Symmetries of Algebras

Volume 1

by

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To Matt & The Zoo

Preface

Welcome! I am happy that you are here, and I am excited to present to you a topic that I have enjoyed over the last several years. My journey towards landing on this topic began in graduate school, where I studied material related to the first chapter of this book (namely, algebras over a field). Later as a faculty member, I encountered the material comprising the remaining chapters here, discovering the world of Algebraic Quantum Symmetry. I've found all of the structures in this field quite beautiful and important in their own way.

This book is geared for newcomers who would love to learn about intriguing algebraic structures in nature beyond their first Abstract Algebra course(s). By now, you might have skimmed the table of contents and thought to yourself, "I know a few of these words, but certainly not all." If you are concerned about this, fret not. To be perfectly honest with you, I did not know most of those words as a student, but this is the way that it is supposed to work. Mathematics is certainly not a 'young man's game'; it is for everyone who simply wants an adventure in discovering new knowledge.

So, I wish to serve as your guide in finding and understanding some fascinating algebraic structures, motivated by the concept of symmetry. Let's proceed!

– C. Walton, 2024

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"You can do everything with tensor categories!" – P. Etingof.

This sentiment inspired the last decade of my research program, and it is my hope that this book contributes as proof of the statement.

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