$
F FAITHFUL: $F_{x, y}$ iNJ. $\forall x, y$
F Fucl: $F_{x, y}$ surj. $\forall x, y$
FFucly FAITHFLL: $F_{X, Y}$ BIJ. $\forall x, y$
F ESS. SURJ:

$$
\forall y \in D, \exists x \in \zeta \geqslant y \cong F(x)
$$

THEOREM
$\zeta \simeq \theta \Longleftrightarrow$ FFncly FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \varphi \rightarrow \theta$

CLAIM: FIS FUCY FAITHFUL AND ESS. SURJ.
TAKE $y \in \theta$. THEN $F G(Y) \stackrel{\Psi_{y}}{\sim} y \Rightarrow F$ ESS.SURJ.
II. EqUivalence of categories

$G F \cong I d_{e} \notin F G \cong I d \theta$


WRITE $\zeta \simeq \theta$

$$
\begin{aligned}
F_{x, y}: H O M_{q}(x, y) & \rightarrow H_{0} M_{\theta}(F(x), F(y)) \\
g & \mapsto F(g)
\end{aligned}
$$

FAITHFUL: $F_{x, y}$ iN. $\forall x, y$
FFucl: $F_{x, y}$ surd. $\forall x, y$
Fancily FAITHFUL: $F_{x, y}$ BIG. $\forall x, y$
FESS. SURE:

$$
\forall y \in B, \exists x \in \zeta \ni y \cong F(x)
$$

THEOREM
$\zeta \simeq \theta \Longleftrightarrow$ FFncly FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \varphi \rightarrow \theta$
$P F /(\Longrightarrow)$ SAy $\exists \phi: I_{d y} \cong G F \neq \psi: F G \cong I d \theta$ As IN
CLAIM: FIG FUMY FAITHFUL AND BS. SURD.
TAKE $y \in \theta$. THEN $F G(y) \stackrel{\Psi_{y}}{\sim} y \Rightarrow F$ ESS.SURJ.
NOW $\forall g: X \rightarrow X^{\prime} \in \mathscr{C}$, GET $X \xrightarrow{g} X^{\prime}$
THIS IMPLIES Fig FAITHFUC $\quad \phi_{x} \cong 2 \cong \uparrow \phi_{x^{\prime}}^{-1}$

$$
G F(x) \underset{G F(g)}{\longrightarrow} G F\left(x^{\prime}\right)
$$

II. EqUivalence of categories

$G F \cong I d_{e} \ddagger F G \cong I d \theta$


WRITE $\zeta \simeq \theta$

$$
\begin{aligned}
& F_{x, y}: H O M_{e}(x, y) \rightarrow H_{0} \mu_{\theta}(F(x), F(y)) \\
& g \mapsto F(g)
\end{aligned}
$$

F Fattheuc: $F_{x, y}$ iNJ. $\forall x, y$
F Fucl: $F_{x, y}$ surj. $\forall x, y$
FFncly FAITHFUL: $F_{x, y}$ BIJ. $\forall x, y$
F ESS. SURJ:

$$
\forall y \in D, \exists x \in \zeta \rightarrow y \cong F(x)
$$

THEOREM
$\zeta \simeq \theta \Longleftrightarrow$ FFucly FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \varphi \rightarrow \theta$
$P F /(\Longrightarrow)$ SAy $\exists \phi: I_{d y} \cong G F \neq \psi: F G \cong I d \theta$ As IN
CLAIM: FIS FUCY FAITHFUL AND ESS. SURJ.
TAKE $y \in \theta$. THEN $F G(y) \stackrel{\psi y}{\sim} y \Rightarrow F$ ESS.SURJ.
Now $\forall g: X \rightarrow X^{\prime} \in \mathscr{C}, G E T \quad X \xrightarrow{g} X^{\prime}$
THIS IMPLIES FIS FAITHFUC $\quad \phi_{x} \triangleq 2 \cong \uparrow \phi_{x^{\prime}}^{-1}$
II. EqUivalence of categories

$G F \cong I d_{e} \ddagger F G \cong I d \theta$


WRITE $\zeta \simeq \theta$

$$
\begin{aligned}
& F_{x, y}: H O M_{e}(x, y) \rightarrow H_{0} \mu_{\theta}(F(x), F(y)) \\
& g \mapsto F(g)
\end{aligned}
$$

F Fattheuc: $F_{x, y}$ iNJ. $\forall x, y$
FFucl: $F_{x, y}$ surj. $\forall x, y$
FFncly FAITHFUL: $F_{x, y}$ BIJ. $\forall x, y$
F ESS. SURJ:

$$
\forall y \in B, \exists x \in \zeta \Rightarrow y \cong F(x)
$$

THEOREM
$\zeta \simeq \theta \Longleftrightarrow$ FFncly FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \varphi \rightarrow \theta$

CLAIM: FIS FUCLY FAITHFUL AND ESS. SURD.
TAKE $y \in \theta$. THEN $F G(y) \stackrel{\psi y}{\sim} y \Rightarrow F$ ESS.SURJ.
NOW $\forall g: X \rightarrow X^{\prime} \in \mathcal{C}, G E T$

$$
X \xrightarrow{\text { THEN }}=\tilde{g}{ }^{\prime} X^{\prime}
$$

$$
\begin{aligned}
& \text { This imples Fis faithenc }
\end{aligned}
$$

II. EqUivalence of categories

$G F \cong I d_{e} \ddagger F G \cong I d \theta$


WRITE $\zeta \simeq \theta$

$$
\begin{aligned}
F_{x, y}: & H_{0} M_{q}(x, y) \\
& \rightarrow H H_{0} M_{\theta}(F(x), F(y)) \\
& \mapsto(g)
\end{aligned}
$$

F FAITHFUL: $F_{x, y}$ iN. $\forall x, y$
F Full: $F_{x, y}$ surd. $\forall x, y$
Fancily FAITHFUL: $F_{x, y}$ BIG. $\forall x, y$
FESS. SUR:

$$
\forall y \in B, \exists x \in \zeta \ni y \cong F(x)
$$

THEOREM
$\zeta \simeq \theta \Leftrightarrow$ FFucly Faithful, ESS. SURJECTIVE FUNCTOR $F: \varphi \rightarrow \theta$
$P F /(\Longrightarrow)$ SAy $\exists \phi: I_{d y} \cong G F \neq \psi: F G \cong I d \theta$ As IN
CLAIM: FIG FUMY FAITHFUL AND BS. SURD.
TAKE $y \in \theta$. THEN $F G(y) \stackrel{\psi y}{\sim} y \Rightarrow F$ ESS.SURJ.
NOW $\forall g: X \rightarrow X^{\prime} \in \mathscr{C}, G E T$


$$
\left[\begin{array}{rl}
F(g)=F(\tilde{g}) & \Rightarrow G F(g)=G F(\tilde{g}) \\
& \Rightarrow g=\tilde{g}
\end{array}\right]
$$

SWAPPING $\&$ WITH $\psi \Rightarrow G$ IS FAITHFUL.

$$
\underset{G F(x)}{\longrightarrow} \underset{G F(g)}{G F\left(x^{\prime}\right)}
$$

II. EqUIVALENCE OF CATEGORIES


$$
\begin{aligned}
& F_{x, y}: H O M_{e}(x, y) \rightarrow H_{0} \mu_{\theta}(F(x), F(y)) \\
& g \mapsto F(g)
\end{aligned}
$$

F FAITHFUL: $F_{x, y}$ iN. $\forall x, y$
F Full: $F_{x, y}$ surd. $\forall x, y$
FFucly FAITHFLL: $F_{X, Y}$ BID. $\forall x, y$
FESS. SUR:

$$
\forall y \in D, \exists x \in \zeta \rightarrow \quad y \cong F(x)
$$

THEOREM
$\zeta \simeq \theta \Leftrightarrow$ Fully FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \varphi \rightarrow \theta$

