MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #9

LAST TIME

- FUNCTORS
- BIFUNCTORS & MULTIFUNCTORS
- NATURAL TRANSFORMATIONS
- COMPOSITIONS OF NATURAL TRANSFORMATIONS

TOPICS:

I. ISOMORPHISM OF CATEGORIES (§2.4.1)

II. EQUIVALENCE OF CATEGORIES (§§2.4.2–2.4.3)

III. MORITA EQUIVALENCE (§2.4.3)

≡ RECALL ≡

A CATEGORY $\mathscr{C}$

CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS

$\mathrm{Hom}_{\mathscr{C}}(X,Y)$

$\forall X, Y \in \mathscr{C}$.

(c) $id_X : X \to X$

$\forall X \in \mathscr{C}$.

(d) $gf : W \to Y$

$\forall f : W \to X$

$g : X \to Y$.

SATISFYING

ASSOCIATIVITY

$(hg)f = h(gf)$

UNITALITY

$id_X f = f$, $g\, id_X = g$

≡ RECALL ≡

A CATEGORY $\mathcal{C}$

CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ $\forall X,Y \in \mathcal{C}$.

(c) $\mathrm{id}_X : X \to X$ $\forall X \in \mathcal{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

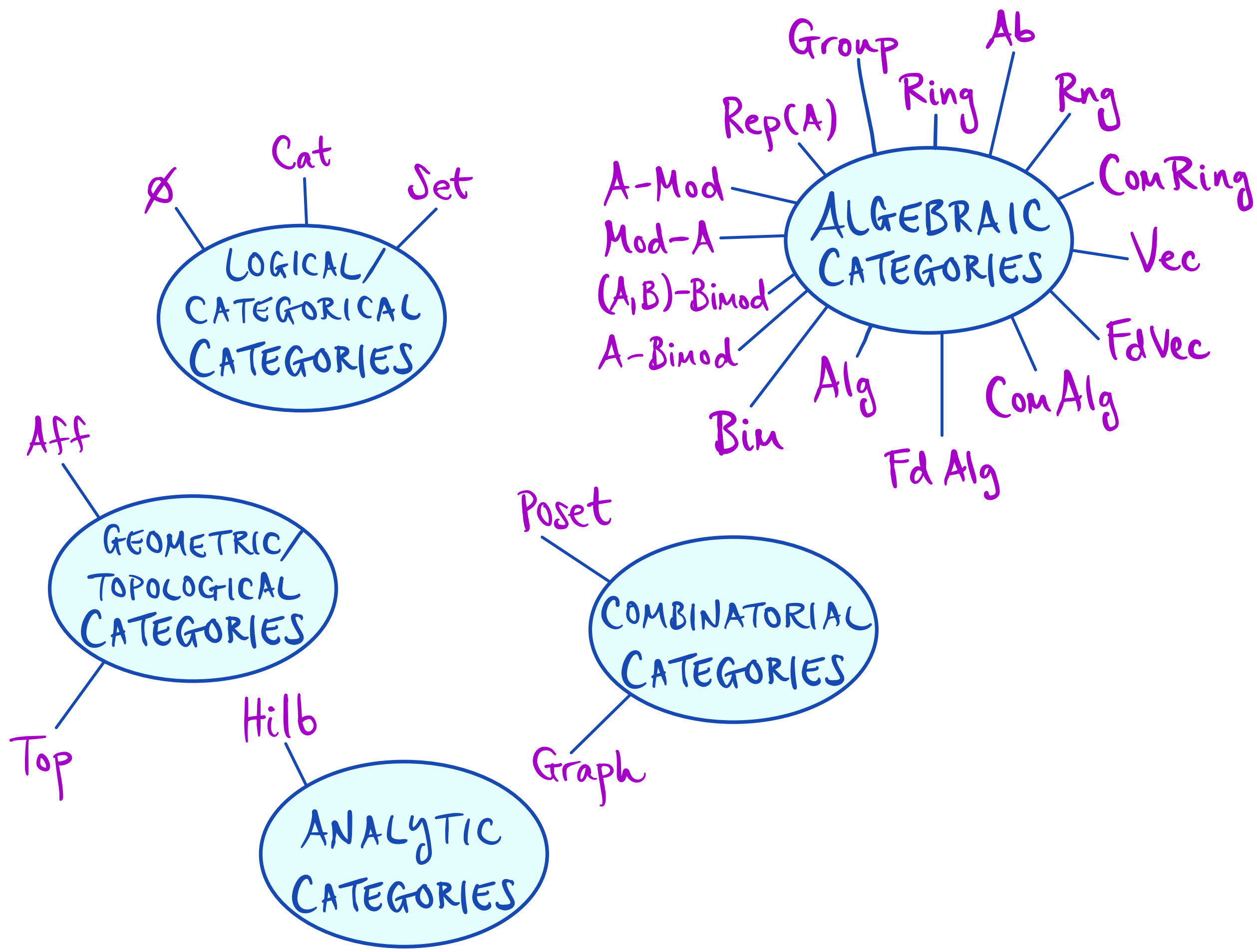
SATISFYING

ASSOCIATIVITY

$(hg)f = h(gf)$

UNITALITY

$\mathrm{id}_X f = f, \; g\, \mathrm{id}_X = g$



≡ RECALL ≡

A CATEGORY $\mathcal{C}$ CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ $\forall X, Y \in \mathcal{C}$.

(c) $\mathrm{id}_X : X \to X$ $\forall X \in \mathcal{C}$.

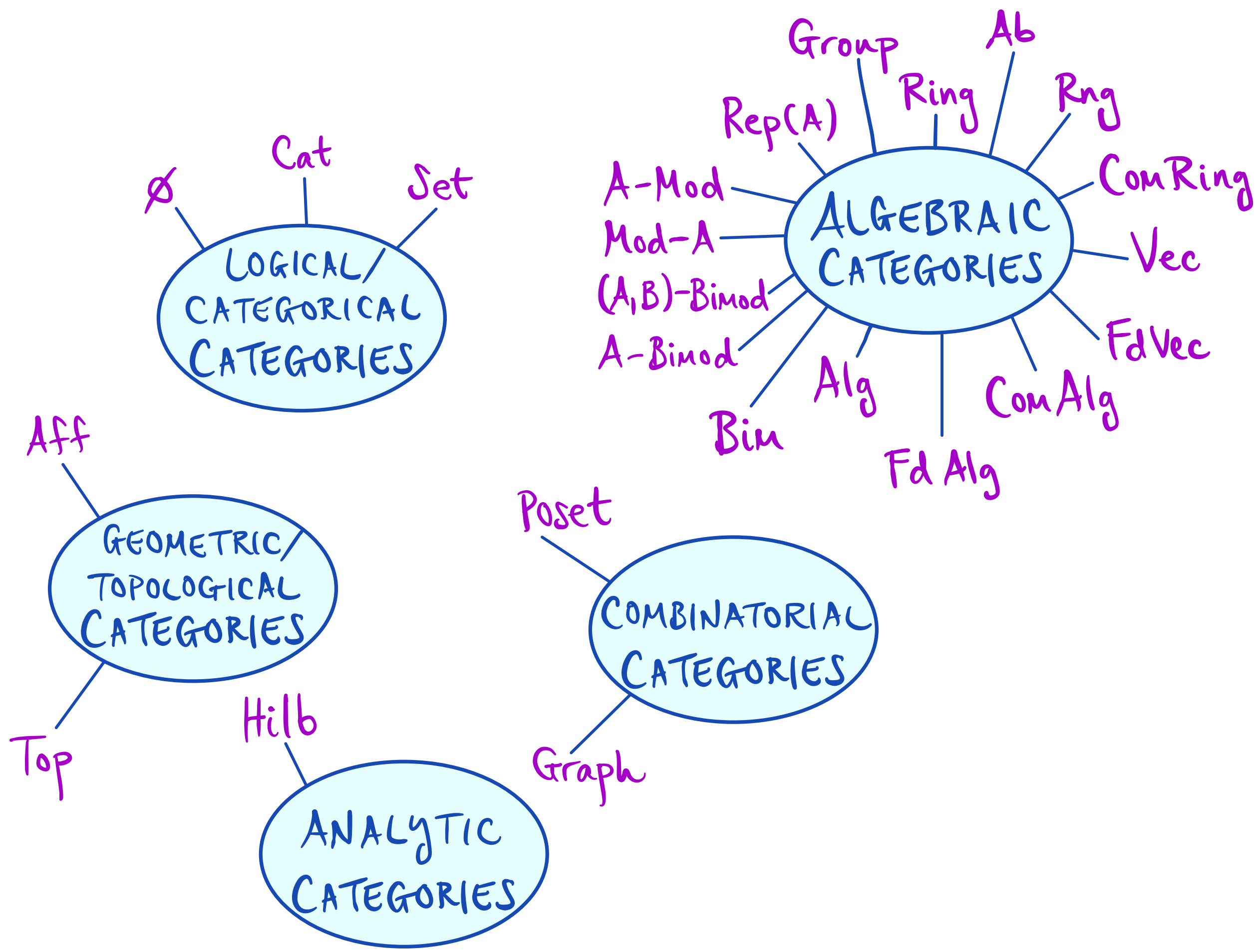
(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY
$(hg)f = h(gf)$

UNITALITY
$\mathrm{id}_X f = f$, $g\, \mathrm{id}_X = g$

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?



≡ RECALL ≡

A **CATEGORY** $\mathcal{C}$

CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ $\forall X,Y \in \mathcal{C}$.

(c) $id_X : X \to X$ $\forall X \in \mathcal{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY

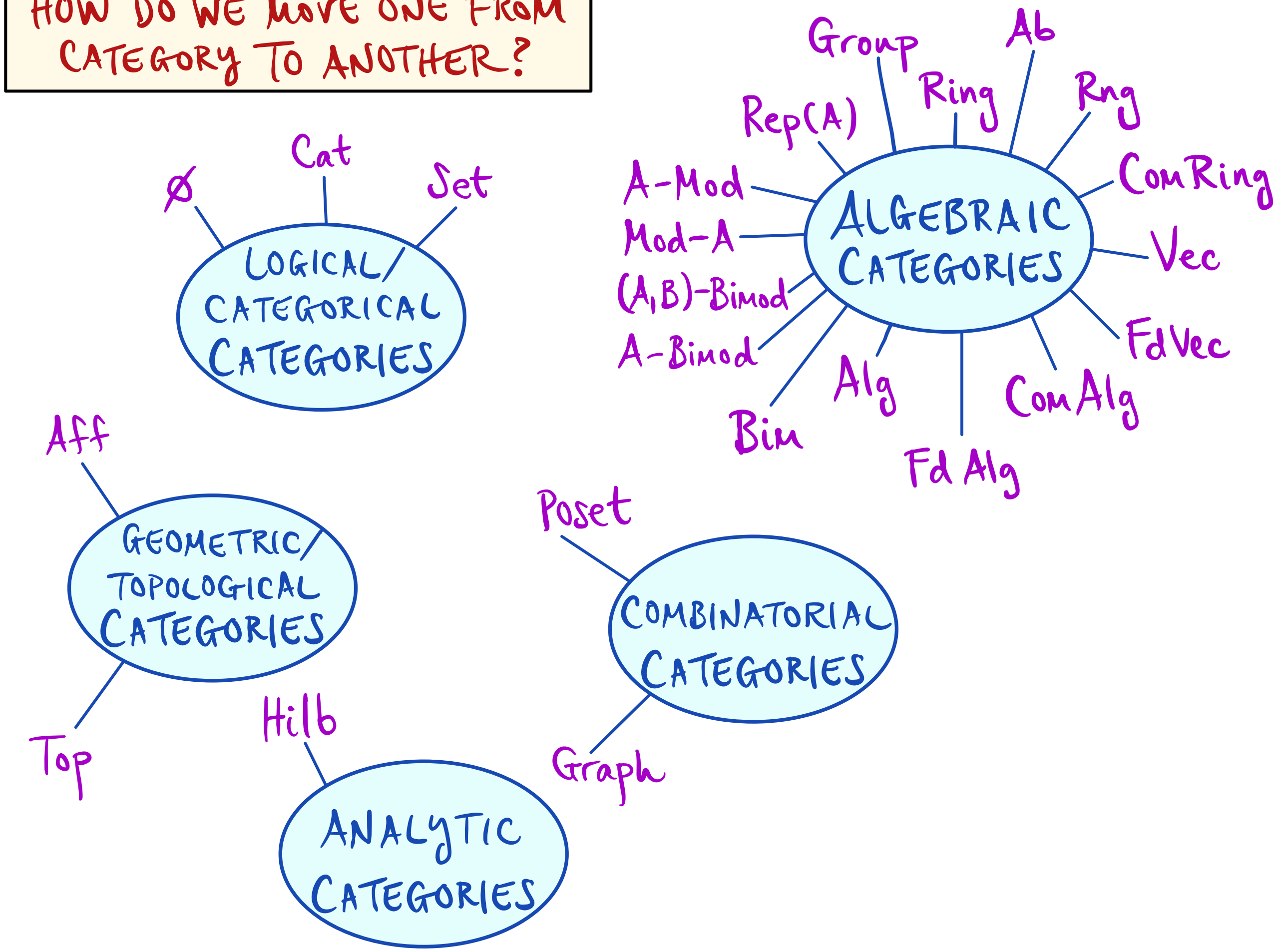
$(hg)f = h(gf)$

UNITALITY

$id_X f = f, \; g\, id_X = g$

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

HOW DO WE MOVE ONE FROM CATEGORY TO ANOTHER?



≡ RECALL ≡

A FUNCTOR $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT)

CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$

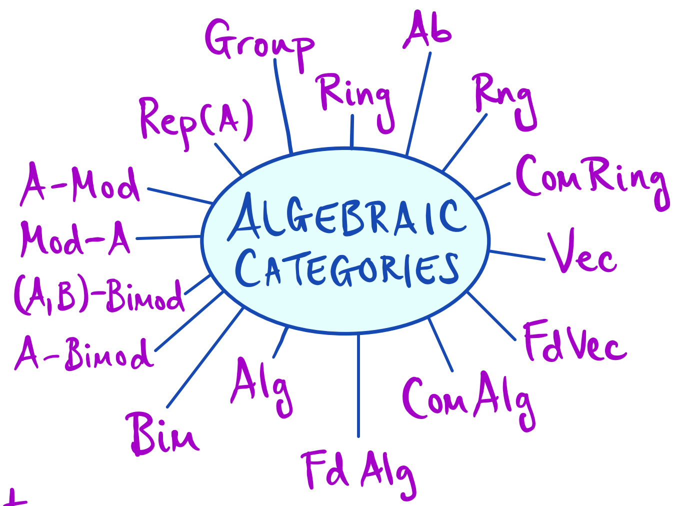
(b) $F(g): F(X) \to F(Y) \in \mathcal{D}$ (RESP., $F(g): F(Y) \to F(X) \in \mathcal{D}$) $\forall g: X \to Y \in \mathcal{C}$.

