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Chelsea Walton

# Tensor Algebras in Finite Tensor Categories

OSU summer school

joint w/ Pavel Etingof & Ryan Kinser, coming soon

**Broad Interest**: To study (quantum) symmetries of algebras A

say over a field $\Bbbk$: $A = (\underset{\Bbbk\text{-vs}}{A}, \underbrace{m: A\otimes A \to A, u: \Bbbk \to A}_{\text{maps in } \mathrm{Vec}_{\Bbbk}})$

satisfying associativity & unit axioms

I. "symmetries" = group actions on A by automorphisms

[Lie alg actions by derivations]

II "quantum symmetries" = actions of a Hopf algebra H in the sense that

A is an H-module algebra

$\Updownarrow$

$A, m, u \in \mathrm{Rep}(H)$

$\Updownarrow$

$\rightsquigarrow \quad A \in \mathrm{Alg}(\underbrace{\mathrm{Rep}(H)})$ ← monoidal category

III. "quantum symmetries of algs in general" $\equiv$ algebras in monoidal categorys

3 questions:
what to act on?
what to act with?
how to act?

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**This talk** $A = \Bbbk Q$ path algebra of a quiver

$Q = (Q_0, Q_1, s,t: Q_1 \to Q_0)$ directed graph

$\Bbbk Q = \bigoplus_{\text{path } p \in Q} \Bbbk x_p$ as a $\Bbbk$-vs,

with multiplication $x_p x_q = \delta_{t(p), s(q)} x_{pq}$

Ex

| Q | $\Bbbk Q$ |
|---|---|
| $\bullet$ | $\Bbbk$ |
| $\bullet \circlearrowright x$ | $\Bbbk[x]$ |
| $x_2 \circlearrowleft \bullet \circlearrowright x_1$<br>$\vdots \circlearrowright x_n$ | $\Bbbk\langle x_1, \dots, x_n\rangle$ |
| $\bullet \to \bullet$ | $\begin{pmatrix} \Bbbk & \Bbbk \\ 0 & \Bbbk \end{pmatrix}$ |
| $d\{\bullet \overset{\to}{\underset{\to}{\vdots}} \bullet$ | $\begin{pmatrix} \Bbbk & \Bbbk^d \\ 0 & \Bbbk \end{pmatrix}$ |

$\dim \Bbbk Q < \infty \iff Q$ is finite $|Q_0|, |Q_1| < \infty$ & acyclic

More generally $A = T_B(V)$ tensor algebra

$B \in \mathrm{Alg}(\mathrm{Vec}_{\Bbbk})$ $\quad V \in \mathrm{Bimod}_{\Bbbk}(B)$.

$\Bbbk Q = T_{\Bbbk Q_0}(\Bbbk Q_1)$

- $\Bbbk Q_0$ = commutative, semisimple $\Bbbk$-algebra $\cong \Bbbk^{|Q_0|}$
- $\Bbbk Q_1 \in \mathrm{Bimod}_{\Bbbk}(\Bbbk Q_0)$

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(Q.) Symmetries are going to be grade / degree-preserving.

Ex. $G$ finite group

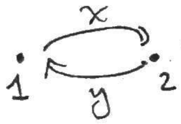
degree preserving $G$-actions on $\Bbbk Q$

$\Uparrow$ since $\Bbbk Q$ is generated in degree 0 & 1

$G$-actions on $Q$ by graph automorphisms

Ex. $G = \mathbb{Z}_2 = \langle g \mid g^2 = 1 \rangle$

Q:



$G \curvearrowright Q$ by $1 \overset{g}{\longleftrightarrow} 2$

$x \longleftrightarrow y$

induces $G \curvearrowright \Bbbk Q$.

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Q. Symmetries are going to be in context III., motivated by context II in the semisimple case....