CLAIM: FIN FUMY FAITHFUL AND BS. SURD.
TAKE $y \in \theta$. THEN $F G(y) \stackrel{\psi y}{\sim} y \Rightarrow F$ ESS.SURJ.
NOW $\forall g: X \rightarrow X^{\prime} \in \mathcal{C}$, GET


THIS IMPLIES FIS FAITHFUL $\phi_{x} \cong 2 \cong \uparrow \phi_{x^{\prime}}^{-1}$

$$
\left[\begin{array}{rl}
F(g)=F(\tilde{g}) & \Rightarrow G F(g)=G F(\tilde{g})] \quad G F(x) \underset{G F(g)}{\longrightarrow} \underset{G F}{ } \underset{G F\left(X^{\prime}\right)}{ } \\
& \Rightarrow g=\tilde{g})
\end{array}\right.
$$

SWAPPING $\phi$ WITH $\psi \Rightarrow G$ IS FAITHFUL.
TAKE $h: F(x) \rightarrow F\left(x^{\prime}\right) \in \theta$.
BuILD $g: X \xrightarrow{\phi_{x}} G F(x) \xrightarrow{G(h)} G F\left(X^{\prime}\right) \xrightarrow{\phi_{x^{\prime}}^{-1}} X^{\prime} \in \zeta$.
II. equivalence of categories


$$
\begin{aligned}
& F_{x, y}: H O M_{e}(x, y) \rightarrow H_{0} \mu_{\theta}(F(x), F(y)) \\
& g \mapsto F(g)
\end{aligned}
$$

FFAITHFUL: $F_{x, y}$ iNJ. $\forall x, y$
F Fucl: $F_{x, y}$ surj. $\forall x, y$
FFucly FAITHFUC: $F_{x, y}$ BIJ, $\forall x, y$
F ESS. SURJ:

$$
\forall y \in D, \exists x \in \zeta \geqslant y \equiv F(x)
$$

THEOREM
$\zeta \simeq \theta \Longleftrightarrow$ FFncly FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \varphi \rightarrow \theta$

CLAIM: FIS FUCY FAITHFUL AND ESS. SURJ.
TAKE $y \in \theta$. THEN $F G(y) \stackrel{\psi_{y}}{\sim} y \Rightarrow F$ ESS.SURJ.
Now $\forall g: X \rightarrow X^{\prime} \in \mathscr{C}$, GET

$$
X \xrightarrow{\text { THEN } g=\tilde{g}} X^{\prime}
$$

THis imples Fis FAITHFUC $\quad \phi_{x} \rrbracket^{\cong} 2 \cong \uparrow \phi_{x^{\prime}}^{-1}$

$$
\left[\begin{array}{rl}
F(g)=F(\tilde{g}) & \Rightarrow G F(g)=G F(\tilde{g})] \quad G F(x) \underset{G F(g)}{\longrightarrow}{ }_{V F F}{ }_{G F\left(X^{\prime}\right)} \\
& \Rightarrow g=\tilde{g})
\end{array}\right.
$$

SWAPPING $\phi$ WITH $\psi \Rightarrow G$ IS FAITHFUL.
TAKE $h: F(x) \rightarrow F\left(x^{\prime}\right) \in \theta$.
BUILD $g: X \xrightarrow{\phi_{x}} G F(x) \xrightarrow{G(h)} G F\left(X^{\prime}\right) \xrightarrow{\phi_{x^{\prime}}^{-1}} X^{\prime} \in \zeta$.
THEN $G F(g)=G(h) \cdot G F A I T H F U L \Rightarrow F(g)=h$
\& Fis Fucl.
II. equivalence of categories


$$
\begin{aligned}
& F_{x, y}: H O M_{e}(x, y) \rightarrow H_{0} \mu_{\theta}(F(x), F(y)) \\
& g \mapsto F(g)
\end{aligned}
$$

FAITHFUL: $F_{x, y}$ iN J. $\forall x, y$
FFucl: $F_{x, y}$ surd. $\forall x, y$
FFucly FAITHFLL: $F_{X, Y}$ BID. $\forall x, y$ FESS. SUR:

$$
\forall y \in D, \exists x \in \zeta \ni \quad y \cong F(x)
$$

THEOREM
$\zeta \simeq \theta \Longleftrightarrow$ Foully FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \varphi \rightarrow \theta$

CLAIM: FIN FUMY FAITHFUL AND BS. SURD.
TAKE $y \in \theta$. THEN $F G(y) \stackrel{\psi y}{\sim} y \Rightarrow F$ ESS.SURJ.
NOW $\forall g: X \rightarrow X^{\prime} \in \mathcal{C}$, GET


SWAPPING $\&$ WITH $\psi \Rightarrow G$ IS FAITHFUL.
TAKE $h: F(x) \rightarrow F\left(x^{\prime}\right) \in \theta$.
BUILD $g: X \xrightarrow{\phi_{x}} G F(x) \xrightarrow{G(h)} G F\left(X^{\prime}\right) \xrightarrow{\phi_{x^{\prime}}^{-1}} X^{\prime} \in \zeta$.
THEN $G F(g)=G(h) \cdot G$ FAITHFUL $\Rightarrow F(g)=h$
\& FISFUCL.
II. equivalence of categories
$\zeta$ AND $\theta$
are equivalent if $\exists F: \zeta \rightarrow \theta \notin G: \theta \rightarrow C$
. $\cdot$.

$$
G F \cong I d_{C} \& F G \cong I d \theta
$$



WRITE $\zeta \simeq \theta$
$\begin{aligned} & F_{x, y}: H O M_{e}(x, y) \\ & g \mapsto F(g) \rightarrow \mu_{\theta}(F(x), F(y)) \\ &\end{aligned}$

$$
g \mapsto F(g)^{\circ}
$$

F FATthenc: $F_{x, y}$ inJ. $\forall x, y$
F Fucl: $F_{x, y}$ sures. $\forall x, y$
FFucly FATTHFuc: $F_{x, y}$ BII. $\forall x, y$
F ESS. SuRJ:

$$
\forall y \in D, \exists x \in G \rightarrow y \cong F(x)
$$

THEOREM
$\zeta \simeq \theta \Leftrightarrow$ FFully FAITHFUL, ESS. SuRJEcTive FUNCTOR $F: \varphi \rightarrow \theta$
$\operatorname{PF} /(\Longleftrightarrow$ TAKE $F: \zeta \rightarrow \theta$ Fully FAITHFUL, ESS. SURJ.
II. equivalence of categories

$$
\begin{aligned}
& \zeta \text { AND } \theta \\
& \text { ARE EQUIVALENT IF } \\
& \exists F: \zeta \rightarrow \theta \& G: \theta \rightarrow \zeta \\
& . \ni .
\end{aligned}
$$

$G F \cong I d_{e} \notin F G \cong I d \theta$


WRITE $\zeta \simeq \theta$
$\begin{aligned} F_{x, y}: H O M_{e}(X, y) & \rightarrow H_{0} \mu_{\theta}(F(x), F(y)) \\ g & \mapsto F(g)\end{aligned}$
FAITHFUL: $F_{x, y}$ iN. $\forall x, y$
FFucl: $F_{x, y}$ surd. $\forall x, y$
FFucly FAITHFLL: $F_{X, Y}$ BID. $\forall x, y$
FESS. SUR:

$$
\forall y \in D, \exists x \in \zeta \rightarrow y \cong F(x)
$$

THEOREM
$\zeta \simeq \theta \Longleftrightarrow$ FFncly Faithful, ESS. SURJEctive FUNCTOR $F: \varphi \rightarrow \theta$

PF $/ \Leftarrow$ TAKE $F: \varphi \rightarrow \theta$ FULLY FAITHFUL, ASS. SURD.