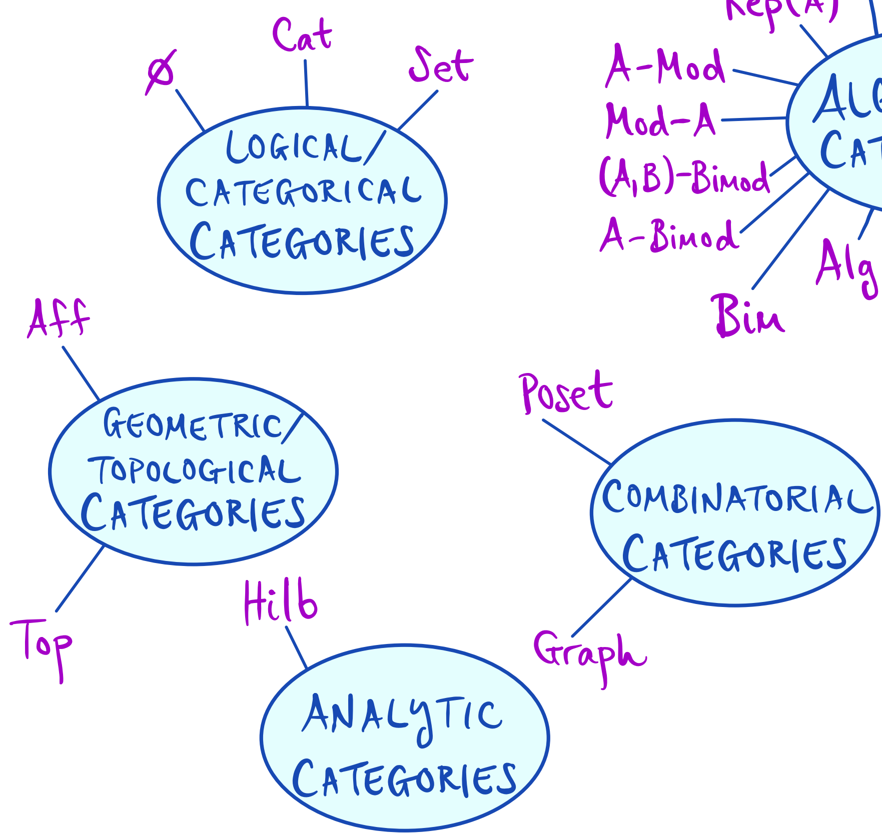
RESPECTING:

IDENTITY & COMPOSED MORPHISMS

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

HOW DO WE MOVE ONE FROM CATEGORY TO ANOTHER?





≡ RECALL ≡

A (resp., contravariant) functor $F : \mathcal{C} \to \mathcal{D}$

consists of:

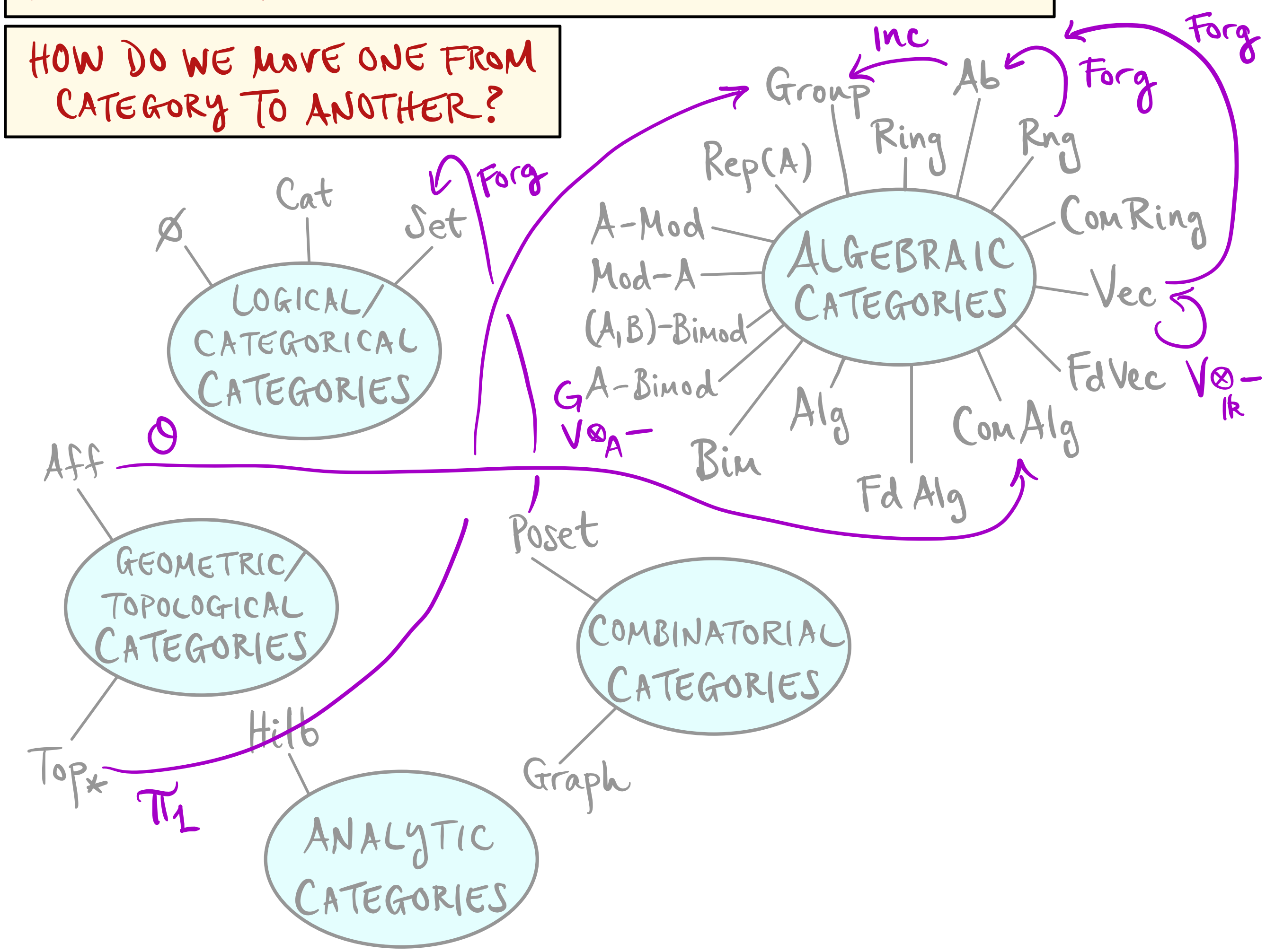
(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g) : F(X) \to F(Y) \in \mathcal{D}$
(resp., $F(g) : F(Y) \to F(X) \in \mathcal{D}$)
$\forall g : X \to Y \in \mathcal{C}$.

Respecting:
identity & composed morphisms

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

HOW DO WE MOVE ONE FROM CATEGORY TO ANOTHER?



≡ RECALL ≡

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

HOW DO WE MOVE ONE FROM CATEGORY TO ANOTHER?

A (v) **FUNCTOR** $F: \mathcal{C} \to \mathcal{D}$ (RESP., CONTRAVARIANT)

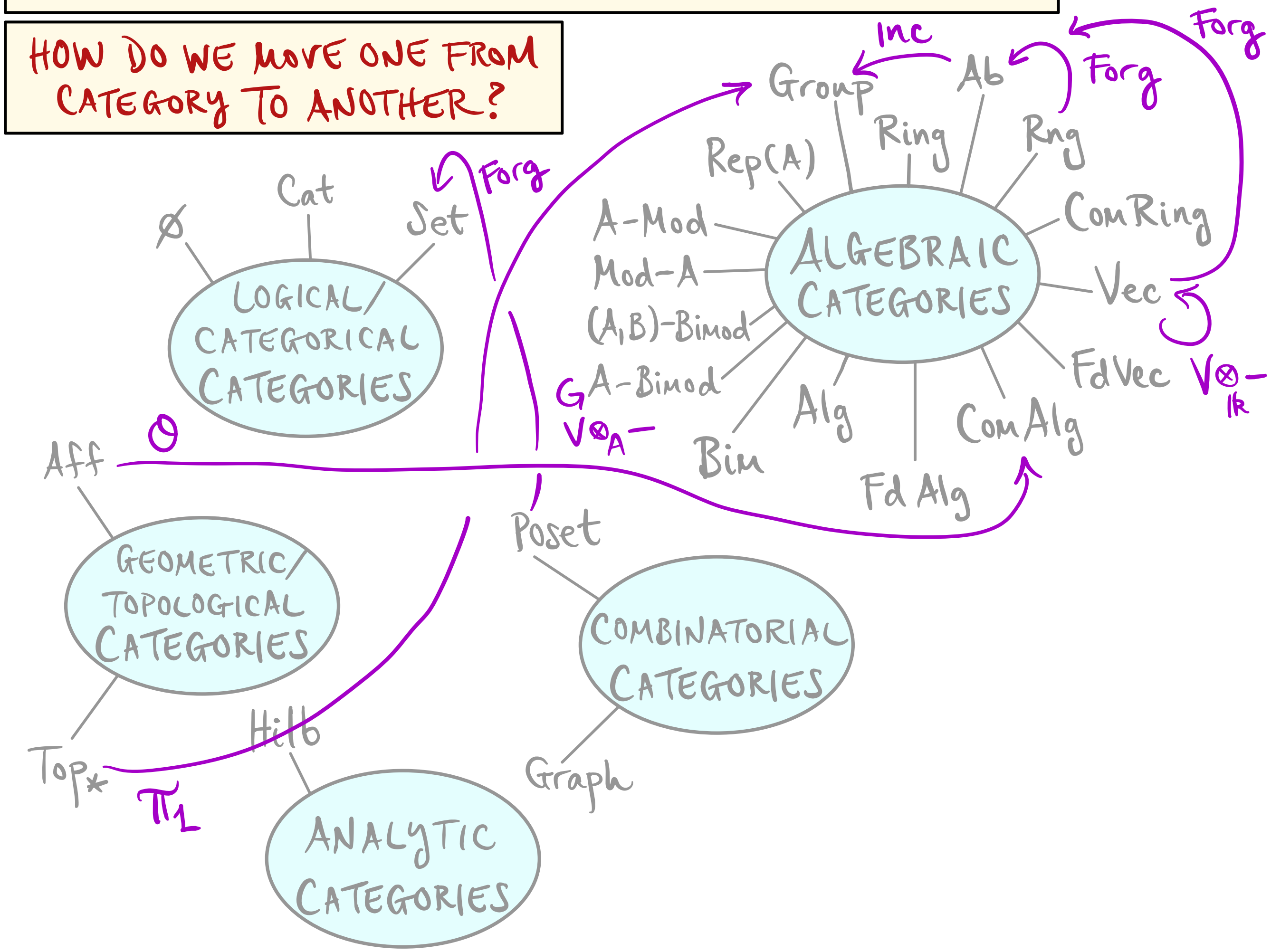
CONSISTS OF:

(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g): F(X) \to F(Y) \in \mathcal{D}$
(RESP., $F(g): F(Y) \to F(X) \in \mathcal{D}$)
$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

IDENTITY & COMPOSED MORPHISMS



$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)),\ g \mapsto F(g)$			F **EMBEDDING**: F FAITH & INJ ON OBJS
F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$	F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$	F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$	F **ESS. SURJ**: $\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni Y \cong F(X)$

≡ RECALL ≡

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

HOW DO WE MOVE ONE FROM CATEGORY TO ANOTHER?

A (covariant) **FUNCTOR** $F: \mathcal{C} \to \mathcal{D}$ (RESP., **CONTRAVARIANT**)

CONSISTS OF:

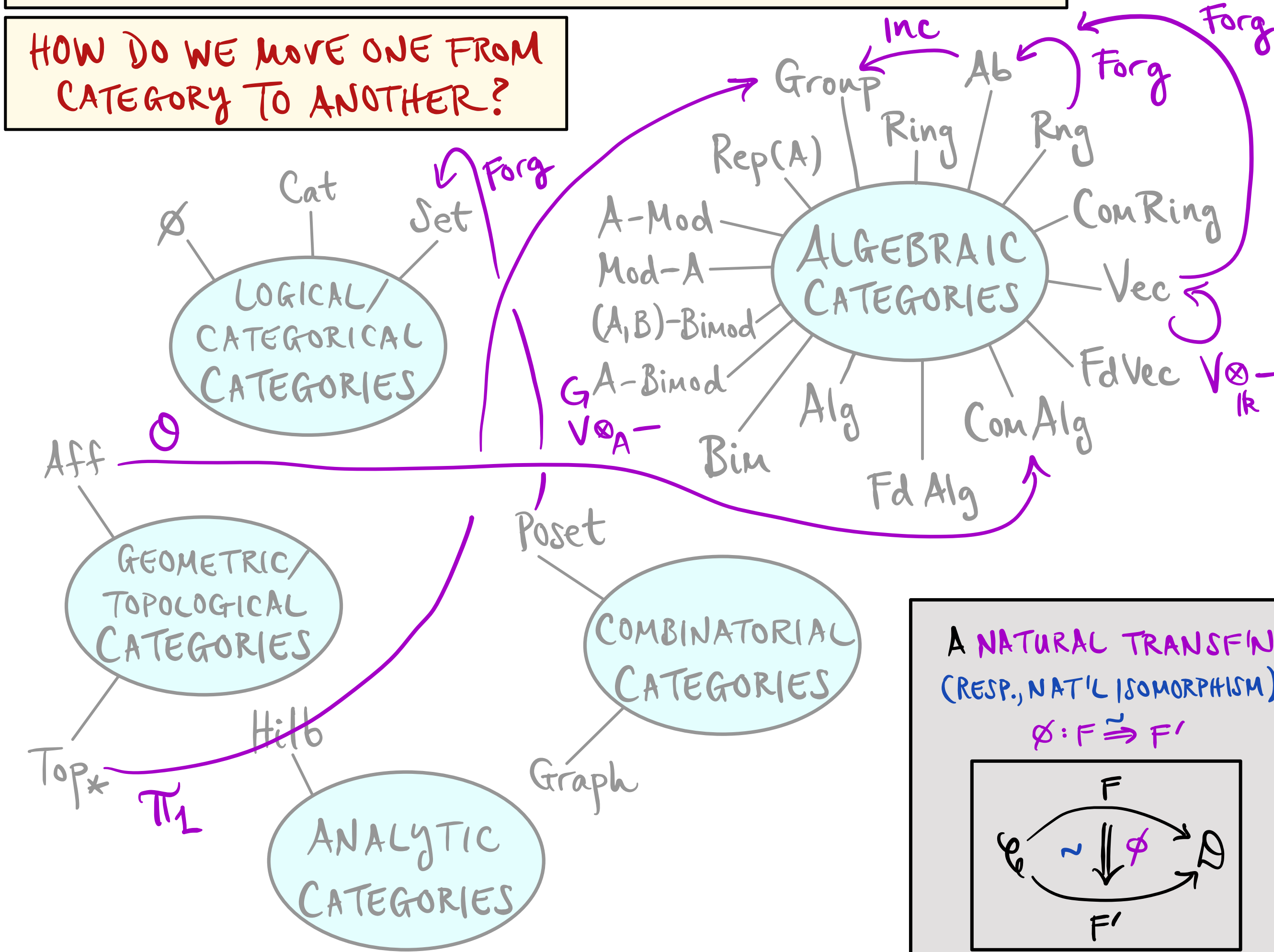
(a) $F(X) \in \mathcal{D} \quad \forall X \in \mathcal{C}$.

