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Symmetries in Algebra

Symmetry is one of the oldest mathematical concepts, and we encountered this notion even as young children. What I aim to do in this talk is axiomatize the concept of symmetry, and illustrate it, using algebraic structures...

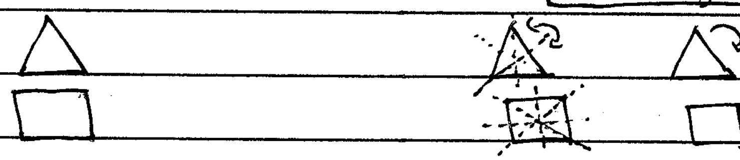
I. Classic Symmetry: captured by groups (set, associative binary operation, identity elt, inverse elts)

Examples: | X object | Collection of symmetries of X |

① regular n-gon, $n \geq 3$. — flips & rotations $\rightsquigarrow$ $\boxed{\text{dihedral group } D_{2n} \text{ of order } 2n}$

n=3

n=4



$120°, 240°, 0°$

$90°, 180°, 270°, 0°$

② collection of m-objects, $m \geq 1$ — permutations of these objects

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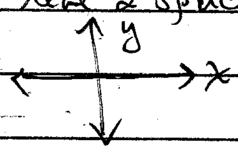
$\rightsquigarrow$ $\boxed{\text{symmetric group } S_m \text{ of order } m!}$

③ $\mathbb{R}^2$, real 2-space



[taking "linear symmetries" those that send lines to lines]

combination of dilations & rotations

$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ $\quad$ $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto R_\theta \begin{pmatrix} x \\ y \end{pmatrix}$

$\alpha \in \mathbb{R} \setminus \{0\}$ stretch by factor of α $\quad$ $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ rotate by θ radians cw

$\rightsquigarrow$ $\boxed{\text{infinite group of invertible } 2\times 2 \text{ matrices: } GL_2(\mathbb{R})}$ $\underbrace{\qquad}_{\text{general linear group}}$

Definition Given an object X, a symmetry of X is a property-preserving transformation from X to itself. The collection of symmetries of X forms a group Sym(X), with operation = composition, identity = "do nothing".

II. Linear Symmetries of Algebras,
[elaborating on example (3)]

Defn: An $\mathbb{R}$-algebra is an $\mathbb{R}$-vector space that admits the structure of a ring, in a compatible fashion.

- $\mathbb{R}$-vector space = (Abelian grp V, scalar mult'n map $*: \mathbb{R}\times V \longrightarrow V$)
- Ring = (Abelian group V, multiplication map $m: V\times V \longrightarrow V$)

$\rightsquigarrow$ $\mathbb{R}$-algebra = (Abelian group V, $*$, m)

Example "Coordinate Algebras" $\mathcal{O}(X)$ of geometric "$\mathbb{R}$-linear objects" X

Ex. $X=\mathbb{R}^2 \rightsquigarrow \mathcal{O}(X)=\mathbb{R}[x,y]$

Ex. $X=\mathbb{R}^n \rightsquigarrow \mathcal{O}(X)=\mathbb{R}[x_1,\dots,x_n]$, in general

Ex. line $y=x$ in $\mathbb{R}^2 \rightsquigarrow \mathcal{O}(X) = \mathbb{R}[x,y]/(\underset{\substack{\uparrow \\ \text{forcing } y=x}}{y-x}) \cong \mathbb{R}[x] = \underset{\substack{\wr\wr \\ \mathcal{O}(y\text{-axis})}}{\mathcal{O}(x\text{-axis})}$

(beginnings of algebraic geometry)

Linear Symmetries of $\mathbb{R}$-algebras via Example.

$\mathbb{R}[x,y] = \dfrac{\mathbb{R}\langle x,y\rangle}{(yx-xy)}$ ← free algebra: elts = $\mathbb{R}$-linear combinations of words in x & y
← yx gets replaced by xy

Linear symmetries of $\mathbb{R}[x,y]$ amount to $\begin{matrix} x \longmapsto \alpha x+\beta y \\ y\longmapsto \gamma x+\delta y\end{matrix}$, $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in GL_2(\mathbb{R})$

that send $\mathbb{R}\langle x,y\rangle$ to itself (automatic ✓)

