

Dec 9, 2020

Bird's eye view of

"Modular categories and TQFTs beyond semisimplicity"
by Christian Blanchett & Marco De Renzi arXiv: 2011.12931.

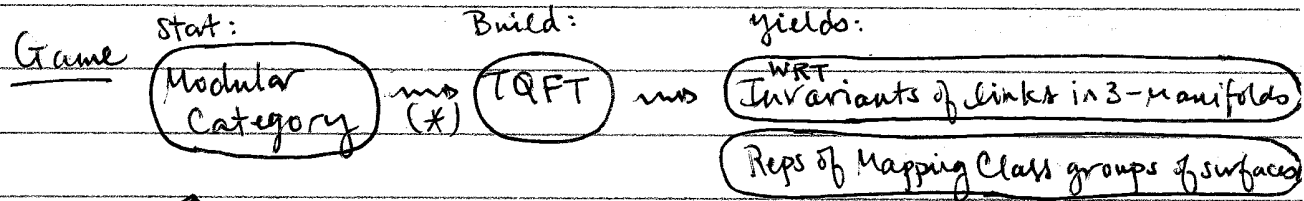
1985: Vaughan Jones ["A polynomial invariant for knots via von Neumann algebras"]

1988-1989: Edward Witten ["Topological Quantum Field Theory" & "QFT and the Jones polynomial"]

— laid the foundation of Quantum Topology —

1991: Nikolai Reshetikin & Vladimir Turaev ["Invariants of 3-manifolds via link polynomials and quantum groups"]
provided a rigorous mathematical framework for Witten's invariants of 3-dim'l manifolds

1994: Turaev ["Quantum Invariants of Knots and 3-manifolds" (text)]
introduced modular categories as the main algebraic structure for the study above.



Main Example: semisimple quotients of rep categories of $U_q^{res}(sl_2)$ for q a root of 1

1995-1996: Mark Hennings ["Invariants of links and 3-manifolds obtained from Hopf algebras"]

Volodymyr Lyubashenko ["Invariants of 3-mflds ... roots of unity"]

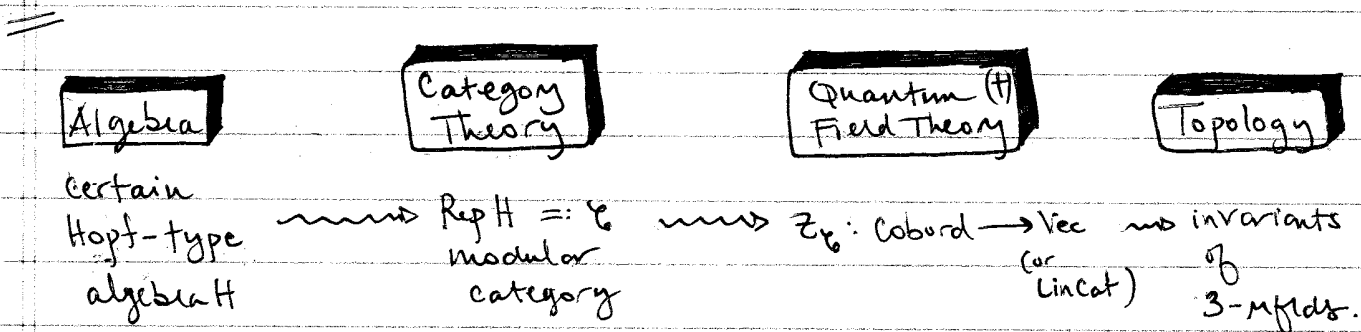
used nonsemisimple constructions to get 3-mfld inv. & rep. class groups

2009: Turaev, Nathan Geer, and Bertrand Patureau
 ["modified quantum dimensions and re-normalized link invariants"]
 developed technology of nonsemisimple quantum
 constructions, used to build TQFTs &
 Homology QFTs (HQFTs) from nonsemisimple mod. categories.

↑
 this generalized step (*) above

See upcoming talk in James Zhang's Seattle
 Noncom. Algebra Day (SNAD) (December 19, 2020)
 for a brief survey & new results towards (**)
 in the nonsemisimple setting

no joint work with Robert Langitz: 2010.11872

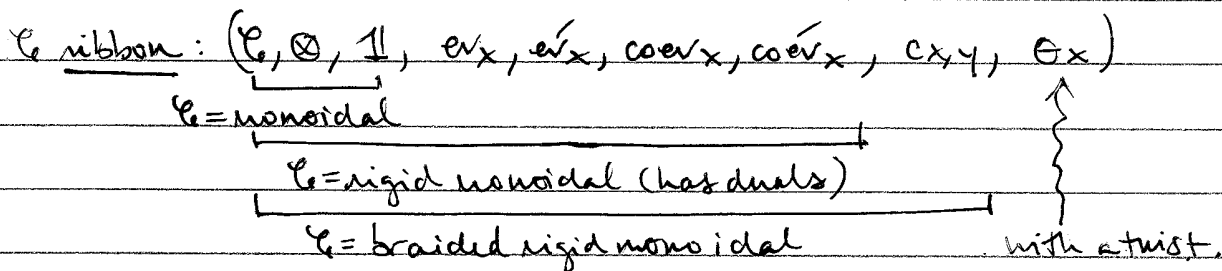


(†) For a fuller picture of QFT, see Sati-Schreiber: 1109.0955.