(b) $F(g): F(X) \to F(Y) \in \mathcal{D}$
(RESP., $F(g): F(Y) \to F(X) \in \mathcal{D}$)
$\forall g: X \to Y \in \mathcal{C}$.

RESPECTING:

IDENTITY &

COMPOSED MORPHISMS



A **NATURAL TRANSF'N**
(RESP., NAT'L ISOMORPHISM)
$\phi: F \overset{\sim}{\Rightarrow} F'$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F'"'] \arrow[r, phantom, "\sim\ \Downarrow \phi"] & \mathcal{D} \end{tikzcd}$$

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\phi_X"', "\sim"] \arrow[dr, phantom, "\circlearrowright"] & F(Y) \arrow[d, "\phi_Y", "\sim"'] \\ F'(X) \arrow[r, "F'(f)"] & F'(Y) \end{tikzcd}$$

$\forall f: X \to Y \in \mathcal{C}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)),\ g \mapsto F(g)$			F **EMBEDDING**: F FAITH & INJ ON OBJS
F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$	F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$	F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$	F **ESS. SURJ**: $\forall Y \in \mathcal{D}, \exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$

WHEN ARE TWO CATEGORIES CONSIDERED THE SAME?

NOW SOME ANSWERS

I. ISOMORPHISM OF CATEGORIES

$\mathscr{C}, \mathscr{D}$ CATEGORIES

$\mathscr{C}$ AND $\mathscr{D}$

ARE SAID TO BE

ISOMORPHIC

IF $\exists$ FUNCTORS

$F: \mathscr{C} \longrightarrow \mathscr{D}$ & $G: \mathscr{D} \longrightarrow \mathscr{C}$

SUCH THAT

$GF = \mathrm{Id}_{\mathscr{C}}$

& $FG = \mathrm{Id}_{\mathscr{D}}$

WRITE $\mathscr{C} \cong \mathscr{D}$

I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE SAID TO BE

ISOMORPHIC

IF $\exists$ FUNCTORS

$F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

SUCH THAT

$GF = \mathrm{Id}_{\mathcal{C}}$

& $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

EXAMPLE: G = GROUP

GET: $\boxed{G\text{-Mod} \cong \mathrm{Rep}(G)}$

I. ISOMORPHISM OF CATEGORIES

$\mathscr{C}, \mathscr{D}$ CATEGORIES

$\mathscr{C}$ AND $\mathscr{D}$

ARE SAID TO BE

ISOMORPHIC

IF $\exists$ FUNCTORS

$F: \mathscr{C} \to \mathscr{D}$ & $G: \mathscr{D} \to \mathscr{C}$

SUCH THAT

$GF = \mathrm{Id}_{\mathscr{C}}$

& $FG = \mathrm{Id}_{\mathscr{D}}$

WRITE $\mathscr{C} \cong \mathscr{D}$

EXAMPLE: G = GROUP

GET: $\boxed{G\text{-Mod} \cong \mathrm{Rep}(G)}$

$F: G\text{-Mod} \longrightarrow \mathrm{Rep}(G)$

$$(\underset{\mathrm{Vec}}{\underset{\cap}{V}}, \triangleright: G \times V \to V) \longmapsto (V, \rho_V: G \to GL(V))$$

$$g \mapsto \begin{bmatrix} V \to V \\ v \mapsto g \triangleright v \end{bmatrix}$$

$F': \mathrm{Rep}(G) \longrightarrow G\text{-Mod}$

$$(\underset{\mathrm{Vec}}{\underset{\cap}{V}}, \rho: G \to GL(V)) \longmapsto (V, \triangleright_V: G \times V \to V)$$

$$(g, v) \mapsto \rho_g(v)$$

I. Isomorphism of Categories

$\mathcal{C}, \mathcal{D}$ categories

$\mathcal{C}$ and $\mathcal{D}$ are said to be **isomorphic** if $\exists$ functors $F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ such that

$$GF = \mathrm{Id}_{\mathcal{C}}$$

$$\& \quad FG = \mathrm{Id}_{\mathcal{D}}$$

Write $\mathcal{C} \cong \mathcal{D}$

Example: G = group

Get: $\boxed{G\text{-Mod} \cong \mathrm{Rep}(G)}$

$F: G\text{-Mod} \longrightarrow \mathrm{Rep}(G)$

$$(\underset{\in \mathrm{Vec}}{V}, \underbrace{\triangleright}_{\equiv\ \text{action}\ \equiv} : G \times V \to V) \longmapsto (V, \underbrace{\rho_V}_{\equiv\ \text{group homom.}\ \equiv} : G \to GL(V))$$

$$g \mapsto \begin{bmatrix} V \to V \\ v \mapsto g \triangleright v \end{bmatrix}$$

$(gh) \triangleright v = g \triangleright (h \triangleright v)$

$$\rho(gh)(v) = (gh) \triangleright v = g \triangleright (h \triangleright v)$$

$$\rho(g)\rho(h)(v) = g \triangleright (\rho(h)(v))$$

$F': \mathrm{Rep}(G) \longrightarrow G\text{-Mod}$

$$(\underset{\in \mathrm{Vec}}{V}, \rho: G \to GL(V)) \longmapsto (V, \triangleright_V : G \times V \to V)$$

$$(g, v) \mapsto \rho_g(v)$$

I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE SAID TO BE ISOMORPHIC IF $\exists$ FUNCTORS $F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$ SUCH THAT

$GF = \mathrm{Id}_{\mathcal{C}}$

& $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

EXAMPLE: G = GROUP

GET: $\boxed{G\text{-Mod} \cong \mathrm{Rep}(G)}$

$F: G\text{-Mod} \longrightarrow \mathrm{Rep}(G)$

$$\Big(\underset{\mathrm{Vec}}{\underset{\cap}{V}},\ \underbrace{\triangleright : G\times V \to V}_{\text{ACTION}}\Big) \longmapsto \Big(V,\ \underbrace{\rho_V : G \to GL(V)}_{\text{GROUP HOMOM.}}\Big)$$

$$g \mapsto \left[\begin{array}{c} V \to V \\ v \mapsto g\triangleright v \end{array}\right]$$

$(gh)\triangleright v = g\triangleright(h\triangleright v)$

$$\rho(gh)(v) = (gh)\triangleright v = g\triangleright(h\triangleright v)$$

$$\rho(g)\rho(h)(v) = g\triangleright(\rho(h)(v))$$

$F': \mathrm{Rep}(G) \longrightarrow G\text{-Mod}$

$$\Big(\underset{\mathrm{Vec}}{\underset{\cap}{V}},\ \rho: G \to GL(V)\Big) \longmapsto \Big(V,\ \triangleright_V : G\times V \to V\Big)$$

$$(g,v) \mapsto \rho_g(v)$$

CHECK $F'F = \mathrm{Id}_{G\text{-Mod}}$ & $FF' = \mathrm{Id}_{\mathrm{Rep}(G)}$

I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE SAID TO BE ISOMORPHIC IF $\exists$ FUNCTORS $F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ SUCH THAT

$$GF = \mathrm{Id}_{\mathcal{C}}$$

$$\& \ FG = \mathrm{Id}_{\mathcal{D}}$$

WRITE $\mathcal{C} \cong \mathcal{D}$

UPGRADE OF EXER. 1.13

EXAMPLE: G = GROUP

GET: $\boxed{G\text{-}\mathrm{Mod} \cong \mathrm{Rep}(G)}$

$$F: G\text{-}\mathrm{Mod} \longrightarrow \mathrm{Rep}(G)$$

$$(\underset{\substack{\cap \\ \mathrm{Vec}}}{V}, \overbrace{\triangleright : G \times V \to V}^{\equiv \text{ ACTION} \equiv}) \longmapsto (V, \overbrace{\rho_V : G \to GL(V)}^{\equiv \text{ GROUP HOMOM.} \equiv})$$

$$g \mapsto \begin{bmatrix} V \to V \\ v \mapsto g \triangleright v \end{bmatrix}$$

$$(gh) \triangleright v = g \triangleright (h \triangleright v)$$

$$\rho(gh)(v) = (gh) \triangleright v = g \triangleright (h \triangleright v)$$

$$\rho(g)\rho(h)(v) = g \triangleright (\rho(h)(v))$$

$$F': \mathrm{Rep}(G) \longrightarrow G\text{-}\mathrm{Mod}$$

$$(\underset{\substack{\cap \\ \mathrm{Vec}}}{V}, \rho: G \to GL(V)) \longmapsto (V, \triangleright_V : G \times V \to V)$$

$$(g, v) \mapsto \rho_g(v)$$

CHECK $F'F = \mathrm{Id}_{G\text{-}\mathrm{Mod}}$ & $FF' = \mathrm{Id}_{\mathrm{Rep}(G)}$

I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE SAID TO BE **ISOMORPHIC** IF $\exists$ FUNCTORS $F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ SUCH THAT

$$GF = \mathrm{Id}_{\mathcal{C}} \quad \& \quad FG = \mathrm{Id}_{\mathcal{D}}$$

WRITE $\mathcal{C} \cong \mathcal{D}$

EXERCISE 2.30: SHOW –

$G\text{-Mod} \cong \mathrm{Rep}(G)$

$\cong \mathrm{Rep}(\Bbbk G)$

$\cong \Bbbk G\text{-Mod}$

UPGRADE OF EXER. 1.13

EXAMPLE: G = GROUP

GET: $\boxed{G\text{-Mod} \cong \mathrm{Rep}(G)}$

$$F: G\text{-Mod} \longrightarrow \mathrm{Rep}(G)$$

$$(\underset{\mathrm{Vec}}{\underset{\cap}{V}},\ \underbrace{\triangleright: G \times V \to V}_{\equiv\ \text{ACTION}\ \equiv}) \longmapsto (V,\ \underbrace{\rho_V : G \to GL(V)}_{\equiv\ \text{GROUP HOMOM.}\ \equiv})$$

$$g \mapsto \begin{bmatrix} V \to V \\ v \mapsto g \triangleright v \end{bmatrix}$$

$$(gh) \triangleright v = g \triangleright (h \triangleright v)$$

$$\rho(gh)(v) = (gh) \triangleright v = g \triangleright (h \triangleright v)$$

$$\rho(g)\rho(h)(v) = g \triangleright (\rho(h)(v))$$

$$F': \mathrm{Rep}(G) \longrightarrow G\text{-Mod}$$

$$(\underset{\mathrm{Vec}}{\underset{\cap}{V}},\ \rho: G \to GL(V)) \longmapsto (V,\ \triangleright_V : G \times V \to V)$$

$$(g, v) \mapsto \rho_g(v)$$

CHECK $F'F = \mathrm{Id}_{G\text{-Mod}}$ & $FF' = \mathrm{Id}_{\mathrm{Rep}(G)}$

I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE SAID TO BE **ISOMORPHIC** IF $\exists$ FUNCTORS $F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ SUCH THAT

$$GF = \mathrm{Id}_{\mathcal{C}}$$
$$\& \; FG = \mathrm{Id}_{\mathcal{D}}$$

WRITE $\mathcal{C} \cong \mathcal{D}$

EXERCISE 2.30: SHOW –

$G\text{-Mod} \cong \mathrm{Rep}(G)$

$\cong \mathrm{Rep}(\Bbbk G) \quad \ni (V, \underbrace{\rho: \Bbbk G \to \mathrm{End}_{\Bbbk}(V)}_{\text{ALG. MORPHISM}})$

$\cong \Bbbk G\text{-Mod} \quad \ni (V, \triangleright: \Bbbk G \times V \to V)$

UPGRADE OF EXER. 1.13

EXAMPLE: G = GROUP

GET: $\boxed{G\text{-Mod} \cong \mathrm{Rep}(G)}$

$F: G\text{-Mod} \longrightarrow \mathrm{Rep}(G)$

$$(\underset{\mathrm{Vec}}{\underset{\cap}{V}}, \underbrace{\triangleright: G \times V \to V}_{\text{ACTION}}) \longmapsto (V, \underbrace{\rho_V: G \to GL(V)}_{\text{GROUP HOMOM.}})$$

$$g \mapsto \begin{bmatrix} V \to V \\ v \mapsto g \triangleright v \end{bmatrix}$$

$(gh) \triangleright v = g \triangleright (h \triangleright v)$

$$\rho(gh)(v) = (gh) \triangleright v = g \triangleright (h \triangleright v)$$
$$\rho(g)\rho(h)(v) = g \triangleright (\rho(h)(v))$$

$F': \mathrm{Rep}(G) \longrightarrow G\text{-Mod}$

$$(\underset{\mathrm{Vec}}{\underset{\cap}{V}}, \rho: G \to GL(V)) \longmapsto (V, \triangleright_V: G \times V \to V)$$
$$(g, v) \mapsto \rho_g(v)$$

CHECK $F'F = \mathrm{Id}_{G\text{-Mod}}$ & $FF' = \mathrm{Id}_{\mathrm{Rep}(G)}$

I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE ISOMORPHIC IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

s.t.

$GF = Id_{\mathcal{C}}$ & $FG = Id_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{A}$ = FULL SUBCATEGORY OF $\text{FdVec}_{\Bbbk}$

ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

PERHAPS FdVec & $\mathcal{A}$ ARE THE "SAME" AS

EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE ISOMORPHIC IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$ $GF = Id_{\mathcal{C}}$ & $FG = Id_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

CONSIDER FdVec $/ \Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\text{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

PERHAPS FdVec & $\mathcal{S}$ ARE THE "SAME" AS EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

TRY:

$F: \text{FdVec} \longrightarrow \mathcal{S}$ & $G: \mathcal{S} \longrightarrow \text{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **ISOMORPHIC** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ s.t.

$$GF = \mathrm{Id}_{\mathcal{C}} \ \& \ FG = \mathrm{Id}_{\mathcal{D}}$$

WRITE $\mathcal{C} \cong \mathcal{D}$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

PERHAPS FdVec & $\mathcal{S}$ ARE THE "SAME" AS EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

TRY:

$$F: \mathrm{FdVec} \longrightarrow \mathcal{S} \qquad \& \qquad G: \mathcal{S} \longrightarrow \mathrm{FdVec}$$

$$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V} \qquad\qquad \Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V}$

I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE ISOMORPHIC IF $\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ s.t. $GF = \mathrm{Id}_{\mathcal{C}}$ & $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

PERHAPS FdVec & $\mathcal{S}$ ARE THE "SAME" AS EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

TRY:

$$F : \mathrm{FdVec} \longrightarrow \mathcal{S} \qquad V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$$

&

$$G : \mathcal{S} \longrightarrow \mathrm{FdVec} \qquad \Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\therefore$ FdVec $\not\cong \mathcal{S}$

I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **ISOMORPHIC** IF

$\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$

$GF = \mathrm{Id}_{\mathcal{C}}$ & $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

PERHAPS FdVec & $\mathcal{S}$ ARE THE "SAME" AS EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

TRY:

$F: \mathrm{FdVec} \longrightarrow \mathcal{S}$ & $G: \mathcal{S} \longrightarrow \mathrm{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore$ FdVec $\ncong \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"...

