PBW DEFORMATIONS OF BRAIDED PRODUCTS

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'Doubled' algebras: What are they and Why care?

An algebra $\mathscr{D}$ has a *doubled structure* if $\mathscr{D} \cong A \otimes H \otimes B$ as vector spaces, where

(i) H is a Hopf algebra that is a subalgebra of $\mathscr{D}$,

(ii) H acts on algebras A, B, and

(iii) A and B are compatible in some sense.

In this case, we call $A \otimes H \otimes B$ a *triangular decomposition* (or PBW decomposition) of $\mathscr{D}$.

Model: the decomposition of $U(\mathfrak{g})$, for $\mathfrak{g}$ a f.dim'l s.s. Lie algebra, as $U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)$ via the classical PBW theorem, where $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is the triangular decomposition of $\mathfrak{g}$.

‘Doubled’ algebras: What are they and Why care?

Key Examples...

$\mathscr{D} \cong A \otimes H \otimes B$, as vec spaces	H	A	B
n-th Weyl algebra	a field k	$k[x_1, \ldots, x_n]$	$k[y_1, \ldots, y_n]$, $y_i = \frac{\partial}{\partial x_i}$
rational (degenerate) DAHA $=$ rational Cherednik algebra	$\mathbb{C}\Gamma$, $\Gamma \leq GL_n(\mathbb{C})$ cpx. ref. group	$\mathbb{C}[x_1, \ldots, x_n]$	$\mathbb{C}[y_1, \ldots, y_n]$
trigonometric (deg.) DAHA	$\mathbb{C}\Gamma$	$\mathbb{C}[x_1, \ldots, x_n]$	$\mathbb{C}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$

[DAHA = double affine Hecke algebra]

...whose ring theory and representation theory has been of great interest (especially in the 15 years). These structures have been used to attack/ understand:

- Heisenberg’s Uncertainty Principle in quantum mechanics,
- MacDonald’s conjectures in combinatorics,
- generalized Calogero-Moser systems (integrable systems), and
- multivariable special functions in harmonic analysis.

Poincaré-Birkhoff-Witt (PBW) deformations

Let $\mathcal{D} = \bigcup_{i\geq 0} F_i$ be a filtered algebra with $\{0\} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq \mathcal{D}$.

We say that $\mathcal{D}$ is a Poincaré-Birkhoff-Witt (PBW) deformation of a $\mathbb{N}$-graded algebra R if $\mathrm{gr}_F \mathcal{D} = \bigoplus_{i\geq 0} F_i/F_{i-1}$ is isomorphic to R, as a $\mathbb{N}$-graded algebra.

Take U a finite dimensional H-module.

Take $J \subseteq U \otimes U$ an H-submodule so that
$R = T(U)/(J)$ is an $\mathbb{N}$-graded H-module algebra (H-action preserves grading of R).

Assign the elements of H degree 0, and elements of U degree 1.

Take a $\Bbbk$-bilinear map $\kappa : J \longrightarrow H \oplus (U \otimes H)$,
the sum of its degree 0 and degree 1 parts, $J \to H$ and $J \to U \otimes H$.

A **PBW Theorem** is a set of necessary and sufficient conditions on κ for

$$\mathcal{D} := \mathcal{D}_{R,\kappa} = \frac{T(U)\#H}{(r - \kappa(r))_{r\in J}}$$

to be a PBW deformation of $R\#H$.

Poincaré-Birkhoff-Witt (PBW) deformations

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In this case,

$U = V \quad\quad \rightarrow \mathcal{D}$ is a [insert adjectives] Hecke algebra

$U = V \oplus V' \rightarrow \mathcal{D}$ is a [insert adjectives] Cherednik algebra/ DAHA

W-Witherspoon (2014) established a **PBW Theorem** under the conditions above in the case when R **is Koszul**.

Aim: Find PBW deformations $\mathcal{D}$ of $(A \otimes^* B)\#H$ with $A = T(V)/(I)$, $B = T(V')/(I')$

— Get triangular decomposition: $\mathcal{D} \cong A \otimes H \otimes B$ as vector spaces

Bazlov-Berenstein's *braided doubles*

Bazlov-Berenstein (2009) studied when algs of the form $\mathcal{D} = \dfrac{T(V \oplus V')\#H}{(I, I', \text{mixed relations})}$

have triangular decomposition $A \otimes H \otimes B$ (PBW theorem)

