UW Seattle

Aug 8, 2015

Title of preprint: “Maps from the enveloping algebra of the positive Witt algebra to regular algebras”

Homomorphisms from enveloping alg of positive Witt algebra

joint work in preparation with Sue Sierra

$\boxed{\text{char } k = 0}$

arXiv:1508.05294

0. Motivation

Definition $\boxed{\text{Virasoro algebra}}$ (V) = Lie algebra with basis $\{e_n\}_{n \in \mathbb{Z}} \cup \{c\}$ and Lie bracket $[e_n, c] = 0$, $[e_n, e_m] = (m-n)e_{n+m} + \frac{c}{12}(m^3 - m)\delta_{n+m,0}$

$\boxed{\text{Witt algebra}}$ (W) = Lie algebra with basis $\{e_n\}_{n \in \mathbb{Z}}$ and Lie bracket $[e_n, e_m] = (m-n)e_{n+m}$

$\boxed{\text{positive Witt algebra}}$ (W_+) = Lie subalgebra of W generated by $\{e_n\}_{n \geq 1}$.

[Sierra-W] (2013) $U(W_+)$, $U(W)$, $U(V)$ are not left nor right Noetherian.

Method: used 'geometric map' $\rho : U(W_+) \longrightarrow k(X)[t; \tau]$ skew polyn'l ring

$$e_1 \longmapsto t$$
$$e_2 \longmapsto ft^2$$

where
$$\left[\begin{array}{l} X = \mathbb{V}(xz - y^2) \subseteq \mathbb{P}^3_{[w:x:y:z]} \\ \tau \in \mathrm{Aut}(X), \quad \tau[w:x:y:z] = [w-2x+2z : z : -y-2z : x+4y+4z] \\ f = \frac{w+12x+22y+8z}{12x+6y} \in k(X) \end{array}\right.$$

and showed (through several reductions) that $\rho(U(W_+))$ is not left/right Noetherian

$\rightsquigarrow$ same is true for $U(W_+)$, $U(W)$, $U(V)$.

But $\boxed{\rho}$ is not so easy to understand!

Outline of talk

1. Manipulate ρ to get alg. maps λ_a $a\in k$ and ϕ from $U(W_+)$ to AS-reg algs
2. Connection between ρ, λ_a, ϕ
3. $\ker \lambda_a$, $\operatorname{im} \lambda_a$
4. $\ker \phi$, $\operatorname{im} \phi$
5. Elementary proof that $U(W_+)$ is not left/right Noetherian

1 $\boxed{\rho \rightsquigarrow \lambda_a, \phi}$

Consider $\alpha \in \operatorname{Aut}(X)$, $\alpha[w:x:y:z] = [w:x:-x-y:x+2y+z]$

For $a\in k$, take $\psi_a : \mathbb{P}^1 \to X$, $[x:y] \mapsto [2x^2-(6a+2)y^2 : x^2 : xy : y^2]$.

Prop Get algebra homomorphism, for $a \in k$,

$$\boxed{\lambda_a} := \psi_a^* \alpha^* \rho : U(W_+) \longrightarrow B(\mathbb{P}^1, \mathcal{L}, \sigma)$$

twisted homogeneous coor. ring

$\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}([1:1])$, $\sigma[x:y] = [x+y:y]$

$$\| $$

$$\bigoplus_{n\geq 0} H^0(\mathbb{P}^1, \mathcal{L}_n)t^n$$

$\mathcal{L}$ twisted by σ as usual

$$\|$$

$$\boxed{\frac{k\langle u,v\rangle}{(uv-vu-v^2)} =: R}$$

Jordan plane, AS regular of dim 2

Note that $U(W_+)$ is gen in degrees 1 & 2

$$\boxed{e_1} \longmapsto \qquad t = \boxed{u}$$

$$\boxed{e_2} \longmapsto (\psi_a^* \alpha^* f)t^2 = \frac{-y^2+xy+ay^2}{(y-x)x} t^2 = \boxed{uv - av^2}$$

with $v = \left(\frac{y}{y-x}\right)t$.