I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **ISOMORPHIC** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ s.t. $GF = Id_{\mathcal{C}}$ & $FG = Id_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

SKELETON OF $\mathcal{C}$ = FULL SUBCATEGORY $Skel(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $Obj(\mathcal{C})$

CONSIDER FdVec /$\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $FdVec_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

PERHAPS FdVec & $\mathcal{S}$ ARE THE "SAME" AS EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

TRY:

$F: FdVec \longrightarrow \mathcal{S}$ & $G: \mathcal{S} \longrightarrow FdVec$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\quad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore$ FdVec $\ncong \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"...

I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **ISOMORPHIC** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ s.t. $GF = \mathrm{Id}_{\mathcal{C}}$ & $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

SKELETON OF $\mathcal{C}$ = FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

PERHAPS FdVec & $\mathcal{S}$ ARE THE "SAME" AS EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

TRY:

$F: \mathrm{FdVec} \longrightarrow \mathcal{S}$ & $G: \mathcal{S} \longrightarrow \mathrm{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore \mathrm{FdVec} \ncong \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"...

I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **ISOMORPHIC** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ s.t. $GF = \mathrm{Id}_{\mathcal{C}}$ & $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

SKELETON OF $\mathcal{C}$ = FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

CONSIDER FdVec / $\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S} = \mathrm{Skel}(\mathrm{FdVec})$

PERHAPS FdVec & $\mathcal{S}$ ARE THE "SAME" AS EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

TRY:

$F: \mathrm{FdVec} \longrightarrow \mathcal{S}$, $V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ & $G: \mathcal{S} \longrightarrow \mathrm{FdVec}$, $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

<u>BUT</u> $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore \mathrm{FdVec} \ncong \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"...

I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE ISOMORPHIC IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ s.t. $GF = \mathrm{Id}_{\mathcal{C}}$ & $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

SKELETON OF $\mathcal{C}$ = FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

EXER. 2.33

$\mathrm{Skel}(\mathcal{C}) \cong \mathcal{C} \Leftrightarrow \mathrm{Skel}(\mathcal{C}) = \mathcal{C}$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathsf{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S} = \mathrm{Skel}(\mathsf{FdVec})$

PERHAPS FdVec & $\mathcal{S}$ ARE THE "SAME" AS EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

TRY:

$F: \mathsf{FdVec} \longrightarrow \mathcal{S}$ & $G: \mathcal{S} \longrightarrow \mathsf{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore \mathsf{FdVec} \not\cong \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"...

I. ISOMORPHISM OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **ISOMORPHIC** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ s.t. $GF = \mathrm{Id}_{\mathcal{C}}$ & $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

SKELETON OF $\mathcal{C}$ = FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

EXER. 2.33

$\mathrm{Skel}(\mathcal{C}) \cong \mathcal{C} \Leftrightarrow \mathrm{Skel}(\mathcal{C}) = \mathcal{C}$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S}$ = Skel(FdVec)

PERHAPS FdVec & $\mathcal{S}$ ARE THE "SAME" AS EVERY F.D. VECTOR SPACE IS $\cong \Bbbk^{\oplus n}$ FOR SOME $n \in \mathbb{N}$.

TRY:

$F: \mathrm{FdVec} \longrightarrow \mathcal{S}$ & $G: \mathcal{S} \longrightarrow \mathrm{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\quad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore$ FdVec $\not\cong \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"... $\simeq$ EQUIV.

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE ISOMORPHIC IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$. $GF = \mathrm{Id}_{\mathcal{C}}$ & $FG = \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \cong \mathcal{D}$

SKELETON OF $\mathcal{C}$ = FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

EXER. 2.33

$\mathrm{Skel}(\mathcal{C}) \cong \mathcal{C} \iff \mathrm{Skel}(\mathcal{C}) = \mathcal{C}$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S} = \mathrm{Skel}(\mathrm{FdVec})$

$F: \mathrm{FdVec} \longrightarrow \mathcal{S}$ & $G: \mathcal{S} \longrightarrow \mathrm{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\quad\quad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore \mathrm{FdVec} \ncong \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"... EQUIV. $\simeq$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

SKELETON OF $\mathcal{C}$

= FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$

ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

EXER. 2.33

$\mathrm{Skel}(\mathcal{C}) \cong \mathcal{C} \iff \mathrm{Skel}(\mathcal{C}) = \mathcal{C}$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$

$\mathcal{S} = \mathrm{Skel}(\mathrm{FdVec})$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$F : \mathrm{FdVec} \longrightarrow \mathcal{S}$ & $G : \mathcal{S} \longrightarrow \mathrm{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore \mathrm{FdVec} \not\cong \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"... $\simeq$ EQUIV.

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$. $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{D} \end{tikzcd}$$

$G \equiv$ "QUASI-INVERSE" OF F

SKELETON OF $\mathcal{C}$ $\equiv$ FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

EXER. 2.33

$\mathrm{Skel}(\mathcal{C}) \cong \mathcal{C} \iff \mathrm{Skel}(\mathcal{C}) = \mathcal{C}$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S} = \mathrm{Skel}(\mathrm{FdVec})$

$F: \mathrm{FdVec} \longrightarrow \mathcal{S}$, $V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ & $G: \mathcal{S} \longrightarrow \mathrm{FdVec}$, $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$
$\cong V$

$\therefore \mathrm{FdVec} \ncong \mathcal{S}$

WILL WEAKEN NOTION OF "SAMENESS"... $\simeq$ EQUIV.

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$ $GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

SKELETON OF $\mathcal{C}$ = FULL SUBCATEGORY $Skel(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $Obj(\mathcal{C})$

EXER. 2.33

$Skel(\mathcal{C}) \simeq \mathcal{C} \Leftrightarrow Skel(\mathcal{C}) = \mathcal{C}$

CONSIDER FdVec / $\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $FdVec_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S} = Skel(FdVec)$

$F: FdVec \longrightarrow \mathcal{S}$ & $G: \mathcal{S} \longrightarrow FdVec$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore FdVec \neq \mathcal{S}$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$. $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

SKELETON OF $\mathcal{C}$ = FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

EXER. 2.33

$\mathrm{Skel}(\mathcal{C}) \cong \mathcal{C} \iff \mathrm{Skel}(\mathcal{C}) = \mathcal{C}$

CONSIDER FdVec $/\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S} = \mathrm{Skel}(\mathrm{FdVec})$

$F: \mathrm{FdVec} \longrightarrow \mathcal{S}$ & $G: \mathcal{S} \longrightarrow \mathrm{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

BUT $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \neq V$

$\cong V$

$\therefore \mathrm{FdVec} \not\cong \mathcal{S}$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ $\ni$. $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

SKELETON OF $\mathcal{C}$ $\equiv$ FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

EXER. 2.33

$\mathrm{Skel}(\mathcal{C}) \cong \mathcal{C} \iff \mathrm{Skel}(\mathcal{C}) = \mathcal{C}$

CONSIDER FdVec / $\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S}$ = Skel(FdVec)

$F : \mathrm{FdVec} \longrightarrow \mathcal{S}$, $V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ & $G : \mathcal{S} \longrightarrow \mathrm{FdVec}$, $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

& $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \cong V$

$\therefore$ FdVec $\simeq \mathcal{S}$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "GF"] \arrow[r, bend right=50, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "FG"] \arrow[r, bend right=50, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

SKELETON OF $\mathcal{C}$

= FULL SUBCATEGORY

$Skel(\mathcal{C})$ OF $\mathcal{C}$

ON ISOCLASSES OF $Obj(\mathcal{C})$

EXER. 2.33

$Skel(\mathcal{C}) \cong \mathcal{C} \iff Skel(\mathcal{C}) = \mathcal{C}$

CONSIDER $FdVec$ $\quad / \Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $FdVec_{\Bbbk}$

$\parallel$

$Skel(FdVec)$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$F : FdVec \longrightarrow \mathcal{S}$ & $G : \mathcal{S} \longrightarrow FdVec$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

& $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \cong V$

$\therefore FdVec \simeq \mathcal{S}$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ s.t. $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

SKELETON OF $\mathcal{C}$ = FULL SUBCATEGORY $\mathrm{Skel}(\mathcal{C})$ OF $\mathcal{C}$ ON ISOCLASSES OF $\mathrm{Obj}(\mathcal{C})$

EXER. 2.33

$\mathrm{Skel}(\mathcal{C}) \cong \mathcal{C} \Leftrightarrow \mathrm{Skel}(\mathcal{C}) = \mathcal{C}$

$\mathrm{Skel}(\mathcal{C}) \simeq \mathcal{C}$ ALWAYS

$\mathcal{C} \simeq \mathcal{D} \Leftrightarrow \mathrm{Skel}(\mathcal{C}) \cong \mathrm{Skel}(\mathcal{D})$

CONSIDER FdVec / $\Bbbk$ FIELD

TAKE $\mathcal{S}$ = FULL SUBCATEGORY OF $\mathrm{FdVec}_{\Bbbk}$ ON OBJECTS $\{\Bbbk^{\oplus n}\}_{n \in \mathbb{N}}$

$\mathcal{S}$ = Skel(FdVec)

$F: \mathrm{FdVec} \longrightarrow \mathcal{S}$ & $G: \mathcal{S} \longrightarrow \mathrm{FdVec}$

$V \mapsto \Bbbk^{\oplus \dim_{\Bbbk} V}$ $\qquad$ $\Bbbk^{\oplus n} \mapsto \Bbbk^{\oplus n}$

HERE, $FG(\Bbbk^{\oplus n}) = F(\Bbbk^{\oplus n}) = \Bbbk^{\oplus n}$.

& $GF(V) = G(\Bbbk^{\oplus \dim_{\Bbbk} V}) = \Bbbk^{\oplus \dim_{\Bbbk} V} \cong V$

$\therefore \mathrm{FdVec} \simeq \mathcal{S}$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

LIKE TWO STRUCTURES ARE THE SAME

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$, $\mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ s.t. $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

$\exists$ MUTUALLY INVERSE STRUCTURE MAPS BETWEEN THEM

$\Updownarrow$

LIKE TWO STRUCTURES ARE THE SAME

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

> $\mathcal{C}$ AND $\mathcal{D}$
> ARE **EQUIVALENT** IF
> $\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$
> $\ni$
> $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$
> WRITE $\mathcal{C} \simeq \mathcal{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

$\exists$ MUTUALLY INVERSE STRUCTURE MAPS BETWEEN THEM

$\Updownarrow$

LIKE TWO STRUCTURES ARE THE SAME

$\Updownarrow$

$\exists$ A BIJECTIVE STRUCTURE MAP FROM ONE TO THE OTHER

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}, \mathcal{D}$ CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ s.t. $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$\exists$ NATURAL ISOMORPHISMS:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

$\exists$ MUTUALLY INVERSE STRUCTURE MAPS BETWEEN THEM

$\Updownarrow$

LIKE TWO STRUCTURES ARE THE SAME

$\Updownarrow$

$\exists$ A BIJECTIVE STRUCTURE MAP FROM ONE TO THE OTHER

A CHARACTERIZATION IN TERMS OF...

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**: $\forall Y \in \mathcal{D}, \exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$

II. EQUIVALENCE OF CATEGORIES

$\mathscr{C}$ AND $\mathscr{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathscr{C} \to \mathscr{D}$ & $G : \mathscr{D} \to \mathscr{C}$

$\ni$.

$GF \cong \mathrm{Id}_{\mathscr{C}}$ & $FG \cong \mathrm{Id}_{\mathscr{D}}$

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathscr{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathscr{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathscr{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{D} \end{tikzcd}$$

WRITE $\mathscr{C} \simeq \mathscr{D}$

$$F_{X,Y} : \mathrm{Hom}_{\mathscr{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(F(X), F(Y))$$
$$g \longmapsto F(g)$$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X, Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X, Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X, Y$

F **ESS. SURJ**:

$\forall Y \in \mathscr{D}$, $\exists X \in \mathscr{C} \ni$. $Y \cong F(X)$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$

$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X, Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X, Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X, Y$

F **ESS. SURJ**:

$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni. Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$

$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:

$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni. Y \cong F(X)$

THEOREM $\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \mathcal{C} \to \mathcal{D}$

PF/ ($\Rightarrow$) SAY $\exists\, \phi: \mathrm{Id}_{\mathcal{C}} \overset{\sim}{\Rightarrow} GF$ & $\psi: FG \overset{\sim}{\Rightarrow} \mathrm{Id}_{\mathcal{D}}$ AS IN DEFN

CLAIM: F IS FULLY FAITHFUL AND ESS. SURJ.

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ $\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$$
$$g \longmapsto F(g)$$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:
$\forall Y \in \mathcal{D},\ \exists X \in \mathcal{C} \ni.\ Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

PF/ ($\Rightarrow$) SAY $\exists\, \phi : \mathrm{Id}_{\mathcal{C}} \overset{\sim}{\Rightarrow} GF$ & $\psi : FG \overset{\sim}{\Rightarrow} \mathrm{Id}_{\mathcal{D}}$ AS IN DEFN

CLAIM: F IS FULLY FAITHFUL AND ESS. SURJ.

TAKE $Y \in \mathcal{D}$. THEN $FG(Y) \xrightarrow[\sim]{\psi_Y} Y \Rightarrow F$ ESS. SURJ.

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$ ∋. $GF\cong Id_{\mathcal{C}}$ & $FG\cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

$F_{X,Y}: Hom_{\mathcal{C}}(X,Y)\longrightarrow Hom_{\mathcal{D}}(F(X),F(Y))$
$g\mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**: $\forall Y\in\mathcal{D},\ \exists X\in\mathcal{C}$ ∋. $Y\cong F(X)$

THEOREM $\mathcal{C}\simeq\mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ ($\Rightarrow$) SAY $\exists\,\phi: Id_{\mathcal{C}}\overset{\sim}{\Rightarrow} GF$ & $\psi: FG\overset{\sim}{\Rightarrow} Id_{\mathcal{D}}$ AS IN DEFN

CLAIM: F IS FULLY FAITHFUL AND ESS. SURJ.