& sends the 1-dimensional relation space $\mathbb{R}(yx-xy)$ to itself

have to check: $yx-xy \longmapsto (\gamma x+\delta y)(\alpha x+\beta y)-(\alpha x+\beta y)(\gamma x+\delta y)$

$= \underbrace{(\beta\gamma-\alpha\delta)}_{\in \mathbb{R}\setminus\{0\}}(yx-xy) \in \mathbb{R}(yx-xy)$ ✓

$\therefore$ $\mathrm{Sym}_{\text{linear}}(\underbrace{\mathbb{R}[x,y]}_{\mathcal{O}(\mathbb{R}^2)}) = GL_2(\mathbb{R}) = \mathrm{Sym}_{\text{linear}}(\mathbb{R}^2)$.

↑ no coincidence

Let's $\overbrace{\text{deform}}$ $\mathbb{R}[x,y]$ — keep $\mathbb{R}$-vs structure the same, alter multiplication (ring structure)

$\mathbb{R}_q[x,y] = \dfrac{\mathbb{R}\langle x,y\rangle}{(yx-qxy)}$ "skew polynomial alg" "q-polynomial alg." $=$ "$\mathcal{O}(\underbrace{\mathbb{R}_q^2}_{\uparrow})$" "quantum $\mathbb{R}^2$"

$q \in \mathbb{R}\setminus\{0\}$ $\quad(yx-qxy) \leftarrow$ yx gets replaced with qxy

Linear symmetries of $\mathbb{R}_q[x,y]$ amount to $\begin{array}{l} x \longmapsto \alpha x+\beta y \\ y \longmapsto \gamma x + \delta y \end{array}$, $\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in GL_2(\mathbb{R})$

that send relation space $\mathbb{R}(yx-qxy)$ to itself:

Check: $yx - qxy \longmapsto (\gamma x+\delta y)(\alpha x+\beta y) - q(\alpha x+\beta y)(\gamma x+\delta y)$

$$= (1-q)\alpha\gamma\, x^2 + (\beta\gamma - q\alpha\delta)xy + (\alpha\delta - q\beta\gamma)yx + (1-q)\beta\delta\, y^2$$

$$\in \mathbb{R}(yx-qxy) \iff \begin{cases} q=1 \\ q=-1, \ \alpha=\delta=0 \text{ or } \beta=\gamma=0 \\ q\neq \pm 1, \ \beta=\gamma=0. \end{cases}$$

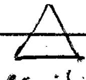
$$\therefore \ \mathrm{Sym}_{\text{linear}}(\mathbb{R}_q[x,y]) = \begin{cases} GL_2(\mathbb{R}), & q=1 \\ \text{anti/diag. matrices}, & q=-1 \\ \text{diag. matrices}, & q\neq \pm 1 \end{cases}$$

Even though $\mathbb{R}[x,y] = \mathcal{O}(\mathbb{R}^2)$ $\underline{\underline{\text{deforms nicely}}}$ to $\mathbb{R}_q[x,y] = \mathcal{O}(\mathbb{R}_q^2)$ $(q\neq 0,1)$,

$\mathrm{Sym}_{\text{linear}}(\mathbb{R}[x,y]) = GL_2(\mathbb{R})$ $\underline{\underline{\text{drops abruptly}}}$ to subgrp of anti/diag. matrices (for $q \neq 0,1$).

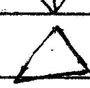
<u>Need framework to handle symmetries of small alterations of objects/algs</u>

Even for triangles, get



equilateral — D_6 dihedral grp of order 6

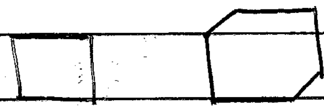
—pull a corner out ever so slightly— $\Downarrow$ $\quad\Downarrow$



scalene — $\langle e \rangle$ trivial group — Dramatic drop in symmetries —

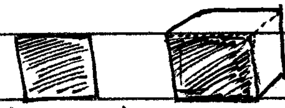
III. Broadening perspective: symmetries beyond groups.

Sometimes there's more to objects than what we can see, so need a broader view in studying their symmetries.