Finite-dim'l Hopf algebras : two important subclasses.

| Semisimple: | pointed |
|---|---|
| as an algebra: | as a coalgebra |
| module ≅ ⊕ simple mod. | simple comodules are 1-dim'l |
| Such $H$ are typically studied | " ——— " |
| with group-theoretic techniques — | w/ Lie-theoretic techniques |
| e.g. normal Hopf subalgs, exts etc .. | eg. "Cartan data is used for classification purposes" |
| Initial Goal<br>$H_{ss} \curvearrowright \Bbbk Q$. | Kinser–W (2016) studied actions of fin-dim'l pted $H$ on $\Bbbk Q$. |

Category Theory provides a beautiful framework for not only achieving our goal of analyzing

degree preserving $\underbrace{\text{actions on semisimple Hopf algs}}_{\downarrow}$ on $\underbrace{\Bbbk Q}_{\downarrow}$

it yielded a fast & clean way to generalize

tensor algebras

in

finite tensor categories.

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As usual for categorification results, the work is <u>done in the set-up</u> so that the theorem "falls out"

<u>Set</u>-up $\boxed{\text{Take } \mathcal{C} \text{ a finite multitensor category}}$

Ex. $\underline{\mathcal{C} = \mathrm{Vec}}$, $\underline{\mathrm{Vec}_G}$, $\underline{\mathrm{Vec}_G^{\omega}}$, $\underline{\mathrm{Rep}(G)}$, $\underline{\mathrm{Rep}(H)}$ when $H$ semisimple.

$\omega \in H^3(G, \Bbbk^{\times})$

$G$ = finite group

$H$ = finite-dim'l Hopf alg.

$\mathcal{C}$ fusion

$\mathcal{C}$ = "<u>group-theoretical fusion category</u>"

$= \mathrm{Bimod}_{\mathrm{Vec}_G^{\omega}}(\Bbbk_{\alpha} K)$

$\nwarrow$ twisted group algebra

$K \leq G$, $\alpha \in Z^2(K, \Bbbk^{\times})$

Let's consider reps of $\mathcal{C}$ and reps of algebras in $\mathcal{C}$ —

$\boxed{\mathcal{M}}$ = left $\mathcal{C}$-module category : abelian $\Bbbk$-linear category $\mathcal{M}$ with $\otimes : \mathcal{C} \times \mathcal{M} \longrightarrow \mathcal{M}$ compatible w/ structure of $\mathcal{C}$

$\mathcal{M}$ is $\boxed{\text{exact}}$ if $\forall$ projective objects $P \in \mathcal{C}$, $\forall M \in \mathcal{M}$, get $P \otimes M$ is projective

E.g. $\underline{\mathcal{M} \text{ is semisimple.}} \longrightarrow$ (all objects are projective)

$\boxed{\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})}$ = category of right exact $\mathcal{C}$-module endofunctors of $\mathcal{M}$.

Take an $\boxed{\text{algebra } A \text{ in } \mathcal{C}}$

Get $\boxed{\mathrm{Mod}_{\mathcal{C}}(A)}$ = category of right $A$-modules in $\mathcal{C}$:

$(M, \rho_M : M \otimes A \longrightarrow M)$ compatible with structure of $\mathcal{C}$.

In fact $\mathrm{Mod}_{\mathcal{C}}(A)$ is a left $\mathcal{C}$-module category:

$$\otimes : \mathcal{C} \times \mathrm{Mod}_{\mathcal{C}}(A) \longrightarrow \mathrm{Mod}_{\mathcal{C}}(A)$$

$$\big(X, (M, \rho_M)\big) \longmapsto (X \otimes M, \ \mathrm{id}_X \otimes \rho_M)$$

<u>Theorem</u> [Ostrik] Take any exact $\mathcal{C}$-module category $\mathcal{M}$. Then $\exists$ algebra $A$ s.t. $\mathcal{M} \sim \mathrm{Mod}_{\mathcal{C}}(A)$ as $\mathcal{C}$-module categorys.