TAKE $Y\in\mathcal{D}$. THEN $FG(Y)\overset{\psi_Y}{\underset{\sim}{\to}} Y \Rightarrow F$ ESS. SURJ.

NOW $\forall g: X\to X'\in\mathcal{C}$, GET

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\phi_X"', "\cong"] \arrow[dr, phantom, "2"] & X' \\ GF(X) \arrow[r, "GF(g)"'] & GF(X') \arrow[u, "\phi_{X'}^{-1}"', "\cong"] \end{tikzcd}$$

THIS IMPLIES F IS FAITHFUL

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$ ∋.

$GF\cong \mathrm{Id}_{\mathcal{C}}$ & $FG\cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y)\longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g\longmapsto F(g)$

F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$

F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$

F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$

F ESS. SURJ: $\forall Y\in\mathcal{D},\ \exists X\in\mathcal{C}$ ∋. $Y\cong F(X)$

THEOREM

$\mathcal{C}\simeq\mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ $(\Rightarrow)$ SAY $\exists\, \phi: \mathrm{Id}_{\mathcal{C}} \overset{\sim}{\Rightarrow} GF$ & $\psi: FG \overset{\sim}{\Rightarrow} \mathrm{Id}_{\mathcal{D}}$ AS IN DEFN

CLAIM: F IS FULLY FAITHFUL AND ESS. SURJ.

TAKE $Y\in\mathcal{D}$. THEN $FG(Y)\overset{\psi_Y}{\underset{\sim}{\to}} Y \Rightarrow F$ ESS. SURJ.

NOW $\forall g: X\to X'\in\mathcal{C}$, GET

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\phi_X"', "\cong"] \arrow[dr, phantom, "2"] & X' \\ GF(X) \arrow[r, "GF(g) \overset{\text{IF}}{=} GF(\tilde{g})"'] & GF(X') \arrow[u, "\phi_{X'}^{-1}"', "\cong"] \end{tikzcd}$$

THIS IMPLIES F IS FAITHFUL

$\left[\ F(g)=F(\tilde{g}) \Rightarrow GF(g)=GF(\tilde{g})\ \right]$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$ ∋. $GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "GF"] \arrow[r, bend right=50, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "FG"] \arrow[r, bend right=50, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(F(X),F(Y))$

$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:

$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C}$ ∋. $Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/($\Rightarrow$) SAY $\exists\, \phi: Id_{\mathcal{C}} \overset{\sim}{\Rightarrow} GF$ & $\psi: FG \overset{\sim}{\Rightarrow} Id_{\mathcal{D}}$ AS IN DEFN

CLAIM: F IS FULLY FAITHFUL AND ESS. SURJ.

TAKE $Y \in \mathcal{D}$. THEN $FG(Y) \xrightarrow[\sim]{\psi_Y} Y \Rightarrow F$ ESS. SURJ.

NOW $\forall g: X \to X' \in \mathcal{C}$, GET

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\phi_X"', "\cong"] \arrow[dr, phantom, "2"] & X' \\ GF(X) \arrow[r, "GF(g)"'] & GF(X') \arrow[u, "\phi_{X'}^{-1}"', "\cong"] \end{tikzcd}$$

THEN $g = \tilde{g}$

$GF(g) \overset{\text{IF}}{=} GF(\tilde{g})$

THIS IMPLIES F IS FAITHFUL

$$\left[\begin{array}{l} F(g) = F(\tilde{g}) \Rightarrow GF(g) = GF(\tilde{g}) \\ \qquad\qquad\quad \Rightarrow g = \tilde{g} \end{array}\right]$$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

∋. $GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "GF"] \arrow[r, bend right=60, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=60, "FG"] \arrow[r, bend right=60, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(F(X),F(Y))$

$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:

$\forall Y \in \mathcal{D}$, $\exists X \in \mathcal{C}$ ∋. $Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff$ ∃ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \mathcal{C} \to \mathcal{D}$

PF/ (⇒) SAY $\exists\, \phi: Id_{\mathcal{C}} \xRightarrow{\sim} GF$ & $\psi: FG \xRightarrow{\sim} Id_{\mathcal{D}}$ AS IN DEFN

CLAIM: F IS FULLY FAITHFUL AND ESS. SURJ.

TAKE $Y \in \mathcal{D}$. THEN $FG(Y) \xrightarrow[\sim]{\psi_Y} Y$ ⇒ F ESS. SURJ.

NOW $\forall\, g: X \to X' \in \mathcal{C}$, GET

THEN $g = \tilde{g}$

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\phi_X"', "\cong"] \arrow[dr, phantom, "2"] & X' \\ GF(X) \arrow[r, "GF(g)"'] & GF(X') \arrow[u, "\phi_{X'}^{-1}"', "\cong"] \end{tikzcd}$$

$GF(g) = GF(\tilde{g})$ IF

THIS IMPLIES F IS FAITHFUL

$$\left[\begin{array}{l} F(g) = F(\tilde{g}) \Rightarrow GF(g) = GF(\tilde{g}) \\ \qquad\qquad\quad \Rightarrow g = \tilde{g} \end{array} \right]$$

SWAPPING ϕ WITH ψ ⇒ G IS FAITHFUL.

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ $\ni$. $GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y} : Hom_{\mathcal{C}}(X,Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$
$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X, Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X, Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X, Y$

F **ESS. SURJ**:
$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni. Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

PF/ ($\Rightarrow$) SAY $\exists \phi : Id_{\mathcal{C}} \overset{\sim}{\Rightarrow} GF$ & $\psi : FG \overset{\sim}{\Rightarrow} Id_{\mathcal{D}}$ AS IN DEFN

CLAIM: F IS FULLY FAITHFUL AND ESS. SURJ.

TAKE $Y \in \mathcal{D}$. THEN $FG(Y) \xrightarrow[\sim]{\psi_Y} Y \Rightarrow$ F ESS. SURJ.

NOW $\forall g : X \to X' \in \mathcal{C}$, GET

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\phi_X"', "\cong"] \arrow[dr, phantom, "2"] & X' \\ GF(X) \arrow[r, "GF(g)"'] & GF(X') \arrow[u, "\phi_{X'}^{-1}"', "\cong"] \end{tikzcd}$$

THEN $g = \tilde{g}$ IF $GF(g) = GF(\tilde{g})$

THIS IMPLIES F IS FAITHFUL

$\left[F(g) = F(\tilde{g}) \Rightarrow GF(g) = GF(\tilde{g}) \Rightarrow g = \tilde{g} \right]$

SWAPPING ϕ WITH $\psi \Rightarrow$ G IS FAITHFUL.

TAKE $h : F(X) \to F(X') \in \mathcal{D}$.

BUILD $g : X \xrightarrow{\phi_X} GF(X) \xrightarrow{G(h)} GF(X') \xrightarrow{\phi_{X'}^{-1}} X' \in \mathcal{C}$.

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$ s.t. $GF\cong \mathrm{Id}_{\mathcal{C}}$ & $FG\cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

$F_{X,Y}:\mathrm{Hom}_{\mathcal{C}}(X,Y)\longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g\mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**: $\forall Y\in\mathcal{D},\ \exists X\in\mathcal{C}$ s.t. $Y\cong F(X)$

THEOREM $\mathcal{C}\simeq\mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ ($\Rightarrow$) SAY $\exists\, \phi:\mathrm{Id}_{\mathcal{C}}\overset{\sim}{\Rightarrow} GF$ & $\psi: FG\overset{\sim}{\Rightarrow}\mathrm{Id}_{\mathcal{D}}$ AS IN DEFN

CLAIM: F IS FULLY FAITHFUL AND ESS. SURJ.

TAKE $Y\in\mathcal{D}$. THEN $FG(Y)\xrightarrow[\sim]{\psi_Y} Y \Rightarrow F$ ESS. SURJ.

NOW $\forall\, g: X\to X'\in\mathcal{C}$, GET

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\phi_X"', "\cong"] \arrow[dr, phantom, "2"] & X' \\ GF(X) \arrow[r] & GF(X') \arrow[u, "\phi_{X'}^{-1}"', "\cong"] \end{tikzcd}$$

THEN $g = \tilde{g}$

$GF(g) \overset{\text{IF}}{=} GF(\tilde{g})$

THIS IMPLIES F IS FAITHFUL

$$\left[\begin{array}{l} F(g)=F(\tilde g) \Rightarrow GF(g)=GF(\tilde g) \\ \qquad\qquad\quad\ \Rightarrow g=\tilde g \end{array}\right]$$

SWAPPING ϕ WITH ψ $\Rightarrow$ G IS FAITHFUL.

TAKE $h: F(X)\to F(X')\in\mathcal{D}$.

BUILD $g: X\xrightarrow{\phi_X} GF(X)\xrightarrow{G(h)} GF(X')\xrightarrow{\phi_{X'}^{-1}} X'\in\mathcal{C}$.

THEN $GF(g)=G(h)$. G FAITHFUL $\Rightarrow F(g)=h$

& F IS FULL.

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ $\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \longmapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**: $\forall Y \in \mathcal{D}$, $\exists X \in \mathcal{C}$ $\ni.$ $Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

PF/ ($\Rightarrow$) SAY $\exists\, \phi : \mathrm{Id}_{\mathcal{C}} \overset{\sim}{\Rightarrow} GF$ & $\psi : FG \overset{\sim}{\Rightarrow} \mathrm{Id}_{\mathcal{D}}$ AS IN DEFN

CLAIM: F IS FULLY FAITHFUL AND ESS. SURJ.

TAKE $Y \in \mathcal{D}$. THEN $FG(Y) \xrightarrow[\sim]{\psi_Y} Y$ $\Rightarrow$ F ESS. SURJ.

NOW $\forall g : X \to X' \in \mathcal{C}$, GET

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\phi_X"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & X' \\ GF(X) \arrow[r, "GF(g)"'] & GF(X') \arrow[u, "\phi_{X'}^{-1}"', "\cong"] \end{tikzcd}$$

THEN $g = \tilde{g}$

GF(g) $\parallel$ IF GF($\tilde{g}$)

THIS IMPLIES F IS FAITHFUL

$\left[F(g) = F(\tilde{g}) \Rightarrow GF(g) = GF(\tilde{g}) \Rightarrow g = \tilde{g} \right]$

SWAPPING ϕ WITH ψ $\Rightarrow$ G IS FAITHFUL.

TAKE $h : F(X) \to F(X') \in \mathcal{D}$.

BUILD $g : X \xrightarrow{\phi_X} GF(X) \xrightarrow{G(h)} GF(X') \xrightarrow{\phi_{X'}^{-1}} X' \in \mathcal{C}$.

THEN $GF(g) = G(h)$. G FAITHFUL $\Rightarrow$ $F(g) = h$

& F IS FULL. //

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists\, F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$

$g \mapsto F(g)$

F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$

F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$

F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$

F ESS. SURJ:

$\forall Y \in \mathcal{D},\ \exists X \in \mathcal{C} \ni.\ Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ ($\Leftarrow$) TAKE $F:\mathcal{C}\to\mathcal{D}$ FULLY FAITHFUL, ESS. SURJ.

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$ ∋.

$$GF \cong \mathrm{Id}_{\mathcal{C}} \;\&\; FG \cong \mathrm{Id}_{\mathcal{D}}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \;\&\; \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:
$\forall Y \in \mathcal{D},\ \exists X \in \mathcal{C}$ ∋. $Y \cong F(X)$

THEOREM $\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ ($\Leftarrow$) TAKE $F:\mathcal{C}\to\mathcal{D}$ FULLY FAITHFUL, ESS. SURJ.

$$F \text{ ESS. SURJ} \Rightarrow \forall Y \in \mathcal{D}\ \ \exists \underset{\substack{\| \\ G(Y)}}{z_Y} \in \mathcal{C} \ni.\ F(z_Y) \underset{\substack{\uparrow \\ =:\,\psi_Y}}{\cong} Y$$

LABELLING:

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

∋.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$$
$$g \mapsto F(g)$$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:

$\forall Y \in \mathcal{D},\ \exists X \in \mathcal{C}$ ∋. $Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ ($\Leftarrow$) TAKE $F:\mathcal{C}\to\mathcal{D}$ FULLY FAITHFUL, ESS. SURJ.

F ESS. SURJ $\Rightarrow \forall Y \in \mathcal{D}\ \ \exists \underbrace{Z_Y}_{=: G(Y)} \in \mathcal{C}$ ∋. $F(Z_Y) \underbrace{\cong}_{=: \psi_Y} Y$

F FULLY FAITHFUL $\Rightarrow \forall\ g: Y \to Y' \in \mathcal{D}$

$\exists!$ MORPHISM $G(Y) \underbrace{\to}_{=: G(g)} G(Y') \in \mathcal{C}$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

s.t.

$GF\cong Id_{\mathcal{C}}$ & $FG\cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

$F_{X,Y}:Hom_{\mathcal{C}}(X,Y)\longrightarrow Hom_{\mathcal{D}}(F(X),F(Y))$
$g\mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:
$\forall Y\in\mathcal{D},\ \exists X\in\mathcal{C}$ s.t. $Y\cong F(X)$

THEOREM

$\mathcal{C}\simeq\mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ ($\Leftarrow$) TAKE $F:\mathcal{C}\to\mathcal{D}$ FULLY FAITHFUL, ESS. SURJ.

F ESS. SURJ $\Rightarrow \forall Y\in\mathcal{D}\ \ \exists \underset{\substack{\| \\ G(Y)}}{z_Y}\in\mathcal{C}$ s.t. $F(z_Y)\underset{\substack{\uparrow \\ =:\psi_Y}}{\cong} Y$

F FULLY FAITHFUL $\Rightarrow \forall\ g:Y\to Y'\in\mathcal{D}$
$\exists!$ MORPHISM $G(Y)\underset{\substack{\| \\ G(g)}}{\to} G(Y')\in\mathcal{C}$

NEED TO SHOW

- ASSIGNMENTS $\begin{array}{l} Y\mapsto G(Y) \\ g\mapsto G(g)\end{array}$ MAKE A FUNCTOR $G:\mathcal{D}\to\mathcal{C}$
- $\exists$ NATURAL ISOM $\psi: FG\Rightarrow Id_{\mathcal{D}}$
- $\exists$ NATURAL ISOM $\phi: Id_{\mathcal{C}}\Rightarrow GF$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:
$\forall Y \in \mathcal{D}$, $\exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

PF/ ($\Leftarrow$) TAKE $F:\mathcal{C}\to\mathcal{D}$ FULLY FAITHFUL, ESS. SURJ.

