TQFT seminar

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Dec 9, 2020

Bird's eye view of
"Modular categories and TQFTs beyond semisimplicity"
by Christian Blanchet & Marco De Renzi arXiv: 2011.12931.

1985: Vaughan Jones ["A polynomial invariant for knots via von Neumann algebras"]

1988–1989: Edward Witten ["Topological Quantum Field Theory" & "QFT and the Jones polynomial"]

— laid the foundation of Quantum Topology —

1991: Nikolai Reshetikhin & Vladimir Turaev ["Invariants of 3-manifolds via link poly'ls & quantum grps"]
provided a rigorous mathematical framework
for Witten's invariants of 3-dim'l manifolds

1994: Turaev ["Quantum Invariants of Knots and 3-manifolds" (text)]
introduced modular categories as the main
algebraic structure for the study above.

Game

Start: (Modular Category) $\rightsquigarrow$ (*) — Build: (TQFT) $\rightsquigarrow$ yields: (WRT Invariants of links in 3-manifolds), (Reps of Mapping Class groups of surfaces)

(**)

Main Example: Semisimple quotients of rep categories of $U_q^{res}(sl_2)$ for q a root of 1

1995–1996: Mark Hennings ["Invariants of links and 3-manifolds obtained from Hopf algebras"]
Volodymir Lyubashenko ["Invariants of 3-mflds ... roots of unity"]
used nonsemisimple constructions to get 3-mfld inv. & map. class grp reps.

2009: Turaev, Nathan Geer, and Bertrand Patureau
["Modified quantum dimensions and re-normalized link invariants"]
developed technology of nonsemisimple quantum constructions, used to build TQFTs & Homology QFTs (HQFTs) from nonsemisimple mod. categories.

this generalized step (*) above

See upcoming talk in James Zhang's Seattle Noncom. Algebra Day (SNAD) (December 19, 2020) for a brief survey & new results towards (**) in the nonsemisimple setting
⇝ joint work with Robert Laugwitz: 2010.11872

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$\boxed{\text{Algebra}}$ $\boxed{\text{Category Theory}}$ $\boxed{\text{Quantum Field Theory (†)}}$ $\boxed{\text{Topology}}$

certain Hopf-type algebra H ⇝ $\text{Rep}\,H =: \mathcal{C}$ modular category ⇝ $Z_{\mathcal{C}}$: Cobord $\to$ Vec (or LinCat) ⇝ invariants of 3-mflds.

(†) For a fuller picture of QFT, see Sati-Schreiber: 1109.0955.
In particular, Witten's 1988 work defined topological invariants of closed oriented 3-manifolds in terms of Chern-Simons gauge theory, and predicted the framework of a TQFT.

$\boxed{\text{Semisimple setting}}$: Quick recap

Modular category à la Turaev is a ribbon, fusion category $\mathcal{C}$ with invertible S-matrix ($\leftarrow$ the nondegeneracy condition)

$\mathcal{C}$ ribbon: $(\mathcal{C}, \otimes, \mathbb{1}, ev_X, \widetilde{ev}_X, coev_X, \widetilde{coev}_X, c_{X,Y}, \theta_X)$

$\mathcal{C}$ = monoidal

$\mathcal{C}$ = rigid monoidal (has duals)

$\mathcal{C}$ = braided rigid monoidal

with a twist.

$\mathcal{C}$ fusion = finite & semisimple

(finite # of iso classes of simple objects, enough projectives, objects have finite length, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a finite dim v.s.)

S-matrix $= \left(tr(c_{Y,X} \circ c_{X,Y}) \right)_{X,Y \in \mathrm{Irr}(\mathcal{C})}$.

$\mathcal{C}$ modular $\rightsquigarrow$ Turaev-$RT_{\mathcal{C}}$ topological invariant of closed oriented 3-manifolds

For M: closed oriented 3-manifold

$T \subseteq M$: closed "ribbon graph" (skipping defn)

Get $\boxed{RT_{\mathcal{C}}(M,T) := \mathcal{D}^{-1-l}\, \delta^{-\sigma}\, F_{\mathcal{C}}(L \cup T)}$

is a topological invariant of the pair (M, T)

Here, $L \subseteq S^3$: surgery presentation of M

l, σ: with l components and "signature" σ

$\mathcal{D} \in \Bbbk^{\times}$: a renormalization parameter

$F_{\mathcal{C}}: \mathcal{R}_{\mathcal{C}} \to \mathcal{C}$: RT-functor from categ. of isotopy classes of ribbon graphs of $\mathcal{C}$, $\mathcal{R}_{\mathcal{C}}$, to $\mathcal{C}$

<u>Main Example</u> $\mathcal{C}$ = semisimple quotient of reps of $U_q^{res}(sl_2)$ for q an odd root of unity.

Say $q = e^{2\pi i/r}$ — explained in course videos

Then for $RT_{\mathcal{C}}$, $\mathcal{D} = r^{3/2}$, $\delta = i^{\frac{-r-1}{2}} q^{\frac{r-3}{2}}$

$\boxed{\text{Nonsemisimple setting}}$: <u>Quick recap</u>

<u>Modular category álà Lyubashenko</u> is a finite ribbon category $\mathcal{C}$ with trivial Müger center. (← the nondegeneracy condition, not originally used by Lyubashenko)

<u>Theorem of Shimizu</u> (2019) "Non-deg. conditions for braided finite tensor categ"
For a braided finite tensor category $\mathcal{C}$ over an alg. closed field $\Bbbk$ the following are equivalent

- $\mathcal{C}$ is non-degenerate ← Lyubashenko's definition
- $\mathcal{C}$ is factorizable
- $\mathcal{C}$ is weakly factorizable
- Müger center of $\mathcal{C}$ is trivial ← easiest to use

- <u>Muger center</u> of $\mathcal{C}$, denoted $\mathcal{C}'$ or $Z_2(\mathcal{C})$, is
$$\{ V \in \mathcal{C} \mid c_{V,X} \circ c_{X,V} = id_{X \otimes V} \text{ for all } X \in \mathcal{C} \};$$
it is <u>trivial</u> if equivalent to $Vec_{\Bbbk}$.