We say that $A \in \mathrm{Alg}(\mathcal{C})$ is <u>indecomposable</u> / <u>exact</u> / <u>semisimple</u> if $\mathrm{Mod}_{\mathcal{C}}(A)$ is so.

Say that $A, A' \in \mathrm{Alg}(\mathcal{C})$ are <u>Morita equivalent</u> if $\mathrm{Mod}_{\mathcal{C}}(A) \sim \mathrm{Mod}_{\mathcal{C}}(A')$ as $\mathcal{C}$-mod. categories

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Let's define tensor algebras in $\mathcal{C}$ and cook up a notion of "sameness" so that ① the tensor algebras can be easily classified up to this notion of sameness & so that ② we illustrate our classification in concrete examples .... building the framework.

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$\mathcal{C}$ = multi-fusion

<u>Def'n</u> A <u>$\mathcal{C}$-tensor algebra</u> $T_S(E)$ is an algebra in $\mathrm{Ind}(\mathcal{C})$ where

- $S$ ≡ exact (semisimple) algebra in $\mathcal{C}$ ← base alg
- $E \in \mathrm{Bimod}_{\mathcal{C}}(S)$ ← generating bimodule
- $T_S(E) = S \oplus E \oplus (E \otimes_S E) \oplus E^{\otimes 3} \oplus \cdots$
- $m_{T_S(E)} : E^{\otimes_S m} \otimes_S E^{\otimes_S n} \longrightarrow E^{\otimes_S n+m}$ nat'l maps
- $u_{T_S(E)} : S \hookrightarrow T_S(E)$ "

<u>Def'n</u> $T_S(E)$ is <u>equivalent</u> to $T_{S'}(E')$ if

boils down to Morita equivalence of base alg

$\exists$ equivalence of $\mathcal{C}$-mod. categories $f : \mathcal{M} := \mathrm{Mod}_{\mathcal{C}}(S) \longrightarrow \mathcal{M}' := \mathrm{Mod}_{\mathcal{C}}(S')$

for which the induced map $\hat{f} : \underset{\sim \mathrm{Bimod}_{\mathcal{C}}(S)}{\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})} \longrightarrow \underset{\sim \mathrm{Bimod}_{\mathcal{C}}(S')}{\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}', \mathcal{M}')}$

$$Q \longmapsto f Q f^{-1}$$

yields $\hat{f}(E) \cong E'$ in $\mathrm{Bimod}_{\mathcal{C}}(S')$.

Ex. $\mathcal{C} = \mathrm{Vec}$

Each equiv. class of a Vec-tensor algebra is represented by a path alg $\Bbbk Q$.

[ Vec semisimple $\Longrightarrow$ base alg $S$ = finite dim'l semisimple $\Bbbk$-alg

$\overset{\text{AW-thm}}{\Longrightarrow}$ $S$ is Morita equivalent to $\underbrace{\Bbbk^{\oplus r}}_{\substack{\cong \\ \Bbbk Q_0}}$ $r<\infty$ for $|Q_0| = r$

Can get $E$ as arrow space on $\Bbbk Q_0$ ]

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Theorem [EKW] The collection of equiv. classes of $\overset{\mathcal{C}\text{ multi-fusion}}{\mathcal{C}}$-tensor algebras $T_S(E)$ are parameterized by pairs $(\mathcal{M}, Q)$, where

- $\mathcal{M}$ = exact $\overset{(\text{semisimple})}{\wedge}$ $\mathcal{C}$-mod. category
- $Q$ = nonzero functor in $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$

Pf/ Takes care of the base algebras $S$ up to equivalence, due to Ostrik's Theorem then with $\mathcal{M}$ fixed, choices for $E$ is parametrized by choices of functors in $\mathrm{Fun}_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$.