LABELLING: GOY)
II. equivalence of categories

$$
\begin{gathered}
\zeta \text { AND } \theta \\
\text { ARE EQUIVALENT IF } \\
\text { FF: } \zeta \rightarrow \theta \& G: \theta \rightarrow \zeta
\end{gathered}
$$

. $\rightarrow$.
$G F \cong I d_{e} \ddagger F G \cong I d \theta$


WRITE $\zeta \simeq \theta$
$\begin{aligned} F_{x, y}: & H O_{q}(x, y) \\ g & \rightarrow F(g)\end{aligned}$
F FAITHFUL: $F_{x, y}$ iNJ. $\forall x, y$
FFucl: $F_{x, y}$ surj. $\forall x, y$
FFncly FAITHFUL: $F_{x, y}$ BIJ. $\forall x, y$
F ESS. SURJ:

$$
\forall y \in D, \exists x \in \zeta \Rightarrow y \cong F(x)
$$

THEOREM
$\zeta \simeq \theta \Leftrightarrow$ FFucly Faithful, ESS. SURJECTIVE FUNCTOR $F: \zeta \rightarrow \theta$

PF $/ \Leftarrow$ TAKE $F: \boldsymbol{\varphi}_{\boldsymbol{l}} \rightarrow \theta$ FULLY FAITHFUL, ESS. SURJ.

$$
\begin{aligned}
& F \text { ESS.SURJ } \Rightarrow \forall y \in D \quad \exists z y \in \mathscr{C} \rightarrow . F(z y) \cong y \\
&!! \\
& G(y)=: \Psi_{y}
\end{aligned}
$$

F Fucly falthful $\Rightarrow \forall g: y \rightarrow y^{\prime} \in \theta$ $\exists!$ MORPHISM $G(y) \rightarrow G\left(y^{\prime}\right) \in \zeta$
$G!!)$
II. EqUIVALENCE OF CATEGORIES

II. EqUIVALENCE OF CATEGORIES

II. Equivalence of categories

II. Equivalence of categories

II. Equivalence of categories

$$
\text { DETAILS } \equiv \text { EXR } 2.34
$$


II. EqUIVALENCE OF CATEGORIES

II. equivalence of categories

$G F \cong I d_{e} \& F G \cong I d \theta$


WRITE $\zeta \simeq \theta$

$$
\begin{aligned}
F_{x, y}: H M_{q}(x, y) & \rightarrow H_{0} M_{\theta}(F(x), F(y)) \\
g & \mapsto F(g)
\end{aligned}
$$

F FAITHFUL: $F_{x, y}$ iN. $\forall x, y$
FFucl: $F_{x, y}$ surd. $\forall x, y$
Fancily FAITHFUL: $F_{x, y}$ BIG. $\forall x, y$
FESS. SUR:

$$
\forall y \in D, \exists x \in \zeta \ni \quad y \cong F(x)
$$

THEOREM
$\zeta \simeq \theta \Longleftrightarrow$ FFncly FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \varphi \rightarrow \theta$

TOWARD EXAMPLES -


Ex. $\left.n=2 \quad O\left(\frac{y \mathbb{C}^{2}}{\downarrow} x\right)=\frac{\mathbb{C}[x, y]}{(y)} \cong \mathbb{C} T_{x}\right]$
$(C)(\stackrel{y}{\downarrow} x)=\mathbb{C}[x, y]$
II. equivalence of categories

$G F \cong I d_{e} \& F G \cong I d \theta$


WRITE $\zeta \simeq \theta$

$$
\begin{aligned}
F_{x, y}: & H_{0} M_{q}(x, y) \\
& \rightarrow H H_{0} M_{\theta}(F(x), F(y)) \\
& \mapsto F(g)
\end{aligned}
$$

FFATTHFUL: $F_{x, y}$ iNJ. $\forall x, y$
FFucl: $F_{x, y}$ surj. $\forall x, y$
FFncly FAITHFUL: $F_{x, y}$ BIJ. $\forall x, y$
F ESS. SURJ:

$$
\forall y \in D, \exists x \in \zeta \Rightarrow y \cong F(x)
$$

THEOREM
$\zeta \simeq \theta \Longleftrightarrow$ FFncly FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \varphi \rightarrow \theta$

TOWARD EXAMPLES -


Ex. $\left.n=2 \underset{\text { inccusion }}{ } O\left(\frac{y}{\downarrow} \mathbb{C}^{2} x\right)=\frac{\mathbb{C}[x, y]}{(y)} \cong \mathbb{C} T_{x}\right]$
$\theta\left(\frac{y}{\downarrow} x\right)=\mathbb{C}[x, y]$
II. EqUIVALENCE OF CATEGORIES

$\begin{aligned} & F_{x, y}: H O M_{e}(x, y) \rightarrow \not H_{0} M_{\theta}(F(x), F(y)) \\ & g \mapsto F(g)\end{aligned}$
FFATTHFUL: $F_{x, y}$ inJ. $\forall x, y$
FFucl: $F_{x, y}$ surj. $\forall x, y$
FFncly FAITHFUL: $F_{x, y}$ BIJ. $\forall x, y$
F ESS. SURJ:

$$
\forall y \in D, \exists x \in \zeta \rightarrow y \cong F(x)
$$

THEOREM
$\zeta \simeq \theta \Leftrightarrow$ FFuCLy FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \varphi \rightarrow \theta$

TOWARD EXAMPLES -


Ex. $n=2 \underset{\text { NCcusion }}{2}\left(\frac{y}{\downarrow} \mathbb{C}^{2}\right)=\frac{\mathbb{C}[x, y]}{\Rightarrow(y)} \cong \mathbb{C}[x]$
$C\left(\frac{y}{\downarrow} x\right)=\mathbb{C}[x, y]$
II. EqUIVALENCE OF CATEGORIES

$\begin{aligned} & F_{x, y}: H O M_{e}(x, y) \rightarrow \not H_{0} M_{\theta}(F(x), F(y)) \\ & g \mapsto F(g)\end{aligned}$
FFATTHFUL: $F_{x, y}$ iNJ. $\forall x, y$
FFucl: $F_{x, y}$ surj. $\forall x, y$
FFncly FAITHFUL: $F_{x, y}$ BIJ. $\forall x, y$
F ESS. SURJ:

$$
\forall y \in D, \exists x \in \zeta \rightarrow y \cong F(x)
$$

$$
\frac{\text { THEOREM }}{\zeta \simeq \theta} \Longleftrightarrow \text { FFUCLY FAITHFUC, ESS. SURJECTIVE }
$$



$$
\text { Ex. } n=\underset{\underset{\text { NCcusion }}{2}}{\int}\left(\frac{y \mathbb{C}^{2}}{\downarrow} x\right)=\frac{\mathbb{C}[x, y]}{\Rightarrow(y)} \cong \mathbb{C}[x]
$$

$$
O(\underset{\downarrow}{y} x)=\mathbb{C}[x, y]
$$

II. equivalence of categories

$\begin{aligned} F_{x_{1} y}: H O M_{e}(X, y) & \rightarrow H_{0} \mu_{\theta}(F(x), F(y)) \\ g & \mapsto F(g)\end{aligned}$
F FAITHFUL: $F_{x, y}$ iNJ. $\forall x, y$
FFucl: $F_{x, y}$ surj. $\forall x, y$
FFncly FAITHFUL: $F_{x, y}$ BIJ. $\forall x, y$
F ESS. SURJ:

$$
\forall y \in D, \exists x \in \zeta \ni \quad y \cong F(x)
$$

$$
\begin{gathered}
\text { THEOREM } \\
\zeta \simeq \theta \Leftrightarrow \begin{array}{c}
\text { JFULLY FAITHFUL, ESS. SURJECTIVE } \\
\text { FUNCTOR } F: \zeta \rightarrow \theta
\end{array}
\end{gathered}
$$