Conditions	**Work of Bazlov-Berenstein**
On H	$*$ None $*$
On A	Is denoted $T(V)/(I^-)$ $*$ just an H-mod alg, no other restrictions $*$ e.g. symm alg $S(V)$ or Nichols alg $\mathfrak{B}(V)$ or $\mathbf{q}$-symm alg $S_{\mathbf{q}}(V)$
On B	Is $T(V^*)/(I^+)$ so that $T(V^*)/(I^+)$ and $T(V)/(I^-)$ satisfy "non-deg. Harish-Chandra pairing" e.g. symm alg $S(V^*)$ or Nichols alg $\mathfrak{B}(V^*)$ or $\mathbf{q}$-symm alg $S_{\mathbf{q}}(V^*)$
On product of A and B	Prescribed "mixed" relations: e.g. $[f,v]=0$ or $[f,v]_{\mathbf{q}}=0$, $f \in V^*$ $v \in V$ Is not always an H-module algebra
On deformations of relations of A, B	Relations are not deformed
On deformations of "mixed" relations of product A and B	Deform by elements of degree 0 only: for $\Bbbk$-vs map $\beta : V^* \otimes V \to H$, get, e.g., $[f,v]=\beta(f,v)$ or $[f,v]_{\mathbf{q}}=\beta(f,v)$

Our framework: braided products

Aim: Find PBW deformations $\mathcal{D}$ of $(A \otimes^* B)\#H$,
with $A = T(V)/(I)$ and $B = T(V')/(I')$
Get triangular decomposition: $\mathcal{D} \cong A \otimes H \otimes B$ as vector spaces

Bazlov-Berenstein's framework:

- Profit: No conditions on H, A
- Price: Limitation on B in terms of A
- Price: Mixed relations are prescribed
- Price: Relations of A, B don't get deformed, only mixed relations get deformed

Our framework with braided products:

- Price: Condition on (category of modules over) H
- Price: need A, B Koszul
- Profit: "Mixed" relations are natural
- Profit: Relations of A, B, "mixed" relations can all get deformed

Our framework: braided products

Take H so that there's a full subcategory $\mathcal{C}$ of H-modules equipped with a braiding:

$$\mathrm{c} = \mathrm{c}_{M,N} : M \otimes N \xrightarrow{\sim} N \otimes M, \quad M, N \in \mathcal{C}.$$

Take A, B algebras in $\mathcal{C}$ (an H-module algebra)

Could take $B = A_{\mathrm{c}}^{op}$ the braided-opposite of A, with multip $m_A \circ \mathrm{c}$, an algebra in $\mathcal{C}$.

Can form $A \otimes^{\mathrm{c}} B$ the braided product of A and B, which is $A \otimes B$ as a vector space, with multip given by $(m_A \otimes m_B) \circ (1 \otimes \mathrm{c} \otimes 1)$. This is also an algebra in $\mathcal{C}$.

Proposition: If A (and B) is Koszul, then so are A_{c}^{op} and $A \otimes^{\mathrm{c}} A_{\mathrm{c}}^{op}$ (and $A \otimes^{\mathrm{c}} B$).

Now apply W-Witherspoon's 2014 PBW Theorem for Hopf actions on Koszul algs to get the desired PBW deformations of $(A \otimes^{\mathrm{c}} B)\#H$.

If $(A \otimes^{\mathrm{c}} B)\#H$ doesn't have many PBW deform's, swap c with twisting H-mod map

$$\tau : B \otimes A \longrightarrow A \otimes B$$

to form twisted tensor product $A \otimes^{\tau} B$. Then proceed as above...

Main result: Statement

Theorem. Given certain Hopf algebras H and Koszul algebras A, B as listed below, we find PBW deformations $\mathcal{D}$ of degree 0 of the smash product algebra $(A \otimes^* B)\#H$, where $\otimes^*$ is either

- a braided product $\otimes^c$ (when a categ of H-modules is braided), or
- a twisted tensor product $\otimes^\tau$ (in general).

Here, either
– the Hopf algebra H is non-cocommutative or
– the Koszul algebras A, B are noncommutative.

The parameter space of all of such PBW deformations is computed in the cases denoted by ★ below.

Main result: Examples for H noncocommutative

H	A	B	**Braiding/ Twisting**	**Parameter space of PBW deformations of degree 0 of** $(A \otimes^* B)\# H$
$U_\zeta(\mathfrak{sl}_2)$	$\Bbbk_\zeta[u,v]$	$A_{\mathrm{c}}^{\mathrm{op}} = \Bbbk_\zeta[u,v]$	braiding c	$\Bbbk \times \mathbb{Z}^3 \times \mathbb{N}^4$
$U_q(\mathfrak{sl}_2)$	$\Bbbk_q[u,v]$	$\Bbbk_q[u,v]$	twisting τ	$\Bbbk \times \mathbb{Z}$
$T(2)$	$\Bbbk[u,v]$	$A_{\mathrm{c}}^{\mathrm{op}} = \Bbbk[u,v]$	braiding c	$\Bbbk$ ★