2-D snapshots of objects

doesn't look like symmetries should be same



(front view) (side view)

3-D snapshots of objects

Same object ⇝ same symmetries

Analogy: 2-D snapshot is to groups [capturing classical syms of algebras]

-as- 3D snapshot is to "Hopf algebras" [" quantum " " "]

↑ generalization of a group

Definition An $\mathbb{R}$-Hopf algebra is an $\mathbb{R}$-algebra (V $\mathbb{R}$-vs, $m: V \otimes V \longrightarrow V$ multiplication map) ($\otimes = \otimes_{\mathbb{R}}$.)

that comes equipped with a "comultiplication map" $\Delta: V \longrightarrow V \otimes V$ ← pulls elts apart

and an "antipode" map $S: V \longrightarrow V$. ← plays the role of "inverse"

(also has "unit" and "counit" maps; see literature for full defn)

Ex ① Take G a finite group. Form $\boxed{\mathbb{R}\text{-group algebra } \mathbb{R}G} = \bigoplus_{g \in G} \mathbb{R}g$ as $\mathbb{R}$-vs.

with mult'n determined by $m(g \otimes h) = gh$ (gp operation)

comult'n " " $\Delta(g) = g \otimes g$

antipode " " $S(g) = g^{-1}$.

② Take G an "algebraic group" = "variety" (← zero set of poly'ls in $\mathbb{R}[x_1, \dots, x_n]$) that has structure of a group.

$\mathcal{O}(G)$ = coordinate algebra of G is a Hopf algebra.

Ex. $G = GL_2(\mathbb{R})$: group ✓ variety: poly'ls in $\mathbb{R}[\alpha, \beta, \gamma, \delta, t]$ such that $(\alpha\delta - \beta\gamma)t - 1 = 0$

algebraic group.

(determinant of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is invertible)

In fact $\mathcal{O}(GL_2(\mathbb{R})) = \mathbb{R}[\alpha,\beta,\gamma,\delta,t] / ((\alpha\delta-\beta\gamma)t-1)$ as an $\mathbb{R}$-alg.

& formulas for Δ, S are explicitly known to make it a Hopf alg.

$[\Delta(\alpha) = \alpha\otimes\alpha + \beta\otimes\gamma,\ S(\alpha)=\delta,\ \dots]$

Advantage of using Hopf algebras over group: can deform the former (not the latter)

Ex. For $q \in \mathbb{R}\setminus\{0\}$, $\exists$ Hopf algebra "$\mathcal{O}_q(GL_2(\mathbb{R}))$"

that is a deformation of $\mathcal{O}(GL_2(\mathbb{R}))$

$\mathbb{R}$-vs structure the same; m, Δ, S altered, involves q

[Relation space: $(\alpha\delta - q\beta\gamma)t-1$ & $\alpha,\beta,\gamma,\delta$ "q-commute"]

Putting this together:

"classical limit $q\to 1$"

Classic Setting	$\xrightarrow{\text{deformation}}$	Quantum Setting
real 2-space $\mathbb{R}^2$	$\longrightarrow$	$\mathbb{R}^2_q$ quantum real 2-space
coordinate alg $\mathbb{R}[x,y]$	$\longrightarrow$	$\mathbb{R}_q[x,y]$ coordinate alg.

linear symmetries via group action $GL_2(\mathbb{R}) \times \mathbb{R}[x,y] \to \mathbb{R}[x,y]$ ....> —doesn't deform well here—

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix} \mapsto \begin{pmatrix}\alpha x+\beta y\\ \gamma x+\delta y\end{pmatrix} = A\begin{pmatrix}x\\y\end{pmatrix}$$

for $A = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$

dual view

linear symmetries via Hopf alg. coaction $\mathbb{R}[x,y] \to \mathbb{R}[x,y]\otimes\mathcal{O}(GL_2(\mathbb{R}))$

$$\begin{pmatrix}x\\y\end{pmatrix} \mapsto \begin{pmatrix}x\otimes\alpha + y\otimes\gamma\\ x\otimes\beta + y\otimes\delta\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}\otimes A$$

$\longrightarrow$ deforms well: ☺ get Hopf alg. coaction $\mathbb{R}_q[x,y] \to \mathbb{R}_q[x,y]\otimes\mathcal{O}_q(GL_2(\mathbb{R}))$ —via same map—

So, Hopf algebras provide a framework for "Quantum Symmetry", just like Groups " " " "Classic Symmetry".

→ rooted in quantum mechanics, physical applications — {provided just a sample of Q. Symmetry ☺}

Reference: "An invitation to Noncom. Alg", C. Walton, written for AWM volume for EDGE program ☺ MSRI-program Spring 2020.