$$
A f f e_{c} \simeq(\ldots \operatorname{ConAlga})^{p}
$$

II. equivalence of categories

$\begin{aligned} F_{x, y}: H 0 m_{c}(x, y) & \rightarrow H_{0} \mu_{B}(F(x), F(y)) \\ g & \mapsto F(g)\end{aligned}$
$g \mapsto F(g)$
F FAITHFUL: $F_{x, y}$ ( $N J, \forall x, y$
FFucl: $F_{x, y}$ surd. $\forall x, y$
FFucly FAITH FUC: $F_{x, y} B I J, \forall x, y$
FESS. SUR:

$$
\forall y \in D, \exists x \in \zeta \rightarrow y \cong F(x)
$$

$$
\begin{aligned}
& \text { THEOREM } \\
& \zeta \simeq \theta \Longleftrightarrow \text { FFncly Faithful, ESS. SURJECTIVE } \\
& \text { FUNCTOR } F: \varphi \rightarrow \theta
\end{aligned}
$$



$$
A f f_{C} \simeq\left(\ldots \operatorname{ConA} \mathrm{Al}_{\mathrm{C}}\right)^{\circ}
$$

THESE TYPES OF algebras are FINITELY GENERATED
II. equivalence of categories


WRITE $\zeta \simeq \theta$

$$
\begin{aligned}
F_{x, y}: & H_{0} M_{q}(x, y) \\
& \rightarrow H H_{0} M_{\theta}(F(x), F(y)) \\
& \mapsto(g)
\end{aligned}
$$

F FAITHFUL: $F_{x, y}$ iN. $\forall x, y$
F Full: $F_{x, y}$ surd. $\forall x, y$
FFucly FAITHFLL: $F_{X, Y}$ BID. $\forall x, y$
FESS. SURE:

$$
\forall y \in D, \exists x \in \zeta \rightarrow Y \cong F(x)
$$

$$
\begin{aligned}
& \text { THEOREM } \\
& \zeta \simeq \theta \Leftrightarrow \text { FULLY FAITHFUL, ELS. SURJECTIVE } \\
& \text { FUNCTOR F: } \rightarrow \theta
\end{aligned}
$$



$$
A f f_{\mathbb{C}} \simeq\left(\mathrm{Fg}_{g} \ldots \text { Con Alga }_{\mathrm{c}}\right)^{\circ}
$$

THESE TYPES OF algebras are Finitely generated
II. EqUIVALENCE OF CATEGORIES

II. equivalence of categories

II. equivalence of categories

II. equivalence of categories


WRITE $\zeta \simeq \theta$

$$
\begin{aligned}
& F_{x, y}: H O M_{e}(x, y) \\
& g \rightarrow H_{0} \mu_{\theta}(F(x), F(y)) \\
&
\end{aligned}
$$

F Fattheuc: $F_{x, y}$ iN. $\forall x, y$
FFucl: $F_{x, y}$ surd. $\forall x, y$
Fancily FAITHFUL: $F_{x, y}$ BIG. $\forall x, y$
FESS. SUR:

$$
\forall y \in D, \exists x \in \zeta \rightarrow y \equiv F(x)
$$

$$
\begin{gathered}
\text { THEOREM } \\
\zeta \simeq \theta
\end{gathered} \begin{gathered}
\text { FFUCLY FAITHFUL, EDS. SURJECTIVE } \\
\text { FUNCTOR } F: \zeta \rightarrow \theta
\end{gathered}
$$



$$
A f f_{\mathbb{C}} \simeq\left(\mathrm{FgRed}_{\mathrm{g}} \mathrm{Com} \mathrm{Ilg}_{\mathrm{C}}\right)^{\circ}
$$

Also have

$$
\left.\begin{array}{l}
\text { SOMETHING } \\
\text { ESSE } \\
\text { GEOMETRIC } \\
\mathbb{C}
\end{array}(\operatorname{CoMAlg})\right)^{\circ P}
$$

II. EqUIVALENCE OF CATEGORIES


$$
\begin{aligned}
F_{x, y}: H H_{q}(x, y) & \rightarrow H H_{\theta}(F(x), F(y)) \\
g & \mapsto F(g)
\end{aligned}
$$

FAITHFUL: $F_{x, y}$ iN. $\forall x, y$
FFucl: $F_{x, y}$ surd. $\forall x, y$
Fancily FAITHFUL: $F_{x, y}$ BIG. $\forall x, y$
FESS. SUR:

$$
\forall y \in D, \exists x \in G \rightarrow y \approx F(x)
$$

$$
\begin{aligned}
& \text { THEOREM } \\
& \zeta \simeq \theta
\end{aligned} \begin{gathered}
\text { FULLY FAITHFUL, ELS. SURJECTIVE } \\
\text { FUNCTOR } F: \zeta \rightarrow \theta
\end{gathered}
$$


$A f f_{\mathbb{C}} \simeq\left(\mathrm{Fg}_{\mathrm{g}} \operatorname{Red} \mathrm{ComAlg}_{4}\right)^{\circ}$
Also have
BEYOND
SCOPE OF
lecture

$$
\text { Scheme }_{\mathbb{C}} \simeq\left(\operatorname{ComAlg}_{\mathbb{C}}\right)^{\circ p}
$$

III. MOrita equivalence


Notion of sameness FOR IR-ALGEBRAS
III. MOrita equivalence


Notion of sameness FOR IR-ALGEBRAS
$A=B$
equality OF ALGEBRAS
III. MORITA equivalence


NOTION OF sAMENESS FOR $\mathbb{k}$-ALGEBRAS

$$
A=B
$$

Equality
of algebras

$$
\left(\begin{array}{c}
A \cong B \\
\text { WEAKEN }
\end{array} \longrightarrow \begin{array}{c}
\text { ISOMORPHISM } \\
\text { OF ALGEBRAS }
\end{array}\right.
$$

III. MORITA equivalence

take lk-algs $A \notin B$
A is morita equiv. to B
IF $A-M_{\text {od }} \simeq B-M_{o d}$.
That is, $A \notin$ b have the same rep thy

NOTION OF SAMENESS FOR IR-ALGEBRAS

$$
A=B
$$

EQUALITY
OF ALGEBRAS

$$
\left(\begin{array}{l}
\text { WEAKEN }
\end{array} \underset{\substack{\text { ISOMORPHISM } \\
\text { OF ALGEBRAS }}}{A \cong B}\right.
$$

 "equivalence" OF ALGEBRAS
III. MOrita equivalence


TAKE $\mathbb{R}$-AlaS $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B-M o d$.
That is, A \& b have
the same rep thy

MORITA'S THEOREM A IS MORITA EQUIVALENT TOB $\Leftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A .}$.
$P \otimes_{B} Q \cong$ Ares As A-Bimoduces \& $Q \otimes_{A} P \cong$ Dreg $A S$-Bimodules.
III. MOrita equivalence


TAKE $\mathbb{R}$-AlES $A \notin B$
A IS MORITA EQuIV. To B
IF $A-M o d \simeq B-M o d$.
That is, A \& b have
the same rep thy

MORITA'S THEOREM A IS MORITA EQUIVALENT TOB $\Leftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A .}$.

$$
\begin{aligned}
& P \otimes_{B} Q \cong \text { Areg AS } A \text {-Bimodules } \\
& \& Q \otimes_{A} P \cong \text { reg AS B-BIMODuLES. }
\end{aligned}
$$

$$
\begin{aligned}
P F / \Longleftrightarrow \text { TAKE } & F:=Q \otimes_{A}-: A-M o d \rightarrow B-M o d \\
& G:=P \otimes_{B}-: B-M o d \rightarrow A-M o d
\end{aligned}
$$