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Why bother with this high-falutin tech? ① Computations/Formulas can be nasty and unintuitive

② $\exists$ many $\mathcal{C}$ for which $\mathcal{C}$-mod. categories $\mathcal{M}$ are concretely & completely understood in terms of group-theoretic or graph-theoretic data

$G$ finite grp
$H$ finite dim'l Hopf alg

Ex. Vec, $\underbrace{\mathrm{Vec}_G}$, $\underbrace{\mathrm{Vec}_G^{\omega}}$, $\underbrace{\mathrm{Rep}(G)}$, $\mathrm{Rep}(H)$, $\mathrm{Bimod}_{\mathrm{Vec}_G^{\omega}}(\Bbbk_\alpha K)$
$K \leq G$, $\alpha \in Z^2(K, \Bbbk^{\times})$

$\mathrm{Corep}(\mathcal{O}_q(SL_2))$

all fusion

Theorem [Ostrik, Natale]

Indecomposable $\overset{\text{semisimple}}{\wedge}$ module categories over or

are parameterized by $(L, \psi)$ where

$L \leq G$, $\psi \in Z^2(L, \Bbbk^{\times})$ s.t. $d\psi = \omega|_L$.

so that \omega|_L is trivial

Ostrik's Theorem
$\rightsquigarrow$ This also parametrizes indecomposable semisimple algs in these categories

Example: $\mathcal{C} = \mathrm{Rep}(\mathbb{Z}_2)$ $\quad \mathbb{Z}_2 = \langle g \mid g^2 = 1 \rangle$

To get all $\mathcal{C}$-tensor algebras up to equivalence, say with $S$ indecomposable, note

$$\{(L, \psi)\} = \{(\langle e \rangle, 1), \ (\mathbb{Z}_2, 1)\} \quad \text{two choices}$$

{ Get by recipe in paper } $\quad S \cong \Bbbk_{triv} \oplus \Bbbk_{sgn}$ $\qquad\qquad$ $S \cong \Bbbk_{triv}$

↑ ↗ 1-diml reps of $\mathbb{Z}_2$

Get realization of path alg. in this case:

Ex. $Q_0$: $\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, dashed, leftrightarrow, "g"'] \end{tikzcd}$ $\qquad\qquad$ $Q_0$: $\begin{tikzcd} \underset{1}{\bullet} \arrow[loop right, dashed, "g"] \end{tikzcd}$

To add arrows —

{ Get by result of Ostrik } $\quad |\mathrm{Irr}(\mathrm{Bimod}_{\mathcal{C}}(S))| = 2$ $\qquad\qquad$ $|\mathrm{Irr}(\mathrm{Bimod}_{\mathcal{C}}(S))| = 2$

$\rightsquigarrow$ $T_S(E) = \Bbbk Q$ for $Q$: $\begin{tikzcd} \bullet \arrow[r, bend left, "x"] \arrow[r, dashed, leftrightarrow] & \bullet \arrow[l, bend left, "y"] \end{tikzcd}$ $\qquad\qquad$ $T_S(E) = \Bbbk Q$ for $Q$ $\begin{tikzcd} \bullet \arrow[loop right, "x"] \end{tikzcd}$ $= \Bbbk[x]$

—OR—

$Q$: $\begin{tikzcd} \bullet \arrow[loop left, "x"] \arrow[r, dashed, leftrightarrow] & \bullet \arrow[loop right, "y"] \end{tikzcd}$ $\qquad\qquad$ $g \cdot x = \begin{cases} x \\ \text{or} \\ -x \end{cases}$

for $g \cdot x = y$, $g \cdot y = x$

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Example $\mathcal{C} = \mathrm{Rep}(H_8)$ $\sim$ $\mathrm{Bimod}_{\mathrm{Vec}^{\omega}_{D_8}}(\Bbbk\mathbb{Z}_2)$ group-theoretical

↑ Kac-Paljutkin Hopf algebra $\qquad$ nontrivial

$\exists$ six indecomp. $\mathcal{C}$-module categories: parameterized by $L \leq D_8$ $\omega|_L$ trivial.

$\rightsquigarrow$ Have explicit description of indecomposable $H_8$-algebras $S$ in paper & many other calculations...