F ESS. SURJ $\Rightarrow \forall Y \in \mathcal{D}$ $\exists \underbrace{Z_Y}_{G(Y)} \in \mathcal{C}$ s.t. $F(Z_Y) \underset{=:\psi_Y}{\cong} Y$

F FULLY FAITHFUL $\Rightarrow \forall g: Y \to Y' \in \mathcal{D}$
$\exists!$ MORPHISM $G(Y) \underset{G(g)}{\longrightarrow} G(Y') \in \mathcal{C}$

NEED TO SHOW

- ASSIGNMENTS $Y \mapsto G(Y)$, $g \mapsto G(g)$ MAKE A FUNCTOR $G:\mathcal{D}\to\mathcal{C}$
  F FAITHFUL
- $\exists$ NATURAL ISOM $\psi: FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$
- $\exists$ NATURAL ISOM $\phi: \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$\qquad g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:
$\qquad \forall Y \in \mathcal{D}, \exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

PF/ ($\Leftarrow$) TAKE $F : \mathcal{C} \to \mathcal{D}$ FULLY FAITHFUL, ESS. SURJ.

F ESS. SURJ $\Rightarrow \forall Y \in \mathcal{D} \;\; \exists \underset{\substack{\| \\ G(Y)}}{Z_Y} \in \mathcal{C}$ s.t. $F(Z_Y) \underset{\substack{\uparrow \\ =: \psi_Y}}{\cong} Y$

F FULLY FAITHFUL $\Rightarrow \forall\, g : Y \to Y' \in \mathcal{D}$
$\qquad \exists!$ MORPHISM $G(Y) \underset{\substack{\| \\ G(g)}}{\to} G(Y') \in \mathcal{C}$

NEED TO SHOW

- ASSIGNMENTS $\begin{array}{l} Y \mapsto G(Y) \\ g \mapsto G(g) \end{array}$ MAKE A FUNCTOR $G : \mathcal{D} \to \mathcal{C}$
  $\quad$ F FAITHFUL
- $\exists$ NATURAL ISOM $\psi : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ WITH COMPONENTS ψ_Y
- $\exists$ NATURAL ISOM $\phi : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$

II. EQUIVALENCE OF CATEGORIES

$\mathscr{C}$ AND $\mathscr{D}$ ARE **EQUIVALENT** IF

$\exists F:\mathscr{C}\to\mathscr{D}$ & $G:\mathscr{D}\to\mathscr{C}$ ∋.

$GF\cong \mathrm{Id}_{\mathscr{C}}$ & $FG\cong \mathrm{Id}_{\mathscr{D}}$

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathscr{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathscr{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathscr{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{D} \end{tikzcd}$$

WRITE $\mathscr{C}\simeq\mathscr{D}$

$F_{X,Y}:\mathrm{Hom}_{\mathscr{C}}(X,Y)\longrightarrow \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y))$
$g\mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**: $\forall Y\in\mathscr{D},\ \exists X\in\mathscr{C}$ ∋. $Y\cong F(X)$

THEOREM
$\mathscr{C}\simeq\mathscr{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathscr{C}\to\mathscr{D}$

PF/ ($\Leftarrow$) TAKE $F:\mathscr{C}\to\mathscr{D}$ FULLY FAITHFUL, ESS. SURJ.

F ESS. SURJ $\Rightarrow \forall Y\in\mathscr{D}\ \exists \underset{\substack{\shortparallel\\ G(Y)}}{Z_Y}\in\mathscr{C}$ ∋. $F(Z_Y)\underset{=:\,\psi_Y}{\cong} Y$

F FULLY FAITHFUL $\Rightarrow \forall\ g: Y\to Y'\in\mathscr{D}$
$\exists!$ MORPHISM $G(Y)\underset{\substack{\shortparallel\\ G(g)}}{\longrightarrow} G(Y')\in\mathscr{C}$

NEED TO SHOW

- ASSIGNMENTS $Y\mapsto G(Y)$, $g\mapsto G(g)$ MAKE A FUNCTOR $G:\mathscr{D}\to\mathscr{C}$
  F FAITHFUL
- $\exists$ NATURAL ISOM $\psi: FG\Rightarrow \mathrm{Id}_{\mathscr{D}}$ WITH COMPONENTS ψ_Y
- $\exists$ NATURAL ISOM $\phi:\mathrm{Id}_{\mathscr{C}}\Rightarrow GF$
  USING COMPONENTS $\psi^{-1}_{F(X)}: F(X)\longrightarrow FGF(X)$ & F FULLY FAITHFUL

II. EQUIVALENCE OF CATEGORIES

DETAILS ≡ EXER 2.34

$\mathscr{C}$ AND $\mathscr{D}$ ARE EQUIVALENT IF

$\exists F:\mathscr{C}\to\mathscr{D}$ & $G:\mathscr{D}\to\mathscr{C}$ $\ni$.

$GF \cong \mathrm{Id}_{\mathscr{C}}$ & $FG \cong \mathrm{Id}_{\mathscr{D}}$

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathscr{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathscr{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathscr{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{D} \end{tikzcd}$$

WRITE $\mathscr{C} \simeq \mathscr{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathscr{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y))$
$\qquad g \mapsto F(g)$

F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$

F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$

F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$

F ESS. SURJ: $\forall Y\in\mathscr{D}$, $\exists X\in\mathscr{C}$ $\ni$. $Y\cong F(X)$

THEOREM $\mathscr{C}\simeq\mathscr{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathscr{C}\to\mathscr{D}$

PF/ ($\Leftarrow$) TAKE $F:\mathscr{C}\to\mathscr{D}$ FULLY FAITHFUL, ESS. SURJ.

F ESS. SURJ $\Rightarrow$ $\forall Y\in\mathscr{D}$ $\exists Z_Y\in\mathscr{C}$ $\ni$. $F(Z_Y)\cong Y$

$Z_Y \overset{!!}{=} G(Y)$, $\quad F(Z_Y)\xrightarrow{\cong} Y =: \psi_Y$

F FULLY FAITHFUL $\Rightarrow$ $\forall$ $g: Y\to Y' \in\mathscr{D}$
$\exists!$ MORPHISM $G(Y)\to G(Y')\in\mathscr{C}$ $\overset{!!}{=} G(g)$

NEED TO SHOW

- ASSIGNMENTS $Y\mapsto G(Y)$, $g\mapsto G(g)$ MAKE A FUNCTOR $G:\mathscr{D}\to\mathscr{C}$
  ↰ F FAITHFUL
- $\exists$ NATURAL ISOM $\psi: FG \Rightarrow \mathrm{Id}_{\mathscr{D}}$ WITH COMPONENTS ψ_Y
- $\exists$ NATURAL ISOM $\phi: \mathrm{Id}_{\mathscr{C}} \Rightarrow GF$
  USING COMPONENTS $\psi^{-1}_{F(X)}: F(X)\to FGF(X)$ & F FULLY FAITHFUL ///

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "GF"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "FG"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$$
$$g \longmapsto F(g)$$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:
$\forall Y \in \mathcal{D}$, $\exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$

THEOREM

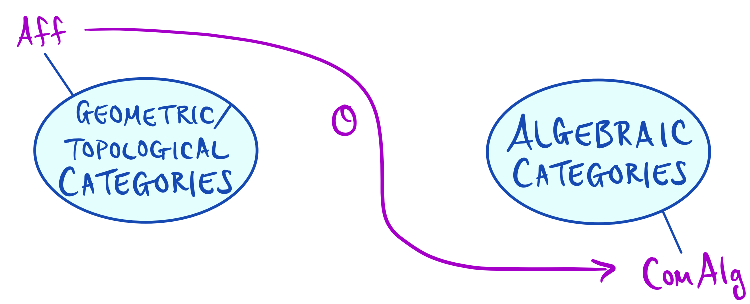
$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

TOWARD EXAMPLES –

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO 0

$$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$$
$$X \longmapsto \mathcal{O}(X) = \frac{\mathbb{C}[x_1,\dots,x_n]}{\left(\text{IDEAL OF POLY'LS DEFINING } X\right)}$$

AFFINE VARIETY — COORDINATE ALGEBRA OF X



II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

∋.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \longmapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:
$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C}$ ∋. $Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

TOWARD EXAMPLES –

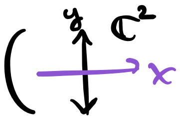
$\mathcal{O} : \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

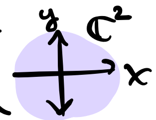
$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1,\dots,x_n]}{\left(\begin{array}{c}\text{IDEAL OF} \\ \text{POLY'LS} \\ \text{DEFINING } X\end{array}\right)}$

AFFINE VARIETY — COORDINATE ALGEBRA OF X

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO 0

Ex. $n=2$

$\mathcal{O}($  $) = \dfrac{\mathbb{C}[x,y]}{(y)} \cong \mathbb{C}[x]$

$\mathcal{O}($  $) = \mathbb{C}[x,y]$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ s.t.

$$GF \cong \mathrm{Id}_{\mathcal{C}} \ \& \ FG \cong \mathrm{Id}_{\mathcal{D}}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**: $\forall Y \in \mathcal{D}, \exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \mathcal{C} \to \mathcal{D}$

TOWARD EXAMPLES –

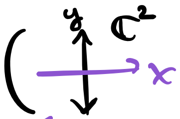
$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1, \dots, x_n]}{(\text{IDEAL OF POLY'LS DEFINING } X)}$

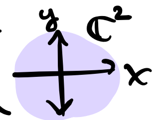
X: AFFINE VARIETY — SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1, \dots, x_n]$ EQUAL TO 0

$\mathcal{O}(X)$: COORDINATE ALGEBRA OF X

Ex. $n=2$

$\mathcal{O}($  $) = \dfrac{\mathbb{C}[x,y]}{(y)} \cong \mathbb{C}[x]$

INCLUSION

$\mathcal{O}($  $) = \mathbb{C}[x,y]$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$

$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:

$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni. Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \mathcal{C} \to \mathcal{D}$

TOWARD EXAMPLES —

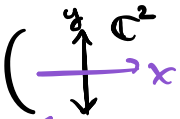
SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO 0

$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

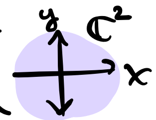
$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1,\dots,x_n]}{(\text{IDEAL OF POLY'LS DEFINING } X)}$

AFFINE VARIETY — COORDINATE ALGEBRA OF X

Ex. $n=2$

$\mathcal{O}\Big($  $\Big) = \dfrac{\mathbb{C}[x,y]}{(y)} \cong \mathbb{C}[x]$

INCLUSION

$\mathcal{O}\Big($  $\Big) = \mathbb{C}[x,y]$

PROJECTION

II. EQUIVALENCE OF CATEGORIES

$\mathscr{C}$ AND $\mathscr{D}$

ARE **EQUIVALENT** IF

$\exists F: \mathscr{C} \to \mathscr{D}$ & $G: \mathscr{D} \to \mathscr{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathscr{C}}$ & $FG \cong \mathrm{Id}_{\mathscr{D}}$

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathscr{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathscr{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathscr{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{D} \end{tikzcd}$$

WRITE $\mathscr{C} \simeq \mathscr{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y))$
$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**:
$\forall Y \in \mathscr{D}$, $\exists X \in \mathscr{C}$ s.t. $Y \cong F(X)$

THEOREM
$\mathscr{C} \simeq \mathscr{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \mathscr{C} \to \mathscr{D}$

≡ CONTRAVARIANT ≡

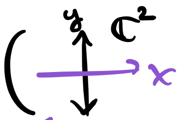
$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1, \dots, x_n]}{(\text{IDEAL OF POLY'LS DEFINING } X)}$

X: AFFINE VARIETY — SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1, \dots, x_n]$ EQUAL TO 0

$\mathcal{O}(X)$: COORDINATE ALGEBRA OF X

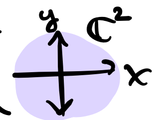
Ex. $n=2$

$\mathcal{O}($

$) = \dfrac{\mathbb{C}[x,y]}{(y)} \cong \mathbb{C}[x]$

INCLUSION

PROJECTION

$\mathcal{O}($

$) = \mathbb{C}[x,y]$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$

$g \longmapsto F(g)$

F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$

F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$

F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$

F ESS. SURJ:

$\forall Y \in \mathcal{D}, \exists X \in \mathcal{C} \ni. Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff$ $\exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \mathcal{C} \to \mathcal{D}$

= CONTRAVARIANT =

$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1, \dots, x_n]}{\left(\text{IDEAL OF POLY'LS DEFINING } X\right)}$

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO 0

AFFINE VARIETY

COORDINATE ALGEBRA OF X

$$\mathrm{Aff}_{\mathbb{C}} \simeq (\quad \dots \mathrm{ComAlg}_{\mathbb{C}})^{op}$$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE EQUIVALENT IF

$\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \mapsto F(g)$

F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$

F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$

F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$

F ESS. SURJ: $\forall Y \in \mathcal{D}, \exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \mathcal{C} \to \mathcal{D}$

≡ CONTRAVARIANT ≡

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO 0

$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1,\dots,x_n]}{(\text{IDEAL OF POLY'LS DEFINING } X)}$

AFFINE VARIETY — COORDINATE ALGEBRA OF X

$$\mathrm{Aff}_{\mathbb{C}} \simeq (\ \ldots \mathrm{ComAlg}_{\mathbb{C}})^{op}$$

THESE TYPES OF ALGEBRAS ARE FINITELY GENERATED

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$

$g \mapsto F(g)$

F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$

F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$

F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$

F ESS. SURJ: $\forall Y \in \mathcal{D}, \exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \mathcal{C} \to \mathcal{D}$

≡ CONTRAVARIANT ≡

$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1, \dots, x_n]}{(\text{IDEAL OF POLY'LS DEFINING } X)}$

AFFINE VARIETY — COORDINATE ALGEBRA OF X

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1, \dots, x_n]$ EQUAL TO 0

$$\mathrm{Aff}_{\mathbb{C}} \simeq (\mathrm{Fg} \dots \mathrm{ComAlg}_{\mathbb{C}})^{op}$$

THESE TYPES OF ALGEBRAS ARE FINITELY GENERATED

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

$\ni.$

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(F(X),F(Y))$
$g \mapsto F(g)$

F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$

F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$

F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$

F ESS. SURJ:
$\forall Y \in \mathcal{D}$, $\exists X \in \mathcal{C} \ni Y \cong F(X)$

THEOREM

$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

= CONTRAVARIANT =

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO 0

$\mathcal{O}: Aff_{\mathbb{C}} \longrightarrow ComAlg_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \mathbb{C}[x_1,\dots,x_n] \big/ \left(\text{IDEAL OF POLY'LS DEFINING } X\right)$

AFFINE VARIETY

COORDINATE ALGEBRA OF X

$$Aff_{\mathbb{C}} \simeq (Fg \dots ComAlg_{\mathbb{C}})^{op}$$

THESE TYPES OF ALGEBRAS ARE FINITELY GENERATED

IN $\mathbb{C}^2_{(x,y)}$:

THE SHAPE $x=y=0$ (ORIGIN)

||

THE SHAPE $x^2=y=0$ (ORIGIN)

II. EQUIVALENCE OF CATEGORIES

$\mathscr{C}$ AND $\mathscr{D}$

ARE EQUIVALENT IF

$\exists F: \mathscr{C} \to \mathscr{D}$ & $G: \mathscr{D} \to \mathscr{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathscr{C}}$ & $FG \cong \mathrm{Id}_{\mathscr{D}}$

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathscr{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathscr{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathscr{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{D} \end{tikzcd}$$

WRITE $\mathscr{C} \simeq \mathscr{D}$

$F_{X,Y}: \mathrm{Hom}_{\mathscr{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(F(X),F(Y))$
$g \mapsto F(g)$

F FAITHFUL: $F_{X,Y}$ INJ. $\forall X,Y$

F FULL: $F_{X,Y}$ SURJ. $\forall X,Y$

F FULLY FAITHFUL: $F_{X,Y}$ BIJ. $\forall X,Y$

F ESS. SURJ: $\forall Y \in \mathscr{D}, \exists X \in \mathscr{C}$ s.t. $Y \cong F(X)$

THEOREM
$\mathscr{C} \simeq \mathscr{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \mathscr{C} \to \mathscr{D}$

≡ CONTRAVARIANT ≡

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO 0

$\mathcal{O}: \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \mathbb{C}[x_1,\dots,x_n] \Big/ \left(\begin{smallmatrix}\text{IDEAL OF}\\ \text{POLY'LS}\\ \text{DEFINING } X\end{smallmatrix}\right)$

AFFINE VARIETY — COORDINATE ALGEBRA OF X

$$\mathrm{Aff}_{\mathbb{C}} \simeq (\mathrm{Fg} \dots \mathrm{ComAlg}_{\mathbb{C}})^{op}$$

THESE TYPES OF ALGEBRAS ARE FINITELY GENERATED

IN $\mathbb{C}^2_{(x,y)}$:

THE SHAPE $x = y = 0$ (ORIGIN) $\overset{\text{BUT}}{}$ $\mathbb{C}[x,y]/(x,y)$

$\|$ $\qquad\qquad\qquad\qquad$ $\not\cong$

THE SHAPE $x^2 = y = 0$ (ORIGIN) $\qquad$ $\mathbb{C}[x,y]/(x^2,y)$ AS ALGS

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

s.t.