III. MOrita equivalence


TAKE $\mathbb{R}$-AlaS $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B-M o d$.
That is, A \& b have the same rep thy

MORITA'S THEOREM A IS MORITA EQUIVALENT TOB $\Leftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A .}$.

$$
\begin{aligned}
& P \otimes_{B} Q \cong \text { Areg AS } A \text {-BImODULES } \\
& \& Q \otimes_{A} P \cong \text { reg AS B-BIMODULES. }
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{PF} / \Leftarrow \\
& \text { TAKE } F:=Q \otimes_{A}-: A-\operatorname{Mod} \rightarrow B-\operatorname{Mod} \\
& G:=P \theta_{B}-: B-M o d \rightarrow A-M o d
\end{aligned}
$$

Now $\forall M \in A-\bmod , G \in T$ :

$$
\begin{aligned}
& G F(M)=G\left(Q \otimes_{A} M\right)=P \otimes_{B}\left(Q \otimes_{A} M\right) \\
& \cong\left(P \otimes_{B} Q\right) \otimes_{A} M \\
& \text { MoDIFy EXER.1.186 }
\end{aligned}
$$

III. MOrita equivalence


TAKE $\mathbb{R}$-AlaS $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B-M o d$.
That is, A \& b have the same rep thy

MORITA'S THEOREM A IS MORITA EQUIVALENT TOB $\Leftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A .}$.

$$
\begin{aligned}
& P \otimes_{B} Q \cong \text { Areg As } A \text {-Bimodules } \\
& \& Q \otimes_{A} P \cong \text { Bred AS } B \text {-Bimoduces. } .
\end{aligned}
$$

$$
\begin{aligned}
P F / \Leftarrow & \\
& G:=Q \otimes_{A}-: A-M o d \rightarrow B-M o d \\
& G: \otimes_{B}-: B-\operatorname{Mod} \rightarrow A-M o d
\end{aligned}
$$

Now $\forall M \in A-m o d, G \in T$ :

$$
\begin{aligned}
G F(M)= & G\left(Q \otimes_{A} M\right)=P \otimes_{B}\left(Q \otimes_{A} M\right) \\
\cong\left(P \otimes_{B} Q\right) \otimes_{A} M & \xlongequal{\imath} \text { Arg } \otimes_{A} M \cong M . \\
& \text { HYOTHESIS } \quad \text { EXER.1.18a }
\end{aligned}
$$

III. MORItA equivalence


TAKE $\mathbb{R}$-AlES $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B$-Mod.
That is, $A \not \& B$ have
the same rep thy

MORITA'S THEOREM A IS MORITA EQUIVALENT TOB $\Leftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A .}$.

$$
\begin{aligned}
& P \otimes_{B} Q \cong \text { Areg AS } A \text {-Bimodules } \\
& \& Q \otimes_{A} P \cong \text { Bred AS } B \text {-BIMODULES. }
\end{aligned}
$$

$$
\begin{aligned}
P F / \Longleftrightarrow \text { TAKE } & F:=Q \otimes_{A}-: A-M o d \rightarrow B-M o d \\
& G:=P \otimes_{B}-: B-M o d \rightarrow A-M o d
\end{aligned}
$$

Now $\forall M \in A-m o d, G \in T$ :

$$
\begin{aligned}
G F(M) & =G\left(Q \otimes_{A} M\right)=P \otimes_{B}\left(Q \otimes_{A} M\right) \\
& \cong\left(P \otimes_{B} Q\right) \otimes_{A} M \cong \operatorname{Areg}_{A} M \cong M .
\end{aligned}
$$

$$
\therefore G F \cong I d_{A-\mu_{0} d}
$$

LIKEWISE, $F G \cong I d_{B-M o d}$.
III. MOrita equivalence


MORITA'S THEOREM $\Longleftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A . \rightarrow}$.

$$
\begin{aligned}
P \otimes_{B} Q \cong & \text { Areg AS } A \text {-Bimodules } \\
\& & Q \otimes_{A} P \cong \text { Bred AS } B \text {-BIMODULES. }
\end{aligned}
$$

PF $/\left(\Longrightarrow\right.$ GIVEN Equivalence $F: A-M_{0 d} \rightarrow B-M_{00}$

$$
\text { TAKE } Q:=F(A \text { Ares }) \in B-M o d
$$

TAKE $\mathbb{R}$-AlaS $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B-M o d$.
That is, A \& b have
the same rep thy
III. MOrita equivalence


TAKE $\mathbb{R}$-AlaS $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B-M o d$.
That is, A \& b have
the same rep thy

MORITA'S THEOREM A IS MORITA EQUIVALENT TOB $\Longleftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A . \rightarrow}$.

$$
P \otimes_{B} Q \cong \text { Areg AS A-Bimoduces }
$$

$$
\notin Q \otimes_{A} P \cong \text { Bred AS B-BIMODULES. }
$$

$\operatorname{PF} /\left(\Longrightarrow\right.$ GIVEN EQUIVALENCE $F: A-M_{0 d} \rightarrow B-M_{0} d$

$$
\text { TAKE } Q:=F\left(A_{A} \text { Ares }\right) \in B-M o d
$$

$$
\begin{aligned}
G E T \quad A^{\circ P} & \cong \operatorname{End}_{A-\bmod }\left(A_{\text {reg }}\right) \\
& \cong \operatorname{End}_{B-\bmod }\left(F\left(_{A} \text { Area }\right)\right) \\
& \cong \operatorname{End}_{B-\bmod ^{2}}\left({ }_{B} Q\right)
\end{aligned}
$$

III. MOrita equivalence


TAKE $\mathbb{R}$-AlES $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B-M o d$.
That is, A \& b have the same rep thy

MORITA'S THEOREM A IS MORITA EQUIVALENT TOB $\Longleftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A . \rightarrow}$.

$$
\begin{aligned}
& P \otimes_{B} Q \cong \text { Areg AS } A \text {-Bimodules } \\
& \& Q \otimes_{A} P \cong \text { reg AS B-BIMODuLES. }
\end{aligned}
$$

PF $/\left(\Longrightarrow\right.$ GIVEN EQUIVALENCE $F: A-M_{0 d} \rightarrow B-M_{0 d}$
TAKE $Q:=F\left({ }_{A}\right.$ Ares $) \in B-\operatorname{Mod}$
GET AP $A^{\circ p} \operatorname{End}_{A-\bmod }\left(A_{\text {reg }}\right)$
EXER. $1.26^{\approx} \operatorname{End}_{B-\bmod }(F($ A Arg $))$
F Fully

$$
\text { FAITHFUL } \cong \operatorname{End}_{B-\operatorname{Mod}^{2}}(B Q)
$$

III. MOrita equivalence


TAKE $\mathbb{R}$-AlES $A \notin B$
A is morita equiv. To b
IF $A-M o d \simeq B-M o d$.
That is, A \& b have
the same rep thy

MORITA'S THEOREM A IS MORITA EQUIVALENT TOB $\Longleftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A . \rightarrow}$.

$$
\begin{aligned}
& P \otimes_{B} Q \cong \text { Areg AS A-BIMODULES } \\
& \& Q \otimes_{A} P \cong \text { Bred AS B-BIMODULES. }
\end{aligned}
$$

PF $/\left(\Longrightarrow\right.$ GIVEN EQUIVALENCE $F: A-M_{0 d} \rightarrow B-M_{0 d}$

$$
\begin{aligned}
& \text { TAKE } Q:=F(A \text { reg }) \in B-\operatorname{Mod} \\
& \text { GET } A^{\circ p} \cong \operatorname{End}_{A-\bmod }\left(A_{\text {reg }}\right) \\
& \begin{aligned}
f( & \left.\cong \operatorname{End}_{B-\bmod ^{\prime}}\left(F_{A_{A}} \text { Ares }\right)\right) \\
& \cong \operatorname{End}_{B-M_{0 d}}\left(B_{B}\right)
\end{aligned}
\end{aligned}
$$