Now consider $i_a : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$, $[x:y] \longmapsto [x:y:ay]$ for $a \in k$.

This induces algebra map

$$\boxed{\pi_a} : \underset{[x:y:z]}{B(\mathbb{P}^2, \mathcal{O}(1), \mu)} \longrightarrow B(\mathbb{P}^1, \mathcal{O}(1), \sigma) = R$$

$$\mu : \begin{bmatrix} x \mapsto x-y \\ y \mapsto y \\ z \mapsto z \end{bmatrix}$$

This map is \eta_a (restricted to S) in the paper.

$\|$

<u>AS-regular of dim 3</u> (Zhang twist of $k[x,y,z]$ by μ)

$$\boxed{\frac{k\langle u,v,w\rangle}{\begin{pmatrix} uv - vu - v^2 \\ uw - wu - vw \\ vw - wv \end{pmatrix}} =: S}$$

$u \mapsto u$

$v \mapsto v$

$w \mapsto av$

<u>Prop</u> Get algebra homomorphism

$$\boxed{\phi : U(W_+) \longrightarrow S, \quad e_n \longmapsto (u - (n-1)w)v^{n-1} \quad n \geq 1}$$

so that $\lambda_a = \pi_a \circ \phi$

---

2. Connection between ρ, λ_a, ϕ

<u>Prop</u>: $\boxed{\ker \rho \overset{(1)}{=} \bigcap_{a \in k} \ker \lambda_a \overset{(2)}{=} \ker \phi}$

Pf/ (sketch) • $h \in U(W_+)_n$ homog. $\Rightarrow$ $\rho(h) = g t^n$ for some $g \in k(X)$

(1) $\supseteq$ • $\lambda_a(h) = 0 \ \forall a \in k \Rightarrow g$ vanishes on open subset of $X \Rightarrow g = 0$

(1) $\subseteq$ • $\lambda_a = \psi_a^* \alpha^* \rho$, so clear.

(2) $\supseteq$ • $\lambda_a = \pi_a \circ \phi$, so clear

(2) $\subseteq$ • Curves $\underset{a \in k}{z = ay}$ cover an open subset of $\mathbb{P}^2$ ///.

3. $\boxed{\ker \lambda_a,\ \mathrm{im}\, \lambda_a}$

<u>Theorem</u> • $\ker \lambda_a = \begin{cases} (e_1e_3 - e_2^2 - e_4) & \text{if } a = 0,1 \\ \text{ideal generated by } \left[\begin{array}{l}\text{one elt of deg 5} \\ \text{two elts of deg 6}\end{array}\right. & \text{if } a \neq 0,1 \end{cases}$

(we have explicit elts — too long to write here)

• $\mathrm{im}\, \lambda_a = \begin{cases} k + uR \quad (\text{right idealizer of } R) & \text{if } a = 0 \\ k + Ru \quad (\text{left idealizer of } R) & \text{if } a = 1 \\ R \text{ in degrees} \geq 4 & \text{if } a \neq 0,1 \end{cases}$

$\boxed{\text{Noetherian}}$

Pf (sketch) / $\lambda_0 : \mathcal{U}(W_+) \to R$, $e_n \mapsto uv^{n-1}$ $\qquad$ $\lambda_1 : \mathcal{U}(W_+) \to R$, $e_n \mapsto (u-(n-1)v)v^{n-1}$

want to save time for §5.

$\mathcal{U}(W_+)$ is generated in degrees 1, 2

$e_1 \mapsto u$ $\qquad$ $e_1 \mapsto u$

$e_2 \mapsto uv$ $\qquad$ $e_2 \mapsto (u-v)v = vu$

$\rightsquigarrow \mathrm{im}\, \lambda_0 = k + uR$ $\qquad$ $\rightsquigarrow \mathrm{im}\, \lambda_1 = k + Ru$

$$\lambda_0(e_1e_3 - e_2^2 - e_4) = u(uv^2) - (uv)(uv) - uv^3$$
$$= u^2v^2 - u(uv - v^2)v - uv^3 = 0 \quad \Rightarrow e_1e_3 - e_2^2 - e_4 \in \ker \lambda_0$$

Tedious to show that $\ker \lambda_0$ is generated by one elt of deg 4 ...

