UCSD Algebra Seminar talk

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The universal enveloping algebra of the Witt alg. is not noetherian

joint w/ Sue Sierra

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/k
$k = \bar{k}$
$\mathrm{Ch}\, k = 0$.

Conjecture A Lie alg L is finite dim'l $\Leftrightarrow$ $U(L)$ is noetherian.
(ACC on left & right ideals)

($\Longrightarrow$) holds easily:
$U(L)$ filtered & $\mathrm{gr}\, U(L) \cong k[x_1, \dots, x_{\dim L}]$ noetherian
$\Rightarrow$ $U(L)$ noetherian

($\Longleftarrow$) believed to be true in general, questioned by many ring theorists.

Consider
Defn The Witt (centerless Virasoro) algebra W
= Lie algebra with basis $\{e_n\}_{n \in \mathbb{Z}}$ w/ bracket $[e_n, e_m] = (m-n)\, e_{n+m}$.
* $U(W)$ is $\mathbb{Z}$-graded w/ $\deg(e_n) = n$
& (a nice ring, no Lie theory needed)

Question (Carolyn Dean & Lance Small) Is $U(W)$ noetherian?

Theorem [Sierra–W] No.

Approach: consider
Defn The positive Witt algebra W_+
= Lie subalgebra of W generated by $\{e_n\}_{n \geq 1}$

$U(W)$ is a flat $U(W_+)$-module
& $M \otimes_{U(W_+)} U(W) \neq 0$ for every nonzero right $U(W_+)$-module M.

Get that $U(W)$ is faithfully flat (or actually free) over $U(W_+)$
$\leadsto$ $U(W_+)$ not noetherian $\Rightarrow$ $U(W)$ not noetherian

— STS $U(W_+)$ is not noetherian —

Note $U(W_+) = \underline{k\langle e_1, e_2 \rangle}$ $\mathbb{N}$-graded, generated in degs 1 & 2

$$\begin{array}{l} \text{deg } 5 \to \\ \\ \text{deg } 7 \to \end{array} \left(\begin{array}{l} [e_1[e_1[e_1,e_2]]] + 6[e_2[e_2,e_1]] \\ \\ [e_1[e_1[e_1[e_1[e_1,e_2]]]]] + 40[e_2[e_2[e_2,e_1]]] \end{array} \right)$$

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We use $\underbrace{\text{NC projective AG}}$ to show that $U(W_+)$ is not noetherian

study of $\mathbb{N}$-graded rings $U = \bigoplus_{i \geq 0} U_i$ w/ $U_0 \cong k$ (U is "connected")
using techniques from projective AG.

<u>Game</u>: use: $U \longrightarrow B(X)$
map

To study   geometric ring

First developed by Artin–Tate–van den Bergh ~1989
to study the ring theoretic properties of the three-dimensional Sklyanin algebra.

Commutative Proj AG

Alg

C com connected graded ring generated in deg 1.

Ex. $C = k[x,y]$

$\longrightarrow$

Geom

$\mathrm{Proj}\, C = X$

$= \{$ homogeneous prime ideals $\mathfrak{p} \not\supseteq C_+ \}$

$X = \mathbb{P}^1$

$= \{ (\alpha y - \beta x) \trianglelefteq k[x,y] \mid (\alpha, \beta) \in k^2 \setminus (0,0) \}$.

$\longleftarrow$

$B(X, \mathcal{L}) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$

$\mathcal{L}$ ample invertible sheaf on X

section ring

$\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(1) \rightsquigarrow B(X, \mathcal{L}) = k[x,y]$

Get homomorphism $C \xrightarrow{\cong} B(X, \mathcal{L})$ & $C_n = B_n$ for $n \gg 0$.

[qgr $C \sim$ qgr $B \sim$ coh X Serre's theorem]

Noncom Proj AG

NC Alg

U nc connected graded ring generated in deg 1

Ex. $U = k\langle x,y \rangle / (yx - qxy)$ $q \in k^*$

skew poly'l ring

$\xrightarrow{\cancel{\mathrm{Proj}}}$ (not enough 2-sided ideals)

"NC" Geom (actually get commutative projective objects)

$X =$ "point scheme" of U

parameterizes "U-point modules"

def'n cyclic graded left U-modules M

w/ Hilb series $H_M(t) = \frac{1}{1-t}$

[same as a coord ring of a pt in com AG]