DEFINE $\quad q \triangleleft a:=f(a)(q)$
$\leadsto \quad B Q \in(B, A)$ - Bipod
III. MOrita equivalence


MORITA'S THEOREM $\Longleftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A . \rightarrow}$.

$$
\begin{aligned}
& P \otimes_{B} Q \cong \text { Areg As A-BIMODULES } \\
& \& Q \otimes_{A} P \cong \text { Bred AS } B \text {-BIMODULES. }
\end{aligned}
$$

PF $/\left(\Longrightarrow\right.$ GIVEN EQuivalence $F: A-M_{0 d} \rightarrow B-M_{00}$

$$
\text { HAVE } Q:=F(A \text { Area }) \in(B, A) \text {-Bimod }
$$

TAKE $\mathbb{R}$-AlES $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B-M o d$.
That is, $A \not \& b$ have
the same rep thy
III. MOrita equivalence


Morita's theorem a is morita equivalent to b $\Longleftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A . \rightarrow}$.

$$
P \otimes_{B} Q \cong \text { Areg AS A-Bimoduces }
$$

$$
\notin Q \otimes_{A} P \cong \text { Bred } A S \text {-BIMODULES. }
$$

PF $/\left(\Longrightarrow\right.$ GIVEN EQUIVALENCE $F: A-M_{0 d} \rightarrow B-M_{0 d}$
have $Q:=F($ Arg $) \in(B, A)$-Bimod
CLAIM: $F \cong Q \otimes_{A}-A S$ FUNCTOR S
take lk-algs $A \notin B$
A IS MORITA EQuIV. To B
IF $A-M o d \simeq B-M o d$.
That is, $A \not \& b$ have
the same rep thy
III. MOrita equivalence


TAKE $\mathbb{R}$-AlES $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B-M o d$.
That is, $A \not \& B$ have the same rep thy

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B $\Longleftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A . \rightarrow}$.

$$
P \otimes_{B} Q \cong \text { Areg AS A-Bimoduces }
$$

$$
\notin Q \otimes_{A} P \cong \text { Brag As } B \text {-BIMODULES. }
$$

PF $/\left(\Longrightarrow\right.$ GIVEN Equivalence $F: A-M_{0 d} \rightarrow B-M_{00}$

$$
\text { HAVE } Q:=F(A \text { Area }) \in(B, A) \text {-Bimod }
$$

CLAIM: $F \cong Q \otimes_{A}-$ AS FUNCTOR S
Pf/ Take Me A-Mod \& Get 180 :

$$
\begin{aligned}
& \sigma_{X}: X \cong H_{M_{A-M o d}}(A, X) \xrightarrow{\cong} \operatorname{Hom}_{B-\operatorname{Mod}}(F(A), F(X))
\end{aligned}
$$

$$
\begin{aligned}
& \text { FAITHFUL }
\end{aligned}
$$

III. MOrita equivalence

$$
\begin{gathered}
\text { C AND } \theta \\
\text { ARE EQUIVALENT IF } \\
\text { FF: } \rightarrow \rightarrow \theta \& G: \theta \rightarrow C \\
\rightarrow \cdot \\
G F \cong I d_{\varphi} \& F G \cong I d \theta \\
\text { WRITE } \zeta \simeq \theta
\end{gathered}
$$

TAKE $\mathbb{R}$-AlES $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B-M o d$.
That is, A \& b have the same rep thy

MORITA'S THEOREM A Is mORITA EQUIVALENT TO B $\Leftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A .}$.

$$
P \otimes_{B} Q \cong \text { Areg AS A-Bimoduces }
$$

$$
\notin Q \otimes_{A} P \cong \text { Bred AS B-BIMODULES. }
$$

PF $/(\Rightarrow)$ GIVEN EQUIVALENCE $F: A-M_{0 d} \rightarrow B-M_{0} d$
have $Q:=F($ Ares $) \in(B, A)$-Bimod
CLAIM: $F \cong Q \otimes_{A}-A S$ FUNCTOR S
Pf/ Take MeA-Mod \& GeT 180 :

$$
\sigma_{X}: X \cong \operatorname{Hom}_{A-\mu_{0 d}}(A, X) \xrightarrow{F} \operatorname{Hom}_{B-\bmod ^{\prime}}(F(A), F(x))
$$

$\begin{cases}\text { TENSOR-HOM ADJUNCTION } & \operatorname{HoM}_{B-\operatorname{Mod}}(Q, F(X) \\ \operatorname{Hom}_{B-\bmod }\left(Q Q_{A} x, y\right) \cong \operatorname{Hom}_{A-\operatorname{Mod}}\left(X, \operatorname{Hom}_{B-\operatorname{Mod}}(Q, y)\right)\end{cases}$
GET 150: $\sigma_{x}^{\prime}: Q \otimes_{A} X \rightarrow F(x)$
III. MOrita equivalence

$$
\begin{gathered}
\text { C AND } \theta \\
\text { ARE EQUIVALENT IF } \\
\text { FF: } \rightarrow \rightarrow \theta \& G: \theta \rightarrow C \\
\rightarrow \cdot \\
G F \cong I d_{\varphi} \& F G \cong I d \theta \\
\text { WRITE } \zeta \simeq \theta
\end{gathered}
$$

TAKE $\mathbb{R}$-AlES $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B-M o d$.
That is, $A \not \ddagger b$ have the same rep thy

MORITA'S THEOREM A Is mORITA EQUIVALENT TO B $\Leftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A .}$.

$$
P \otimes_{B} Q \cong \text { Areg AS A-Bimoduces }
$$

$$
\notin Q \otimes_{A} P \cong \text { Dreg } A S \text { B-BIMODuLES. }
$$

PF $/(\Rightarrow)$ GIVEN EQUIVALENCE $F: A-M_{0 d} \rightarrow B-M_{0} d$

$$
\text { HAVE } Q:=F(A \text { Area }) \in(B, A) \text {-Bimod }
$$

CLAIM: $F \cong Q \otimes_{B}-$ AS FUNCTOR S
Pf/ Take Me A-Mod \& GeT 180 :

$$
\sigma_{X}: X \cong \operatorname{HoM}_{A-\bmod }(A, X) \xrightarrow{F} \operatorname{Hom}_{B-\operatorname{Mod}}(F(A), F(X))
$$

$\left\{\right.$ TENSOR-HOM ADJUNCTION $\quad H O M_{B-M o d} \quad$ " $Q, F(X)$ ) $\operatorname{Hom}_{\beta-\bmod }(Q \in X, y) \cong \operatorname{Hom}_{A-\bmod }\left(X, \operatorname{HoM}_{B-\operatorname{Mod}}(Q, y)\right)$ GET (50: $\sigma_{X}^{\prime}: Q \otimes_{A} X \rightarrow F(X) \leadsto Q \otimes_{A}-\underset{\sim}{\underset{\sim}{\Rightarrow}} F$
III. MOrita equivalence


TAKE $\mathbb{R}$-AlES $A \notin B$
A IS MORITA EQuIV. To B
IF $A-M o d \simeq B-M o d$.
That is, A \& b have
the same rep thy

MORITA'S THEOREM A IS MORITA EQUIVALENT TOB $\Leftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A .}$.

$$
P \otimes_{B} Q \cong \text { Areg AS A-Bimoduces }
$$

$$
\notin Q \otimes_{A} P \cong \text { Bred AS B-BIMODULES. }
$$

PF $/(\Rightarrow)$ GIVEN EQUIVALENCE $F: A-M_{0 d} \rightarrow B-M_{0} d$
Have $Q:=F(A$ Area $) \in(B, A)$-Bipod HAVE $F \cong Q \otimes_{A}$ - AS FUNCTOR S
III. MOrita equivalence