$U_*(\mathfrak{sl}_2)$ = quantized enveloping algebras

$T(2)$ = Sweedler (Hopf) algebra

$\Bbbk_*[u,v]$ = (quantum) polynomial ring

ζ = primitive third root of unity

q = primitive n-th root of unity, $n \geq 3$

Main result: Examples for A, B noncommutative

H	A	B	**Braiding**	**Parameter space of PBW deformations of degree 0 of** $(A \otimes^c B)\#H$
$\Bbbk C_2$	$\Bbbk_J[u,v]$	$A_c^{op} \cong \Bbbk_J[u,v]$	non-trivial	$\Bbbk^3$ ★
$\Bbbk C_2$	$\Bbbk_J[u,v]$	$A_c^{op} \cong \Bbbk_J[u,v]$	trivial	$\Bbbk^3$ ★
$\Bbbk C_2$	$S(a,b,c)$	$A_c^{op} = S(b,a,c)$	non-trivial	$\Bbbk^6$ if $a \neq b$ $\Bbbk^{15}$ if $a = b$ ★
$\Bbbk C_2$	$S(a,b,c)$	$A_c^{op} = S(b,a,c)$	trivial	$\Bbbk^6$ if $a \neq b$ $\Bbbk^{15}$ if $a = b$ ★

$\Bbbk_J[u,v]$ = Jordan plane

$S(a,b,c)$ = three-dimensional Sklyanin algebra

A question to consider

H	A	B	**Braiding**	**Parameter space of PBW deformations of degree 0 of** $(A \otimes^{c} B)\#H$
$\Bbbk C_2$	$\Bbbk_J[u,v]$	$A_c^{op} \cong \Bbbk_J[u,v]$	non-trivial	$\Bbbk^3$ ★
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$\Bbbk C_2$	$S(a,b,c)$	$A_c^{op} = S(b,a,c)$	non-trivial	$\Bbbk^6$ if $a \neq b$ $\Bbbk^{15}$ if $a = b$ ★
$\Bbbk C_2$	$S(a,b,c)$	$A_c^{op} = S(b,a,c)$	trivial	$\Bbbk^6$ if $a \neq b$ $\Bbbk^{15}$ if $a = b$ ★

Question: Is the PBW deformation parameter space of $(A \otimes^{c} B)\#H$ independent of the choice of the braiding c? When H is cocommutative?

Another problem to consider

Theorem. Given certain Hopf algebras H and Koszul algebras A, B, we find PBW deformations $\mathcal{D}$ of degree 0 of the $(A \otimes^* B)\#H$, where $\otimes^*$ is either a braided product $\otimes^{\mathrm{c}}$ or a twisted tensor product $\otimes^{\tau}$.
Here, either
– the Hopf algebra H is non-cocommutative or
– the Koszul algebras A, B are noncommutative.
The parameter space of <u>all</u> of such PBW deformations is computed in some examples denoted by ★ above.

<u>Problem</u>: Extend this work to classify PBW deformations of $(A \otimes^{\mathrm{c}} B)\#H$ (or, in particular, of $(A \otimes^{\mathrm{c}} A_{\mathrm{c}}^{\mathrm{op}})\#H$) <u>of degree 1</u>.

Another direction: q-deformed infinitesimal Cherednik algs

[Etingof-Gan-Ginzburg (2005)] infinitesimal Cherednik algebras (iCa) = degree 0 PBW deformations of $(S(V) \otimes S(V^*))\#U(\mathfrak{g})$, for a Lie algebra $\mathfrak{g}$.

$*$ The iCa arise from continuous Cherednik algebras (cCa) = degree 0 PBW deformations of $(S(V) \otimes S(V^*))\#\mathcal{O}(G)^*$, for G an algebraic group with $\mathfrak{g} = \mathrm{Lie}(G)$.

$*$ The iCa are also realized as certain W-algebras [Losev-Tsymbaliuk (2014)]

Problem (Etingof): Define and study a q-analogue of the infinitesimal Cherednik/ Hecke algebras. (Say, for $\mathfrak{g} = \mathfrak{gl}_n$.) Do the same for corresp. cCa's and W-algs.

Possible approach: Use our framework. For instance...

H	A	B	Param. space of PBW deform'n of deg 0 of $(A \otimes^\tau B)\#H$
$U_q(\mathfrak{sl}_2)$	$k_q[u,v]$	$k_q[u,v]$	$k \times \mathbb{Z}$

...extends to actions of $U_q(\mathfrak{gl}_2)$ also yielding PBW deform. parameter space $k \times \mathbb{Z}$.