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: Hom_{\mathcal{C}}(X,Y) \longrightarrow Hom_{\mathcal{D}}(F(X),F(Y))$

$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**: $\forall Y \in \mathcal{D}, \exists X \in \mathcal{C}$ s.t. $Y \cong F(X)$

THEOREM $\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F:\mathcal{C}\to\mathcal{D}$

≡ CONTRAVARIANT ≡

$\mathcal{O}: Aff_{\mathbb{C}} \longrightarrow ComAlg_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \mathbb{C}[x_1,\dots,x_n] \Big/ \left(\text{IDEAL OF POLY'LS DEFINING } X\right)$

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO 0

AFFINE VARIETY

COORDINATE ALGEBRA OF X

$$Aff_{\mathbb{C}} \simeq (Fg\,\underbrace{Red}_{\text{"REDUCED"}}\,ComAlg_{\mathbb{C}})^{op}$$

THESE TYPES OF ALGEBRAS ARE FINITELY GENERATED

IN $\mathbb{C}^2_{(x,y)}$:

THE SHAPE $x=y=0$ (ORIGIN)

$\|$

THE SHAPE $x^2=y=0$ (ORIGIN)

BUT $\mathbb{C}[x,y]/(x,y)$

$\not\cong$

$\mathbb{C}[x,y]/(x^2,y)$ AS ALGS

MAKES THIS CHOICE TO CORRESP. TO ORIGIN

NO NILPOTENT ELEMENTS

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF

$\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ ∋.

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y}: Hom_{\mathcal{C}}(X,Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$
$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X, Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X, Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X, Y$

F **ESS. SURJ**:
$\forall Y \in \mathcal{D}$, $\exists X \in \mathcal{C}$ ∋. $Y \cong F(X)$

THEOREM
$\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F: \mathcal{C} \to \mathcal{D}$

≡ CONTRAVARIANT ≡

$\mathcal{O}: Aff_{\mathbb{C}} \longrightarrow ComAlg_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1, \dots, x_n]}{\left(\text{IDEAL OF POLY'LS DEFINING } X\right)}$

AFFINE VARIETY — COORDINATE ALGEBRA OF X

SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1, \dots, x_n]$ EQUAL TO 0

$$Aff_{\mathbb{C}} \simeq (FgRedComAlg_{\mathbb{C}})^{op}$$

ALSO HAVE

$$\text{SOMETHING ELSE GEOMETRIC}_{\mathbb{C}} \simeq (ComAlg_{\mathbb{C}})^{op}$$

II. EQUIVALENCE OF CATEGORIES

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF

$\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ $\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$
$g \mapsto F(g)$

F **FAITHFUL**: $F_{X,Y}$ INJ. $\forall X,Y$

F **FULL**: $F_{X,Y}$ SURJ. $\forall X,Y$

F **FULLY FAITHFUL**: $F_{X,Y}$ BIJ. $\forall X,Y$

F **ESS. SURJ**: $\forall Y \in \mathcal{D}$, $\exists X \in \mathcal{C}$ $\ni.$ $Y \cong F(X)$

THEOREM $\mathcal{C} \simeq \mathcal{D} \iff \exists$ FULLY FAITHFUL, ESS. SURJECTIVE FUNCTOR $F : \mathcal{C} \to \mathcal{D}$

= CONTRAVARIANT =

$\mathcal{O} : \mathrm{Aff}_{\mathbb{C}} \longrightarrow \mathrm{ComAlg}_{\mathbb{C}}$

$X \longmapsto \mathcal{O}(X) = \dfrac{\mathbb{C}[x_1,\dots,x_n]}{\left(\text{IDEAL OF POLY'LS DEFINING } X\right)}$

X: AFFINE VARIETY — SHAPE IN $\mathbb{C}^n$ CUT OUT BY SETTING POLY'LS IN $\mathbb{C}[x_1,\dots,x_n]$ EQUAL TO 0

$\mathcal{O}(X)$: COORDINATE ALGEBRA OF X

$$\mathrm{Aff}_{\mathbb{C}} \simeq \left(\mathrm{FgRedComAlg}_{\mathbb{C}}\right)^{op}$$

ALSO HAVE

BEYOND SCOPE OF LECTURE

$$\mathrm{Scheme}_{\mathbb{C}} \simeq \left(\mathrm{ComAlg}_{\mathbb{C}}\right)^{op}$$

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

NOTION OF SAMENESS FOR $\Bbbk$-ALGEBRAS

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

NOTION OF SAMENESS FOR $\Bbbk$-ALGEBRAS

$A = B$

EQUALITY OF ALGEBRAS

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

NOTION OF SAMENESS FOR $\Bbbk$-ALGEBRAS

$A = B$

EQUALITY OF ALGEBRAS

WEAKEN

$A \cong B$

ISOMORPHISM OF ALGEBRAS

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF

$\exists\, F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

NOTION OF SAMENESS FOR $\Bbbk$-ALGEBRAS

$A = B$

EQUALITY OF ALGEBRAS

WEAKEN

$A \cong B$

ISOMORPHISM OF ALGEBRAS

WEAKEN

$A \underset{\text{MORITA}}{\sim} B$

"EQUIVALENCE" OF ALGEBRAS

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists\, F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS MORITA EQUIV. TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\Longleftrightarrow$ $\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni.$

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni.$

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

PF/ $(\Leftarrow)$

TAKE $F := Q \otimes_A - : A\text{-Mod} \longrightarrow B\text{-Mod}$

$G := P \otimes_B - : B\text{-Mod} \longrightarrow A\text{-Mod}$

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni$

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

PF/ ($\Leftarrow$)

TAKE $F := Q \otimes_A - : A\text{-Mod} \longrightarrow B\text{-Mod}$

$G := P \otimes_B - : B\text{-Mod} \longrightarrow A\text{-Mod}$

NOW $\forall M \in A\text{-Mod}$, GET:

$$GF(M) = G(Q \otimes_A M) = P \otimes_B (Q \otimes_A M)$$

$$\cong (P \otimes_B Q) \otimes_A M$$

$\boxed{\text{MODIFY EXER. 1.18b}}$

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF A-Mod $\simeq$ B-Mod.

THAT IS, A & B HAVE THE SAME REP THY

<u>MORITA'S THEOREM</u> A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni.$

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

PF/ ($\Longleftarrow$)

TAKE $F := Q \otimes_A - : A\text{-Mod} \longrightarrow B\text{-Mod}$

$G := P \otimes_B - : B\text{-Mod} \longrightarrow A\text{-Mod}$

NOW $\forall\, M \in A\text{-Mod}$, GET:

$$GF(M) = G(Q \otimes_A M) = P \otimes_B (Q \otimes_A M)$$

$$\cong (P \otimes_B Q) \otimes_A M \underset{\text{HYPOTHESIS}}{\cong} A_{reg} \otimes_A M \underset{\text{EXER. 1.18a}}{\cong} M.$$

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists\, F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$ $\ni$.

$$GF\cong \mathrm{Id}_{\mathcal{C}} \quad \& \quad FG\cong \mathrm{Id}_{\mathcal{D}}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod}\simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\iff$ $\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$.

$P\otimes_B Q\cong A_{reg}$ AS A-BIMODULES

& $Q\otimes_A P\cong B_{reg}$ AS B-BIMODULES.

PF/ ($\Longleftarrow$)

TAKE $F:=Q\otimes_A - : A\text{-Mod}\longrightarrow B\text{-Mod}$

$G:=P\otimes_B - : B\text{-Mod}\longrightarrow A\text{-Mod}$

NOW $\forall\, M\in A\text{-Mod}$, GET:

$$GF(M)=G(Q\otimes_A M)=P\otimes_B(Q\otimes_A M)$$

$$\cong (P\otimes_B Q)\otimes_A M\cong A_{reg}\otimes_A M\cong M.$$

$\therefore\ GF\cong \mathrm{Id}_{A\text{-Mod}}$.

LIKEWISE, $FG\cong \mathrm{Id}_{B\text{-Mod}}$.

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni$.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "GF"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "FG"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

PF/ ($\Rightarrow$) GIVEN EQUIVALENCE $F : A\text{-Mod} \to B\text{-Mod}$

TAKE $Q := F({}_A A_{reg}) \in B\text{-Mod}$

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

$\ni$.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

PF/ ($\Rightarrow$) GIVEN EQUIVALENCE $F: A\text{-Mod} \to B\text{-Mod}$

TAKE $Q := F({}_A A_{reg}) \in B\text{-Mod}$

GET
$$\begin{aligned} A^{op} &\cong \mathrm{End}_{A\text{-Mod}}({}_A A_{reg}) \\ &\cong \mathrm{End}_{B\text{-Mod}}(F({}_A A_{reg})) \\ &\cong \mathrm{End}_{B\text{-Mod}}({}_B Q) \end{aligned}$$

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ s.t. $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

PF/ ($\Rightarrow$) GIVEN EQUIVALENCE $F : A\text{-Mod} \to B\text{-Mod}$

TAKE $Q := F({}_A A_{reg}) \in B\text{-Mod}$

GET $A^{op} \cong \mathrm{End}_{A\text{-Mod}}({}_A A_{reg})$

EXER. 1.26

$\cong \mathrm{End}_{B\text{-Mod}}(F({}_A A_{reg}))$

F FULLY FAITHFUL

$\cong \mathrm{End}_{B\text{-Mod}}({}_B Q)$

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF

$\exists\, F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ ∋.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ ∋.

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

PF/ ($\Rightarrow$) GIVEN EQUIVALENCE $F: A\text{-Mod} \to B\text{-Mod}$

TAKE $Q := F({}_A A_{reg}) \in B\text{-Mod}$

GET $A^{op} \cong \mathrm{End}_{A\text{-Mod}}({}_A A_{reg})$

$\cong \mathrm{End}_{B\text{-Mod}}(F({}_A A_{reg}))$

f

$\cong \mathrm{End}_{B\text{-Mod}}({}_B Q)$

DEFINE $q \triangleleft a := f(a)(q)$

$\rightsquigarrow {}_B Q \in (B, A)\text{-Bimod}$

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists\, F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

s.t.

$GF\cong \mathrm{Id}_{\mathcal{C}}$ & $FG\cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS MORITA EQUIV. TO B IF $A\text{-Mod}\simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

<u>MORITA'S THEOREM</u> A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

$P\otimes_B Q\cong A_{reg}$ AS A-BIMODULES

& $Q\otimes_A P\cong B_{reg}$ AS B-BIMODULES.

PF/ ($\Rightarrow$) GIVEN EQUIVALENCE $F: A\text{-Mod}\to B\text{-Mod}$

HAVE $Q := F({}_A A_{reg}) \in (B,A)\text{-Bimod}$

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists\, F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$. $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$. $P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES & $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

PF/ ($\Rightarrow$) GIVEN EQUIVALENCE $F: A\text{-Mod} \to B\text{-Mod}$

HAVE $Q := F({}_AA_{reg}) \in (B,A)\text{-Bimod}$

CLAIM: $F \cong Q \otimes_A -$ AS FUNCTORS

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$ $\ni$ $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B $\iff$ $\exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ $\ni$ $P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES & $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

PF/ ($\Rightarrow$) GIVEN EQUIVALENCE $F: A\text{-Mod} \to B\text{-Mod}$

HAVE $Q := F({}_A A_{reg}) \in (B,A)\text{-Bimod}$

CLAIM : $F \cong Q \otimes_A -$ AS FUNCTORS

PF/ TAKE $M \in A\text{-Mod}$ & GET ISO:

$$\sigma_X : X \cong \mathrm{Hom}_{A\text{-Mod}}(A, X) \xrightarrow[\cong]{F} \mathrm{Hom}_{B\text{-Mod}}(F(A), F(X))$$
$$\| $$
$$\mathrm{Hom}_{B\text{-Mod}}(Q, F(X))$$

F FULLY FAITHFUL

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS MORITA EQUIV. TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\Longleftrightarrow$ $\exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ $\ni.$

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

PF/ ($\Longrightarrow$) GIVEN EQUIVALENCE $F : A\text{-Mod} \to B\text{-Mod}$

HAVE $Q := F({}_A A_{reg}) \in (B,A)\text{-Bimod}$

CLAIM: $F \cong Q \otimes_A -$ AS FUNCTORS

PF/ TAKE $M \in A\text{-Mod}$ & GET ISO:

$$\sigma_X : X \cong \mathrm{Hom}_{A\text{-Mod}}(A, X) \xrightarrow{F} \mathrm{Hom}_{B\text{-Mod}}(F(A), F(X)) = \mathrm{Hom}_{B\text{-Mod}}(Q, F(X))$$

TENSOR-HOM ADJUNCTION

$$\mathrm{Hom}_{B\text{-Mod}}(Q \otimes_A X, Y) \cong \mathrm{Hom}_{A\text{-Mod}}(X, \mathrm{Hom}_{B\text{-Mod}}(Q, Y))$$

GET ISO: $\sigma'_X : Q \otimes_A X \longrightarrow F(X)$

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

<u>MORITA'S THEOREM</u> A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ $\ni.$

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

PF/ ($\Rightarrow$) GIVEN EQUIVALENCE $F: A\text{-Mod} \to B\text{-Mod}$

HAVE $Q := F({}_A A_{reg}) \in (B,A)\text{-Bimod}$

<u>CLAIM</u>: $F \cong Q \otimes_B -$ AS FUNCTORS

PF/ TAKE $M \in A\text{-Mod}$ & GET ISO:

$$\sigma_X : X \cong \mathrm{Hom}_{A\text{-Mod}}(A, X) \xrightarrow{F} \mathrm{Hom}_{B\text{-Mod}}(F(A), F(X)) = \mathrm{Hom}_{B\text{-Mod}}(Q, F(X))$$

TENSOR-HOM ADJUNCTION

$$\mathrm{Hom}_{B\text{-Mod}}(Q \otimes_A X, Y) \cong \mathrm{Hom}_{A\text{-Mod}}(X, \mathrm{Hom}_{B\text{-Mod}}(Q, Y))$$

GET ISO: $\sigma'_X : Q \otimes_A X \to F(X) \rightsquigarrow Q \otimes_A - \overset{\sigma}{\underset{\sim}{\Rightarrow}} F$

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$
$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\Longleftrightarrow$ $\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni.$

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

PF/ ($\Longrightarrow$) GIVEN EQUIVALENCE $F: A\text{-Mod} \to B\text{-Mod}$

HAVE $Q := F({}_A A_{reg}) \in (B,A)\text{-Bimod}$

HAVE $F \cong Q \otimes_A -$ AS FUNCTORS

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

s.t.