TAKE $\mathbb{R}$-AlaS $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B-M o d$.
That is, A \& b have
the same rep thy

MORITA'S THEOREM A IS MORITA EQUIVALENT TOB $\Longleftrightarrow \exists$ BIMODULES $A P_{B} \notin B Q_{A . \Rightarrow}$.

$$
P \otimes_{B} Q \cong \text { Areg AS A-Bimoduces }
$$

$$
\notin Q \otimes_{A} P \cong \text { Dreg AS B-BIMODULES. }
$$

PF $/(\Rightarrow)$ GIVEN EQUIVALENCE $F: A-M_{0 d} \rightarrow B-M_{0} d$
have $Q:=F(A$ Arg $) \in(B, A)$-Bipod
HAVE $F \cong Q \otimes_{A}$ - AS FUNCTOR S
NOW $\exists G: B-M o d \rightarrow A-M o d$ WITH

$$
\phi: I d_{A-\mu o d} \stackrel{\sim}{\Rightarrow} G F \quad \& \quad \Psi: F G \stackrel{\sim}{\Rightarrow} I d_{B-\mu_{0 d}}
$$

III. MOrita equivalence


TAKE $\mathbb{R}$-AlES $A \notin B$
A IS MORITA EQUIV. To B
IF $A-M o d \simeq B$-Mod.
That is, $A \not \& b$ have the same rep thy

MORITA'S THEOREM A IS MORITA EQUIVALENT TOB $\Longleftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A . \rightarrow}$.

$$
P \otimes_{B} Q \cong \text { Areg AS A-Bimoduces }
$$

$$
\notin Q \otimes_{A} P \cong \text { Dreg AS B-BIMODULES. }
$$

PF $/\left(\Longrightarrow\right.$ GIVEN EQuivalence $F: A-M_{0 d} \rightarrow B-M_{0} d$
have $Q:=F(A$ Arg $) \in(B, A)$-Bimod
HAVE $F \cong Q \otimes_{A}$ - AS FUNCTOR S
NOW $\exists G: B-M o d \rightarrow A-M o d$ WITH
$\phi: I d_{A-M_{0 d}} \stackrel{\sim}{\Rightarrow} G F$ \& $\Psi: F G \stackrel{\sim}{\Rightarrow} I d_{B-\mu_{0 d}}$
have $P:=G\left({ }_{B} B_{\text {reg }}\right) \in(A, B)$-Bimod
HAVE $G \cong P \otimes_{B}$ - AS FUNCTOR S
III. MOrita equivalence


TAKE $\mathbb{R}$-alas $A \notin B$
A IS MORITA EQUIV. To B
IF $A-M o d \simeq B-M o d$.
That is, A \& b have the same rep thy

MORITA'S THEOREM A Is mORITA EQUIVALENT TOB $\Longleftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A . \rightarrow}$.

$$
P \otimes_{B} Q \cong \text { Areg AS A-Bimoduces }
$$

$$
\notin Q \otimes_{A} P \cong \text { Dreg } A S \text {-BIMODULES. }
$$

PF $/\left(\Rightarrow\right.$ GIVEN EQUIVALENCE $F: A-M_{0 d} \rightarrow B-M_{0} d$

$$
\text { HAVE } Q:=F(A \text { Area }) \in(B, A) \text {-Bipod }
$$

HAVE $F \cong Q \otimes_{A}$ - AS FUNCTOR S
NOW $\exists G: B-M o d \rightarrow A-M o d$ WITH
$\phi: I d_{A-M_{0 d}} \stackrel{\sim}{\Rightarrow} G F$ \& $\Psi: F G \stackrel{\sim}{\Rightarrow} I d_{B-\mu_{0 d}}$
have $P:=G\left({ }_{B} B_{\text {reg }}\right) \in(A, B)$-Bipod
HAVE $G \cong P \otimes_{B}$ - AS FUNCTOR S
$G \in T \quad \phi_{A}: A \leadsto G F(A) \cong P \otimes_{B} Q$ AS $A$-BIMODS

- $\Psi_{B}^{-1}: B \simeq F G(B) \cong Q Q_{A} P \quad A S B-B / M O D S / / \pi$
III. MOrita equivalence

DETAILS = EXER.2.35


TAKE $\mathbb{R}$-AlES $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B-M o d$.
That is, A \& b have the same rep thy

MORITA'S THEOREM A IS MORITA EQUIVALENT TOB $\Longleftrightarrow \exists$ BIMODULES $A P_{B} \notin B Q_{A . \Rightarrow}$.
$P \otimes_{B} Q \cong$ Area As $A$-Bimoduces
\& $Q \otimes_{A} P \cong$ Dreg $A S$-Bimoduces.
$P F /\left(\Longrightarrow\right.$ GIVEN EQUIVALENCE $F: A-M_{0 d} \rightarrow B-M_{0} d$
Have $Q:=F(A$ Area $) \in(B, A)$-Bimod
HAVE $F \cong Q \otimes_{A}$ - AS FUNCTOR S
NOW $\exists G: B-M o d \rightarrow A-M o d$ WITH
$\phi: I d_{A-\bmod } \stackrel{\sim}{\Rightarrow} G F$ \& $\Psi: F G \stackrel{\sim}{\Rightarrow} I d_{B-M_{0 d}}$
have $P:=G\left({ }_{B} B_{\text {reg }}\right) \in(A, B)$-Bipod
HAVE $G \cong P \otimes_{B}$ - AS FUNCTOR S
$G \in T \quad \oint_{A}: A \xrightarrow{\leadsto} G F(A) \cong P \otimes_{B} Q$ AS $A$-BIMODS

- $\Psi_{B}^{-1}: B \simeq F G(B) \cong Q Q_{A} P \quad A S B-B / M O D S / / \pi$
III. MOrita equivalence


MORITA'S THEOREM A is MORITA EQUIVALENT TOB $\Longleftrightarrow \exists$ BIMODULES $A P_{B} \notin B Q_{A .7}$.

$$
\begin{aligned}
& P \otimes_{B} Q \cong \text { Areg AS } A \text {-Bimodules } \\
& \& Q \otimes_{A} P \cong \text { reg AS B-BIMODULES. }
\end{aligned}
$$

MAIN EXAMPLE
A is morita equivalent to Matn(A)
take lk-algs $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B-M o d$.
That is, A \& b have
the same rep thy
III. MOrita equivalence


TAKE $\mathbb{R}$-AlaS $A \notin B$
A is morita equiv. To b
IF $A-M o d \simeq B-M o d$.
That is, A \& b have the same rep thy

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B $\Longleftrightarrow \exists$ BIMODULES $A P_{B} \notin B Q_{A .}$.

$$
\begin{aligned}
& P \otimes_{B} Q \cong \text { Areg AS A-BIMODULES } \\
& \& Q \otimes_{A} P \cong \text { Bred AS B-BIMODULES. }
\end{aligned}
$$

MAIN EXAMPLE
A is morita equivalent to Matn(a)
VIA

$$
P=\left\{\left(a_{1}, \ldots, a_{n}\right) \mid a_{i} \in A\right\} \in\left(A, M_{a} t_{n}(A)\right)-\text { Bimod }
$$

$$
Q=\left\{\left.\left(\begin{array}{c}
a_{1} \\
\vdots \\
a_{n}
\end{array}\right) \right\rvert\, a_{i} \in A\right\} \in\left(\operatorname{Mat}_{n}(A), A\right)-B i \operatorname{Mod}
$$

III. MOrita equivalence


TAKE $\mathbb{R}$-AlaS $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B-M o d$.
That is, A \& b have the same rep thy

MORITA'S THEOREM A Is mORITA EqUIVALENT TOB $\Longleftrightarrow \exists$ BIMODULES $A P_{B} \notin B Q_{A . \rightarrow}$.