$\longleftarrow$

$B(X, \mathcal{L}, \sigma) = \bigoplus_{n \geq 0} H^0(X, \mathcal{L}_n)$ (twisted by σ)

$\sigma \in \mathrm{Aut}\, X$

$\mathcal{L} =$ "σ-ample invertible sheaf on X"

twisted homogeneous coordinate ring

$\mathbb{P}^1_{[u:v]}$ $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(1)$ $\sigma[u:v] = [qu:v]$

$\rightsquigarrow B(X, \mathcal{L}, \sigma) = k\langle x,y \rangle / (yx - qxy)$

U point modules

$= \{ U / U(\alpha y - \beta x) \mid [\alpha : \beta] \in \mathbb{P}^1 \}$

$\rightsquigarrow X = \mathbb{P}^1$

Get homomorphism $U \longrightarrow B(X, \mathcal{L}, \sigma)$ & $U_n = B_n$ for $n \gg 0$

[qgr $U \sim$ qgr B]

(no algebraic proof of this to date)

Technique was used to show that the 3-dim'l Sklyanin alg is noetherian

$$S(a,b,c) = \frac{k\langle x,y,z\rangle}{\begin{pmatrix} ayz + bzy + cx^2 \\ azx + bxz + cy^2 \\ axy + byx + cz^2 \end{pmatrix}}$$

$[a:b:c] \in \mathbb{P}^2$ generic

$\rightsquigarrow$ • X = elliptic curve $E \subseteq \mathbb{P}^2$

• $B(E, \mathcal{L}, \sigma)$ is noetherian

• $S/gS \cong B$ $\Longrightarrow$ S is noetherian

g: central, regular deg 3 elt of S

But TTCRs & Veronese subrings of are never generated in deg 1 & 2.

Need another type of geometric rings -

Examples of geometric rings in NCPAG	noetherian?
① <u>TTCR</u> $B(X,\mathcal{L},\sigma) = \bigoplus_{n\geq 0} H^0(X, \mathcal{L}_n)$ ($B^{(n)}$ is generated in deg 1 for some $n \geq 0$.)	always ✓

— Now assume X = projective surface & $\underbrace{|\sigma| = \infty}$ —

as the next two rings have geom data involving a 0-dim'l subscheme of X

& the ring is noetherian when the points of this subscheme

move w/in a σ-orbit generically

so we want the σ-orbit to be large $\rightsquigarrow$ $|\sigma| = \infty$

	noetherian $\Leftrightarrow$
② <u>Naive blowups</u> $B(X,\mathcal{L},\sigma,P) = \bigoplus_{n\geq 0} H^0(X, \mathfrak{m}\mathfrak{m}^{\sigma}\cdots\mathfrak{m}^{\sigma^{n-1}}\mathcal{L}_n)$	P is supported at points w/ dense σ-orbits :
P = 0-dim'l subscheme of X	$\mathcal{O}(p) = \{\sigma^n(p) \mid n \in \mathbb{Z}\}$
$\mathfrak{m}$ = ideal sheaf defining P in X	~~At~~ curve in X
(But $B^{(n)}$ is generated in degree 1 for some $n \geq 0$)	

(3) ADC Rings $B(X, \mathcal{L}, \sigma, Z, \Lambda, \Lambda', \Omega) = \bigoplus_{n \geq 0} H^0(X, \mathcal{A}\mathcal{D}^{\sigma} \cdots \mathcal{D}^{\sigma^{n-1}} \mathcal{C}^n \mathcal{L}_n)$

Z = subscheme of X

Λ, Λ' = 0-dim'l subschemes of X

$\mathcal{A}, \mathcal{D}, \mathcal{C}$ = ideal sheaves defining Λ, Z, Λ' resp.

Ω = curve on X (irrelevant for us here).

noetherian $\Leftrightarrow$ $\Lambda \cup \Lambda'$ is supported at points w/ dense σ-orbits

Useful because $B^{(n)}$ is generated in deg 1 & 2 for some $n > 0$

& and so $U(W_+)$

Aside: These rings (2) & (3) along w/ "idealizers" were studied by my coauthor, Dan Rogalski, Toby Stafford to classify (noncommutative) birationally commutative projective surfaces.