- $\mathcal{C}$ is <u>factorizable</u> if $\mathcal{C} \boxtimes \mathcal{C}^{rev} \simeq Z(\mathcal{C})$ as braided tensor categories
$$V \boxtimes W \longmapsto (V \otimes W, c)$$

$\mathcal{C}^{rev} = (\mathcal{C}, \otimes, \mathbb{1}, c^{-1})$, $\boxtimes$ = "Deligne product",

$Z(\mathcal{C})$ = "Drinfeld center" of $\mathcal{C}$, objects are $(V \in \mathcal{C}, c_{V,-})$ — half-braidings

- $\mathcal{C}$ is <u>weakly-factorizable</u> if

$$\begin{aligned} \Omega_{\mathcal{C}} : \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{F}) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathbb{F}, \mathbb{1}) \\ f &\longmapsto \omega \circ (f \otimes \mathrm{id}) \end{aligned}$$ is bijective.

Here, $\mathbb{F} =$ "coend" $\int^{X \in \mathcal{C}} X^* \otimes X$, a certain univ. object in $\mathcal{C}$

since $\mathcal{C}$ is braided, $\mathbb{F}$ is a Hopf algebra in $\mathcal{C}$.

$\omega_{\mathcal{C}} : \mathbb{F} \otimes \mathbb{F} \longrightarrow \mathbb{1}$, "Hopf pairing"

Indeed, $\mathbb{F} \xrightarrow{\sim} \mathbb{1} \otimes \mathbb{F} \xrightarrow{f \otimes \mathrm{id}} \mathbb{F} \otimes \mathbb{F} \xrightarrow{\omega} \mathbb{1}$.

- $\mathcal{C}$ is <u>non-degenerate</u> if the composition

$$\mathbb{F} \xrightarrow{\mathrm{id} \otimes \mathrm{coev}} \mathbb{F} \otimes \mathbb{F} \otimes \mathbb{F}^* \xrightarrow{\omega \otimes \mathrm{id}} \mathbb{F}^*$$

is an isomorphism in $\mathcal{C}$.

Get that if $\mathcal{C}$ is a ribbon fusion category,

$\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{F})$ & $\mathrm{Hom}_{\mathcal{C}}(\mathbb{F}, \mathbb{1})$ have natural bases

& the S-matrix is the matrix associated to the

linear map $\Omega_{\mathcal{C}}$ with respect to these bases.

So, Lyubashenko modular + semisimple $\Rightarrow$ Turaev modular.

$\mathcal{C}$ Lyubashenko-modular $\rightsquigarrow$ $L'_{\mathcal{C}}$ topological invariant of closed oriented 3-mfld.

For $M =$ closed oriented 3-manifold

$T \subseteq M$: "admissible closed bichrome graph"

Get $\boxed{L'_{\mathcal{C}}(M, T) := \mathcal{D}^{-1-\ell} \delta^{-\sigma} F'_{\Lambda}(L \cup T)}$

is a topological invariant of the pair (M, T)

Here, $L, \ell, \sigma, \mathcal{D}$ are the same as for the semisimple setting.

F'_{Λ} is a variant of the LRT-functor $F_{\Lambda} : \mathcal{R}_{\Lambda} \longrightarrow \mathcal{C}$

from category of isotopy classes of bichrome graphs, $\mathcal{R}_{\Lambda} \subseteq \mathcal{R}_{\mathcal{C}}$

Main Example ~~semisimple quotient of~~ $U_q^{res}(sl_2)$-mod,
for $q = e^{2\pi i/r}$ an odd root of 1

Again for $L'_{\mathcal{C}}$, $\mathcal{D} = r^{3/2}$, $\delta = i^{-\frac{r-1}{2}} q^{\frac{r-3}{2}}$.

In general, one just needs a certain Hopf algebra H to get a modular category

H	$\rightsquigarrow$	H-mod
Hopf algebra		rigid monoidal category] = finite tensor category
finitedimensional		finite
+ quasitriangular		+ braided
+ ribbon (= a quasi Δ Hopf alg equipped with ribbon element)		+ ribbon — ribbon finite tensor category

factorizable $\rightsquigarrow$ modularity
(a certain quasi Δ Hopf alg - see Radford's book §12.4)

Example: $D(H)$, the Drinfeld double of a finite-dim'l Hopf alg. is a finite dim'l quasitriangular Hopf algebra that is factorizable (Radford's text, Ch 13)

Kauffman - Radford (1993) ["A nec. & suf. cond. for ... double to be .. ribbon."] determined the set of ribbon elements of $D(H)$ (could be $\emptyset$)

$\rightsquigarrow$ know explicitly when $D(H)$-mod is modular

Also: If $\mathcal{C} = H$-mod, then $\mathcal{Z}(\mathcal{C}) \sim D(H)$-mod ...

Theorem [Shimizu] $\exists$ explicit condition for which $\mathcal{Z}(\mathcal{C})$ is modular.
1707.0969 (more on this in SNAD talk)