4. $\boxed{\ker \phi,\ \mathrm{im}\, \phi}$

<u>Theorem</u> • $\ker \phi = (e_1e_5 - 4e_2e_4 + 3e_3^2 + 2e_6)$

• $\mathrm{im}\, \phi \simeq \mathrm{nip}$ $\quad$ $(\boxed{\text{not left nor right Noetherian}})$

from Prop in §2.

So $\mathrm{nip} \cong \dfrac{\mathcal{U}(W_+)}{\ker \phi}$ is the main object of study in [SW2013]

5. Elementary proof that $U(W_+)$ is not left/right Noetherian

<u>Theorem</u> $\ker(\lambda_0 : U(W_+) \to R)$ is <u>not</u> fin. gen as a left/right ideal of $U(W_+)$.

Pf/ Recall $\ker\lambda_0 = (e_1e_3 - e_2^2 - e_4)$.

$$\begin{tikzcd}
e_n \arrow[d, mapsto] & & U(W_+) \arrow[dl, "\phi"'] \arrow[dr, "\lambda_0"] & & e_n \arrow[d, mapsto] \\
(u-(n-1)w)v^{n-1} & S \arrow[rr, "\pi_0"] & & R & uv^{n-1}
\end{tikzcd}$$

$$u \longmapsto u$$
$$v \longmapsto v$$
$$w \longmapsto 0$$

<u>Notation</u>: $p := \phi(e_1e_3 - e_2^2 - e_4)$ $\quad B := \operatorname{im}\phi$

$I := (p) = \phi(\ker\lambda_0)$ $\quad Q :=$ subalg of S gen by u, v, vw

<u>Facts</u> Ⓐ $p = v^3w - v^2w^2$

Ⓑ p is a normal elt of S, and of Q

Ⓒ $I = Qp$.

<u>Note</u>: $\ker\lambda_0$ fin. gen as left/right ideal of $U(W_+) \Rightarrow I$ fin. gen. as left/right ideal of B.

$\leadsto$ STS $_BI$ and I_B are <u>not</u> finitely generated

will only show this, statement for I_B holds in similar fashion.

By way of contradiction, suppose that $_BI$ is finitely generated.

$\deg(p)=4 \Rightarrow \exists n \geq 4$ so that $BI_{\leq n} = I$.

$U(W_+)$ generated in degrees $1, 2 \Rightarrow$

$$I_{n+1} = B_1 I_n + B_2 I_{n-1} = u I_n + (u-w) v I_{n-1}$$

Ⓑ Ⓒ $I \doteq Q_p \subseteq \mathcal{S}_p \mathcal{S} \subseteq \mathcal{S}_p \Rightarrow vI \subseteq v\mathcal{S}_p \subseteq \mathcal{S}_p$.

Get $\boxed{I_{n+1} \subseteq u\mathcal{S}_p + (u-w)\mathcal{S}_p = u\mathcal{S}_p + w\mathcal{S}_p.}$ $(*)$

$u\mathcal{S} + w\mathcal{S}$ does not contain a positive power of v by direct computation

$$\Rightarrow \boxed{v^{n-3} p \notin u\mathcal{S}_p + w\mathcal{S}_p} \qquad (**).$$

But $v^{n-3} p \in (Q p)_{n+1} \overset{\textcircled{c}}{=} I_{n+1}$

Now $v^{n-3} p \in I_{n+1} \overset{(*)}{\subseteq} u\mathcal{S}_p + w\mathcal{S}_p$, which contradicts $(**)$ ///.