$$
\begin{aligned}
& P \otimes_{B} Q \cong \text { Areg AS A-BIMODULES } \\
& \& Q \otimes_{A} P \cong \text { Bred AS B-BIMODULES. }
\end{aligned}
$$

MAIN EXAMPLE
A is morita equivalent to Matn(a) VIA

$$
P=\left\{\left(a_{1}, \ldots, a_{n}\right) \mid a_{i} \in A\right\} \in\left(A, M_{a} t_{n}(A)\right)-\text { Bimod }
$$

$$
a^{\prime} \triangleright\left(a_{1}, \ldots, a_{n}\right):=\left(a^{\prime} a_{1}, \ldots, a^{\prime} a_{n}\right)
$$

$$
\underset{\underline{a}}{\left(a_{1}, \ldots, a_{n}\right)} \triangleleft A^{\prime}:=\underline{a} A^{\prime} \text { MATRIX MULTIP'N }
$$

$$
Q=\left\{\left.\left(\begin{array}{c}
a_{1} \\
\vdots \\
a_{n}
\end{array}\right) \right\rvert\, a_{i} \in A\right\} \in\left(\operatorname{Mat}_{n}(A), A\right)-\text { Bimod }
$$

III. MOrita equivalence


WRITE $\zeta \simeq \theta$

TAKE $\mathbb{R}$-AlaS $A \notin B$
A IS MORITA EQuIV. To B
IF $A-M o d \simeq B-M o d$.
That is, A \& b have
the same rep thy

MORITA'S THEOREM A is mORITA EqUivalent To B $\Longleftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A . \rightarrow}$.

$$
\begin{aligned}
& P \otimes_{B} Q \cong \text { Areg As } A \text {-Bimodules } \\
\& & Q \otimes_{A} P \cong \text { Bred AS } B \text {-Bimoduces. }
\end{aligned}
$$

MAIN EXAMPLE
A is morita equivalent to Matn(a)
III. MOrita equivalence


TAKE $\mathbb{R}$-AlaS $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B-M o d$.
That is, A \& b have
the same rep thy

MORITA'S THEOREM A IS MORITA EQUIVALENT TOB $\Leftrightarrow \exists$ BIMODULES $A_{B} \not P_{B} Q_{A .}$.

$$
\begin{aligned}
& P \otimes_{B} Q \cong \text { Areg AS A-BIMODULES } \\
& \& Q \otimes_{A} P \cong \text { Bred AS B-BIMODULES. }
\end{aligned}
$$

MAIN EXAMPLE
A is morita equivalent to Matn(A)

EXER.2.38 $A \sim_{\text {MAR }} B \Rightarrow, Z(A) \cong Z(B)$ CENTER OF A
III. MOrita equivalence


TAKE $\mathbb{R}$-AlaS $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B$-Mod.
That is, A \& b have
the same rep thy

MORITA'S THEOREM A IS MORITA EQUIVALENT TOB $\Longleftrightarrow \exists$ BIMODULES $A P_{B} \notin B Q_{A . \Rightarrow}$.

$$
\begin{aligned}
& P \otimes_{B} Q \cong \text { Areg AS A-BIMODULES } \\
& \& Q \otimes_{A} P \cong \text { Bred AS B-BIMODULES. }
\end{aligned}
$$

MAIN EXAMPLE
A is morita equivalent to Matn(a)
FOR INSTANCE:
$\mathbb{k} \sim_{\text {Mo }} \operatorname{Mat}_{n}(\mathbb{k}) \not \& Z\left(\operatorname{Mat}_{n}(\mathbb{k})\right) \cong \mathbb{k}$

EXER.2.38 $A \sim_{\text {HOR }} B \Rightarrow Z(A) \cong Z(B)$ CENTER OF A
III. MOrita equivalence


TAKE $\mathbb{R}$-AlaS $A \notin B$
A is morita equiv. to b
IF $A-M o d \simeq B$-Mod.
That is, A \&b have
the same rep thy

MORITA'S THEOREM A Is mORITA EqUivalent To B $\Longleftrightarrow \exists$ Bimodules $A P_{B} \notin B Q_{A . \rightarrow}$.

$$
P \otimes_{B} Q \cong \text { Areg AS A-Bimoduces }
$$

$$
\notin Q \otimes_{A} P \cong \text { Bred AS B-BIMODULES. }
$$

MAIN EXAMPLE
A is morita equivalent to Matn(a)
FOR INSTANCE:
$\mathbb{k} \sim_{\text {Mo }} \operatorname{Mat}_{n}(\mathbb{k}) \not \& Z\left(\operatorname{Mat}_{n}(\mathbb{k})\right) \cong \mathbb{k}$

EXER.2.38 $A \sim_{\text {MIR }} B \underset{\neq}{\Rightarrow} \quad Z(A) \cong Z(B)$ in General center of a
III. MOrita equivalence


TAKE $\mathbb{R}$-AlaS $A \notin B$
A is morita equiv. To B
IF $A-M o d \simeq B-M o d$.
That is, A \& b have
the same rep thy

MORITA'S THEOREM A Is mORITA EqUIVALENT TOB $\Longleftrightarrow \exists$ BIMODULES $A P_{B} \notin B Q_{A . \Rightarrow}$.

$$
P \otimes_{B} Q \cong \text { Areg AS A-Bimoduces }
$$

$$
\notin Q \otimes_{A} P \cong \text { Bred AS B-BIMODULES. }
$$

MAIN EXAMPLE
A is morita equivalent to Matn(a)
FOR INSTANCE:
$\mathbb{k} \sim_{\text {Mo }} \operatorname{Mat}_{n}(\mathbb{k}) \not \& Z\left(\operatorname{Mat}_{n}(\mathbb{k})\right) \cong \mathbb{k}$

EXER.2.38 $A \sim_{\text {MAR }} B \underset{\nRightarrow}{\Rightarrow} \quad Z(A) \cong Z(B)$ in General center of a

FOR COMM. $\quad C \sim_{\text {HOR }} C^{\prime} \Leftrightarrow Z(C) \cong Z\left(C^{\prime}\right)$
ALG $C, C^{\prime}$

$$
\equiv \operatorname{SUMMARY} \equiv
$$

Notion of sameness FOR IR-ALGEBRAS

$$
A=B
$$

EqUALITY
of ALGEBRAS

WEAKEN $\longrightarrow A$ MORITA $B$ "equivalence" OF ALGEBRAS
$\equiv \operatorname{SUMMARY} \equiv$

Notion of sameness for categories

$$
\varphi_{e}=D
$$

EQUALITY
of categories

NOTION OF SAMENESS
FOR Ik-AlGEBRAS

$$
A=B
$$

equality
OF ALGEBRAS

of ALGEBRAS

equivalence
WEAKEN $\rightarrow A$ MORITA $B$
"equivalence"
OF ALGEBRAS
$\equiv \operatorname{SUMMARY} \equiv$

Notion of sameness for categories

$$
\varphi_{e}=D
$$

EQUALITY
of Categories

NOTION OF SAMENESS FOR Ik-AlGEBRAS

$$
A=B
$$

equality
OF ALGEBRAS

equivalence
"equivalence"
of categories
of ALGEBRAS

LECTURE \#9

TOPICS:
I. ISOMORPHISM OF CATEGORIES (\$2.4.1)
II. Equivalence of categories (8s2.4.2-2.4.3)
III. morita equivalence
( 82.4 .3 )
NEXT TIME: ADJUNCTION


Available for purchase at :

619 Wreath (at a discount)
https://www.619wreath.com/

Also on Amazon<br>\&<br>Google Play

Lecture \#9 keywords: equivalence of categories, isomorphism of categories, Morita equivalence of algebras, Morita's Theorem, skeleton