$GF\cong \mathrm{Id}_{\mathcal{C}}$ & $FG\cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod}\simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

$P\otimes_B Q\cong A_{reg}$ AS A-BIMODULES

& $Q\otimes_A P\cong B_{reg}$ AS B-BIMODULES.

PF/ ($\Longrightarrow$) GIVEN EQUIVALENCE $F: A\text{-Mod}\to B\text{-Mod}$

HAVE $Q := F({}_A A_{reg}) \in (B,A)\text{-Bimod}$

HAVE $F\cong Q\otimes_A -$ AS FUNCTORS

NOW $\exists\, G: B\text{-Mod}\to A\text{-Mod}$ WITH

$\phi: \mathrm{Id}_{A\text{-Mod}} \overset{\sim}{\Rightarrow} GF$ & $\psi: FG \overset{\sim}{\Rightarrow} \mathrm{Id}_{B\text{-Mod}}$

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE EQUIVALENT IF $\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ $\ni$ $GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS MORITA EQUIV. TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

PF/ ($\Rightarrow$) GIVEN EQUIVALENCE $F : A\text{-Mod} \to B\text{-Mod}$

HAVE $Q := F({}_A A_{reg}) \in (B,A)\text{-Bimod}$

HAVE $F \cong Q \otimes_A -$ AS FUNCTORS

NOW $\exists\, G : B\text{-Mod} \to A\text{-Mod}$ WITH

$\phi : \mathrm{Id}_{A\text{-mod}} \overset{\sim}{\Rightarrow} GF$ & $\psi : FG \overset{\sim}{\Rightarrow} \mathrm{Id}_{B\text{-Mod}}$

HAVE $P := G({}_B B_{reg}) \in (A,B)\text{-Bimod}$

HAVE $G \cong P \otimes_B -$ AS FUNCTORS

III. MORITA EQUIVALENCE

$\mathscr{C}$ AND $\mathscr{D}$

ARE **EQUIVALENT** IF

$\exists\, F: \mathscr{C} \to \mathscr{D}$ & $G: \mathscr{D} \to \mathscr{C}$ $\ni$.

$GF \cong \mathrm{Id}_{\mathscr{C}}$ & $FG \cong \mathrm{Id}_{\mathscr{D}}$

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathscr{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathscr{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathscr{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathscr{D} \end{tikzcd}$$

WRITE $\mathscr{C} \simeq \mathscr{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

PF/ ($\Rightarrow$) GIVEN EQUIVALENCE $F: A\text{-Mod} \to B\text{-Mod}$

HAVE $Q := F({}_A A_{reg}) \in (B,A)\text{-Bimod}$

HAVE $F \cong Q \otimes_A -$ AS FUNCTORS

NOW $\exists\, G: B\text{-Mod} \to A\text{-Mod}$ WITH

$\phi: \mathrm{Id}_{A\text{-Mod}} \overset{\sim}{\Rightarrow} GF$ & $\psi: FG \overset{\sim}{\Rightarrow} \mathrm{Id}_{B\text{-Mod}}$

HAVE $P := G({}_B B_{reg}) \in (A,B)\text{-Bimod}$

HAVE $G \cong P \otimes_B -$ AS FUNCTORS

GET $\phi_A : A \xrightarrow{\sim} GF(A) \cong P \otimes_B Q$ AS A-BIMODS

& $\psi_B^{-1} : B \xrightarrow{\sim} FG(B) \cong Q \otimes_A P$ AS B-BIMODS //

III. MORITA EQUIVALENCE

DETAILS = EXER. 2.35

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F:\mathcal{C}\to\mathcal{D}$ & $G:\mathcal{D}\to\mathcal{C}$

∋.

$GF\cong Id_{\mathcal{C}}$ & $FG\cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C}\simeq\mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod}\simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ ∋.

$P\otimes_B Q\cong A_{reg}$ AS A-BIMODULES

& $Q\otimes_A P\cong B_{reg}$ AS B-BIMODULES.

PF/ ($\Rightarrow$) GIVEN EQUIVALENCE $F: A\text{-Mod}\to B\text{-Mod}$

HAVE $Q:=F({}_A A_{reg})\in(B,A)\text{-Bimod}$

HAVE $F\cong Q\otimes_A -$ AS FUNCTORS

NOW $\exists\, G: B\text{-Mod}\to A\text{-Mod}$ WITH

$\phi: Id_{A\text{-Mod}}\xRightarrow{\sim} GF$ & $\psi: FG\xRightarrow{\sim} Id_{B\text{-Mod}}$

HAVE $P:=G({}_B B_{reg})\in(A,B)\text{-Bimod}$

HAVE $G\cong P\otimes_B -$ AS FUNCTORS

GET $\phi_A: A\xrightarrow{\sim} GF(A)\cong P\otimes_B Q$ AS A-BIMODS

& $\psi_B^{-1}: B\xrightarrow{\sim} FG(B)\cong Q\otimes_A P$ AS B-BIMODS ///

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

∋.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "GF"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "FG"] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS MORITA EQUIV. TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\Longleftrightarrow$ $\exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ ∋.

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

MAIN EXAMPLE

A IS MORITA EQUIVALENT TO $\mathrm{Mat}_n(A)$

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "GF"] \arrow[r, bend right=60, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow \sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=60, "FG"] \arrow[r, bend right=60, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow \sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni.$

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

MAIN EXAMPLE

A IS MORITA EQUIVALENT TO $Mat_n(A)$

VIA

$P = \{(a_1, \dots, a_n) \mid a_i \in A\} \in (A, Mat_n(A))$-Bimod

$$Q = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \middle| \; a_i \in A \right\} \in (Mat_n(A), A)\text{-Bimod}$$

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ $\ni$.

$GF \cong Id_{\mathcal{C}}$ & $FG \cong Id_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "Id_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "Id_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

MAIN EXAMPLE

A IS MORITA EQUIVALENT TO $Mat_n(A)$

VIA

$P = \{(a_1, \dots, a_n) \mid a_i \in A\} \in (A, Mat_n(A))\text{-Bimod}$

$a' \triangleright (a_1, \dots, a_n) := (a'a_1, \dots, a'a_n)$

$\underbrace{(a_1, \dots, a_n)}_{\underline{a}} \triangleleft A' := \underline{a}\, A'$ MATRIX MULTIP'N

$Q = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \middle| \; a_i \in A \right\} \in (Mat_n(A), A)\text{-Bimod}$

LIKEWISE

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE EQUIVALENT IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

s.t.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\mathbb{k}$-ALGS A & B

A IS MORITA EQUIV. TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ s.t.

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

MAIN EXAMPLE

A IS MORITA EQUIVALENT TO $\mathrm{Mat}_n(A)$

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni$.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

MAIN EXAMPLE

A IS MORITA EQUIVALENT TO $\mathrm{Mat}_n(A)$

EXER. 2.38 $A \sim_{MOR} B \implies Z(A) \cong Z(B)$

CENTER OF A

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$ ARE **EQUIVALENT** IF $\exists\, F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$ $\ni$.

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "GF"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "FG"] \arrow[r, bend right, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\sim"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

<u>MORITA'S THEOREM</u> A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_A P_B$ & ${}_B Q_A$ $\ni$.

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

<u>MAIN EXAMPLE</u>

A IS MORITA EQUIVALENT TO $\mathrm{Mat}_n(A)$

FOR INSTANCE:

$\Bbbk \sim_{\mathrm{MOR}} \mathrm{Mat}_n(\Bbbk)$ & $Z(\mathrm{Mat}_n(\Bbbk)) \cong \Bbbk$

<u>EXER. 2.38</u> $A \sim_{\mathrm{MOR}} B \implies Z(A) \cong Z(B)$

CENTER OF A

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists\, F: \mathcal{C} \to \mathcal{D}$ & $G: \mathcal{D} \to \mathcal{C}$

$\ni$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "GF"] \arrow[r, bend right=60, "\mathrm{Id}_{\mathcal{C}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{C} \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=60, "FG"] \arrow[r, bend right=60, "\mathrm{Id}_{\mathcal{D}}"'] \arrow[r, phantom, "\Downarrow\wr"] & \mathcal{D} \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

<u>MORITA'S THEOREM</u> A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni$

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

<u>MAIN EXAMPLE</u>

A IS MORITA EQUIVALENT TO $\mathrm{Mat}_n(A)$

FOR INSTANCE:

$\Bbbk \sim_{MOR} \mathrm{Mat}_n(\Bbbk)$ & $Z(\mathrm{Mat}_n(\Bbbk)) \cong \Bbbk$

<u>EXER. 2.38</u> $A \sim_{MOR} B \;\underset{\not\Leftarrow}{\Longrightarrow}\; Z(A) \cong Z(B)$

IN GENERAL

CENTER OF A

III. MORITA EQUIVALENCE

$\mathcal{C}$ AND $\mathcal{D}$

ARE **EQUIVALENT** IF

$\exists F : \mathcal{C} \to \mathcal{D}$ & $G : \mathcal{D} \to \mathcal{C}$

$\ni.$

$GF \cong \mathrm{Id}_{\mathcal{C}}$ & $FG \cong \mathrm{Id}_{\mathcal{D}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "GF"{name=U}] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{C}}"'{name=D}] & \mathcal{C} \arrow[from=U, to=D, Rightarrow, "\sim"] \end{tikzcd} \quad \& \quad \begin{tikzcd} \mathcal{D} \arrow[r, bend left=50, "FG"{name=U}] \arrow[r, bend right=50, "\mathrm{Id}_{\mathcal{D}}"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\sim"] \end{tikzcd}$$

WRITE $\mathcal{C} \simeq \mathcal{D}$

TAKE $\Bbbk$-ALGS A & B

A IS **MORITA EQUIV.** TO B IF $A\text{-Mod} \simeq B\text{-Mod}$.

THAT IS, A & B HAVE THE SAME REP THY

MORITA'S THEOREM A IS MORITA EQUIVALENT TO B

$\iff \exists$ BIMODULES ${}_AP_B$ & ${}_BQ_A$ $\ni.$

$P \otimes_B Q \cong A_{reg}$ AS A-BIMODULES

& $Q \otimes_A P \cong B_{reg}$ AS B-BIMODULES.

MAIN EXAMPLE

A IS MORITA EQUIVALENT TO $\mathrm{Mat}_n(A)$

FOR INSTANCE:

$\Bbbk \sim_{MOR} \mathrm{Mat}_n(\Bbbk)$ & $Z(\mathrm{Mat}_n(\Bbbk)) \cong \Bbbk$

EXER. 2.38 $A \sim_{MOR} B$ $\underset{\not\Leftarrow}{\Longrightarrow}$ $Z(A) \cong Z(B)$

IN GENERAL CENTER OF A

FOR COMM. ALGS C, C' $\quad C \sim_{MOR} C' \iff Z(C) \cong Z(C')$

≡ SUMMARY ≡

NOTION OF SAMENESS
FOR $\Bbbk$-ALGEBRAS

$A = B$
EQUALITY
OF ALGEBRAS

WEAKEN

$A \cong B$
ISOMORPHISM
OF ALGEBRAS

WEAKEN

$A \underset{\text{MORITA}}{\sim} B$
"EQUIVALENCE"
OF ALGEBRAS

≡ SUMMARY ≡

NOTION OF SAMENESS FOR CATEGORIES	NOTION OF SAMENESS FOR $\Bbbk$-ALGEBRAS
$\mathcal{C} = \mathcal{D}$ EQUALITY OF CATEGORIES	$A = B$ EQUALITY OF ALGEBRAS
WEAKEN	WEAKEN
$\mathcal{C} \cong \mathcal{D}$ ISOMORPHISM OF CATEGORIES	$A \cong B$ ISOMORPHISM OF ALGEBRAS
WEAKEN	WEAKEN
$\mathcal{C} \simeq \mathcal{D}$ EQUIVALENCE OF CATEGORIES	$A \underset{\text{MORITA}}{\sim} B$ "EQUIVALENCE" OF ALGEBRAS

Summary

Notion of sameness for categories

$\mathcal{C} = \mathcal{D}$

EQUALITY OF CATEGORIES

WEAKEN

$\mathcal{C} \cong \mathcal{D}$

ISOMORPHISM OF CATEGORIES

WEAKEN

$\mathcal{C} \simeq \mathcal{D}$

EQUIVALENCE OF CATEGORIES

WEAKEN

NEXT TIME: ADJUNCTION

Notion of sameness for $\Bbbk$-algebras

$A = B$

EQUALITY OF ALGEBRAS

WEAKEN

$A \cong B$

ISOMORPHISM OF ALGEBRAS

WEAKEN

$A \underset{\text{Morita}}{\sim} B$

"EQUIVALENCE" OF ALGEBRAS

MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #9

TOPICS:

I. ISOMORPHISM OF CATEGORIES (§2.4.1)

II. EQUIVALENCE OF CATEGORIES (§§2.4.2–2.4.3)

III. MORITA EQUIVALENCE (§2.4.3)

NEXT TIME: ADJUNCTION

Lecture #9 keywords: equivalence of categories, isomorphism of categories, Morita equivalence of algebras, Morita's Theorem, skeleton