= noetherian, connected graded domains w/ GKdim 3 (NC Krull dim)

so that the graded quotient ring $\simeq D[z^{\pm 1}; \varphi]$ $\left(\begin{array}{l} \varphi \in \mathrm{Aut}\, D \\ zg = g^{\varphi} z \quad \forall g \in D \end{array} \right)$

division ring in general, commutative here

Roughly speaking —

$U(W_+)$ has a homomorphic image R that's "close to" an ADC ring with $\Lambda = \{r\}$ & $\Lambda' = \{s\}$ points having non-dense orbits

$\Rightarrow$ R is not noeth $\Rightarrow$ $U(W_+)$ not noeth $\rightarrow$ $U(W)$ not noeth.

Sketch of proof of theorem ($U(W)$ is not noetherian)

Get homomorphism $\rho: U(W_+) \longrightarrow k(X)[t, \tau]$

$e_1 \longmapsto t$

$e_2 \longmapsto f t^2$

projective surface $X = \mathbb{V}(xz - y^2) \subseteq \mathbb{P}^3_{[w:x:y:z]}$

$\tau \in \mathrm{Aut}\, X$

$f = \dfrac{w + 12x + 22y + 8z}{12x + 6y} \in k(X)$

Let $R =$ image ρ — STS R is not noetherian

Take $R^{(2)}$ veronese subring — STS $R^{(2)}$ is not noetherian

Get $R^{(2)} \subseteq B(X, \mathcal{L}, \tau^2, \{r\}, \{s\})$ non-noetherian ADC ring w/ $\boxed{\begin{matrix}\Phi(r) \\ \cap \\ C_r\end{matrix}}$ $\begin{matrix}\Phi(s) \\ \cap \\ C_s\end{matrix}$ curves:

$$\subseteq T = \bigoplus H^0(Y, \mathcal{I}_r(\mathcal{L}_n \otimes_X \mathcal{O}_Y)) \quad \mathcal{I}_r = 2C_r$$

So $R^{(2)}$ is close to non-noeth ADC ring but we have to be more precise to draw our conclusion

Get T is a finitely generated $R^{(2)}$-module — STS T is not noetherian

T is not noetherian:

* $\mathcal{J}$ = ideal sheaf on Y^{2C_r} defining C_r
* $J = \bigoplus_{n \geq 1} H^0(Y, \underbrace{\mathcal{J}(\mathcal{L}_n \otimes_X \mathcal{O}_Y)}_{\text{globally generated}})$ left ideal of T
* $\mathcal{I}_r \mathcal{J} \subsetneq \mathcal{J}$ because $\mathcal{J}/\mathcal{I}_r\mathcal{J} = \mathcal{J} \otimes_Y \mathcal{O}_r \neq 0$

* $H^0(Y, \mathcal{I}_r \mathcal{J}(\mathcal{L}_{n+\ell} \otimes_X \mathcal{O}_Y)) \subsetneq H^0(Y, \mathcal{J}(\mathcal{L}_{n+\ell} \otimes_X \mathcal{O}_Y))$

$\Rightarrow T_n J_\ell \subsetneq J_{n+\ell} \quad \forall n \geq 1$

$\Rightarrow {}_T J$ not finitely generated

$\Rightarrow T$ is not noetherian ///

$\Rightarrow R^{(2)}, R, U(W_+), U(W)$ are not noetherian ///.

Consequence Conjecture: L finite dim'l $\iff$ $U(L)$ noetherian holds for other ∞-dim'l LAs.

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(1) Virasoro algebra V = Lie algebra gen by $\{e_n\}_{n\in\mathbb{Z}} \cup \{c\}$ w/ bracket $[e_n, e_m] = (m-n)e_{n+m} + \frac{c}{12}(m^3-m)\delta_{n+m,0}$

Prop $U(V)$ not noetherian.
pf/ $U(V)/(c) \cong U(W)$ not noetherian ///.

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(2) $\mathbb{Z}$-graded simple Lie algebras L of polynomial growth. (classified by O. Mathieu)

Prop: $U(L)$ is not noetherian
pf/
L either contains $\begin{cases} W_+ \text{ or} \\ L' \ \infty\text{'dim'l abelian Lie subalg} \end{cases}$
$U(W_+)$ not noeth ✓
$U(L') \cong k[x_1, x_2, \ldots]$ not noetherian
faithful flatness $U(L)$ not noetherian. ///

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Still interested in conjecture
- maybe some hope for completing this for all graded Lie algs
Otherwise, I don't know