In particular, Witten's 1988 work defined topological invariants of closed oriented 3-manifolds in terms of Chern-Simons gauge theory, and predicted the framework of a TQFT.

Semisimple setting: Quick recap

Modular category à la Turaev is a ribbon, fusion category \mathcal{C} with invertible S-matrix (← the nondegeneracy condition)



\mathcal{C} fusion = finite & semisimple
 (finite # of iso classes of simple objects, enough projectives, objects have finite length, $\text{Hom}_{\mathcal{C}}(X, Y)$ is a finite dim v.s.)

S-matrix = $(\text{tr}(c_{y,x} \circ c_{x,y}))_{x,y \in \text{Irr}(\mathcal{C})}$

Turaev-
 \mathcal{C} modular inv. RT \mathcal{C} topological invariant of closed oriented 3-manifolds
 for M : closed oriented 3-manifold

$T \subseteq M$: closed "ribbon graph" (skipping def)

Get $\boxed{\text{RT}_{\mathcal{C}}(M, T) := S^{-1-2} S^{-\sigma} F_{\mathcal{C}}(L \cup T)}$

is a topological invariant of the pair (M, T)

Here, $L \subseteq S^3$: surgery presentation of M

ℓ, σ : with ℓ components and "signature" σ

$S \in \mathbb{R}^X$: a renormalization parameter

$F_{\mathcal{C}}: \mathcal{R}_{\mathcal{C}} \rightarrow \mathcal{C}$: RT-functor from categ. of isotopy classes of ribbon graphs of \mathcal{C} , $\mathcal{R}_{\mathcal{C}}$, to \mathcal{C}

Main Example $\mathcal{C} =$ semisimple quotient of reps of $U_q(\mathfrak{sl}_2)$
for q an odd root of unity.

Say $q = e^{2\pi i/r}$

↑ explained in course videos

Then for $RT\mathcal{C}$, $S = r^{3/2}$, $\delta = i^{-\frac{r-1}{2}} q^{\frac{r-3}{2}}$

Nonsemisimple setting: Quick recap

Modular category à la Lyubashenko is a finite ribbon category \mathcal{C} with trivial Müger center. (← the nondegeneracy condition, not originally used by Lyubashenko)

Theorem of Shimizu (2019) "Non-deg. conditions for braided finite tensor cat'g"

For a braided finite tensor category \mathcal{C} over an alg. closed field \mathbb{k} the following are equivalent

- \mathcal{C} is non-degenerate ← Lyubashenko's definition
- \mathcal{C} is factorizable
- \mathcal{C} is weakly factorizable
- Müger center of \mathcal{C} is trivial ← easiest to use

• Müger center of \mathcal{C} , denoted \mathcal{C}' or $Z_2(\mathcal{C})$, is $\{V \in \mathcal{C} \mid C_{V,X} \circ C_{X,V} = \text{id}_{X \otimes V} \text{ for all } X \in \mathcal{C}\}$; it is trivial if equivalent to $\text{Vec}_{\mathbb{k}}$.

• \mathcal{C} is factorizable if $\mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} = Z(\mathcal{C})$ as braided tensor categories
 $V \boxtimes W \mapsto (V \otimes W, c)$

$\mathcal{C}^{\text{rev}} = (\mathcal{C}, \otimes, \mathbb{1}, c^{-1})$, $\boxtimes =$ "Deligne product",

$Z(\mathcal{C}) =$ "Drinfeld center" of \mathcal{C} , objects are $(V \in \mathcal{C}, C_{V,-})$
↑ half-braidings

• \mathcal{C} is weakly-factorizable if

$$\begin{array}{ccc} \mathcal{R}_{\mathcal{C}}: \text{Hom}_{\mathcal{C}}(\mathbb{1}, F) & \longrightarrow & \text{Hom}_{\mathcal{C}}(F, \mathbb{1}) \\ f & \longmapsto & \omega \circ (f \otimes \text{id}) \end{array} \text{ is bijective.}$$

Here, $F = \text{"counit" } \int^{X \in \mathcal{C}} X^* \otimes X$, a certain univ. object in \mathcal{C}

Since \mathcal{C} is braided, F is a Hopf algebra in \mathcal{C} .

$$\omega: F \otimes F \longrightarrow \mathbb{1}, \text{ "Hopf pairing"}$$

$$\text{Indeed, } F \xrightarrow{\sim} \mathbb{1} \otimes F \xrightarrow{f \otimes \text{id}} F \otimes F \xrightarrow{\omega} \mathbb{1}.$$

• \mathcal{C} is non-degenerate if the composition

$$F \xrightarrow{\text{id} \otimes \text{coev}} F \otimes F \otimes F^* \xrightarrow{\omega \otimes \text{id}} F^*$$

is an isomorphism in \mathcal{C} .

Get that if \mathcal{C} is a ribbon fusion category,

$\text{Hom}_{\mathcal{C}}(\mathbb{1}, F) \neq \text{Hom}_{\mathcal{C}}(F, \mathbb{1})$ have natural bases

& the S -matrix is the matrix associated to the

linear map $\mathcal{R}_{\mathcal{C}}$ with respect to these bases.

So, Lyubashenko modular + semisimple \Rightarrow Turaev modular.

\mathcal{C} Lyubashenko-modular \Rightarrow $L_{\mathcal{C}}$ topological invariant of closed oriented 3-ufld.

For $M =$ closed oriented 3-manifold

$T \subseteq M$: "admissible closed bichrome graph"

$$\text{Get } \boxed{L'_{\mathcal{C}}(M, T) := \delta^{-l} \delta^{-\sigma} F_X(L \cup T)}$$

is a topological invariant of the pair (M, T)

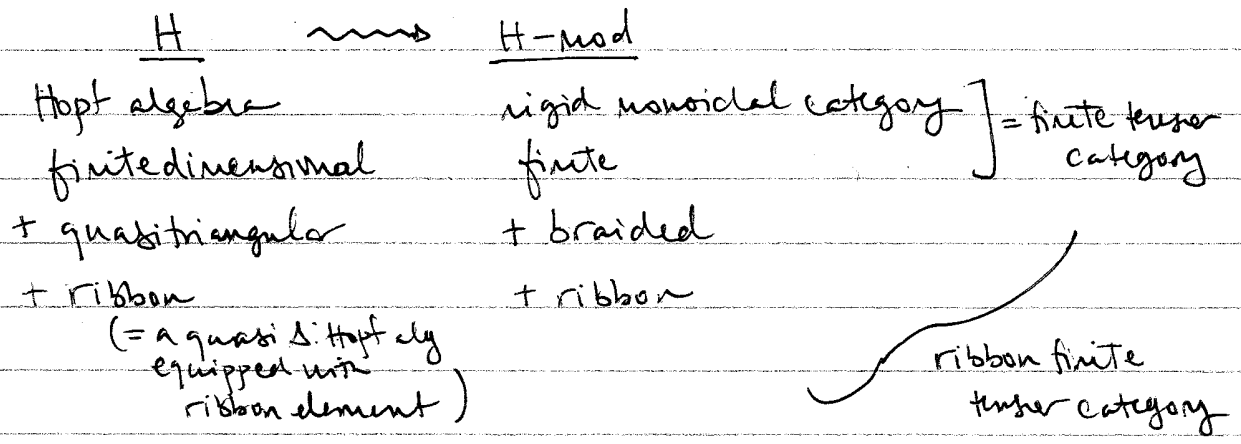
Here, L, l, σ, δ are the same as for the semisimple setting.

F_X is a variant of the LRT-functor $F_X: \mathcal{R}_X \rightarrow \mathcal{C}$

from category of isotopy classes of bichrome graphs, $\mathcal{R}_X \cong \mathcal{R}_{\mathcal{C}}$

Main Example ~~semisimple quotient of~~ $U_q^{res}(sl_2)\text{-mod}$,
 for $q = e^{2\pi i/r}$ an odd root of 1
 Again for U_q , $D = r^{3/2}$, $\delta = i^{-\frac{r-1}{2}} q^{\frac{r-3}{2}}$.

In general, one just needs a certain Hopf algebra H
 to get a modular category



factorizable \rightsquigarrow modularity
 (a certain quasi Δ
 Hopf alg - see Radford's book §12.4)

Example: $D(H)$, the Drinfeld double of a finite-dim'l Hopf alg
 is a finite-dim'l quasitriangular Hopf algebra
 that is factorizable (Radford's text, Ch 13)
 Kauffman-Radford (1993) ["A nec. & suf. cond. for ... double to be ... ribbon-"]
 determined the set of ribbon elements of $D(H)$ (could be \emptyset)
 \rightsquigarrow know explicitly when $D(H)\text{-mod}$ is modular
 Also: If $\mathcal{C} = H\text{-mod}$, then $\mathcal{Z}(\mathcal{C}) \sim D(H)\text{-mod} \dots$

Theorem [Shimizu] Explicit condition for which $\mathcal{Z}(\mathcal{C})$ is modular.
 (1707.0969) (more on this in SNAD talk)