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Algebra Seminar
UIUC
F. Sept 6, 2019

Quantum Symmetry in the context of co/representation categories

Part I. Introduction (Expo talk)

Broad Goal: To understand symmetries of a given object X

(invertible)
property-preserving transformations from X to X
↳ collection of which forms an algebraic structure $\mathrm{Sym}(X)$

Ex. X = regular n-gon, $n \geq 3$ $\rightsquigarrow$ $\mathrm{Sym}(X) = D_{2n}$, dihedral group
$X = \{1, \dots, n\}$, $n \geq 1$ $\rightsquigarrow$ $\mathrm{Sym}(X) = S_n$, symmetric group

$\rightsquigarrow$ A classical framework for symmetries: group actions
$\mathrm{Sym}(X) = \mathrm{Aut}_{*}(X)$, automorphism group
$*$ sometimes conditions are imposed on symmetries
$*$ this yields a subgroup of full automorphism group

Setting X = $\Bbbk$-algebra, $\Bbbk$ = field
$\Bbbk$-vector space !! a $\Bbbk$-vs that has the compatible structure of a ring

$\otimes := \otimes_{\Bbbk}$

$A = (A,\ m: A \otimes A \to A,\ u: \Bbbk \to A)$ multiplication, unit, $\Bbbk$-linear maps, satisfying assoc. & unit conds

$$\begin{tikzcd}
(A\otimes A)\otimes A \arrow[rr, "\text{assoc}"] \arrow[d, "m\otimes id"'] & & A\otimes(A\otimes A) \arrow[d, "id\otimes m"] \\
A\otimes A \arrow[dr, "m"'] & \circlearrowright & A\otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A\otimes A \arrow[dd, "m"] & \\
\Bbbk\otimes A \arrow[ur, "u\otimes id"] \arrow[dr, "\cong"'] & \circlearrowright \quad \circlearrowright & A\otimes\Bbbk \arrow[ul, "id\otimes u"'] \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}$$

We'll see soon that we'll need to work beyond group actions to study symmetries of algebras more faithfully.

Commutative case $A = \Bbbk[v_1, \dots, v_n] = S(V)$ commutative polynomial alg., $\dim_{\Bbbk} V = n$

Note: ① $A = \mathcal{O}(\mathbb{A}^n_{\Bbbk}) = \Bbbk[\mathbb{P}^{n-1}]$
coordinate ring, homog. coord. ring

& symmetries of $A \longleftrightarrow$ symmetries of $\mathbb{A}^n$ or $\mathbb{P}^{n-1}$
so studying Aut A has geometric implications

(2) Computing the automorphism group Aut A is tough in general:

- Aut $\Bbbk[v]$ is affine (also given by $v \longmapsto \alpha v + \beta$; $\alpha \neq 0$, $\alpha, \beta \in \Bbbk$)
- Aut $\Bbbk[v_1, v_2]$ is tame (generated by affine & "triangular" automs)
- Aut $\Bbbk[v_1, v_2, v_3]$ is wild ($\equiv$ not tame, not understood).

So there's interesting work & open problems in studying symmetries of commutative algebras via group actions

(3) make this easier by imposing condition: linearity
namely suppose $v_i \longmapsto \sum_{j=1}^{n} \alpha_{ij} v_j$

$\rightsquigarrow$ $\text{Aut}_{\text{linear}}(\Bbbk[v_1, \dots, v_n]) = GL_n(\Bbbk)$, general linear group

Quantum Symmetry

On Deformations / Noncom. Case

Philosophy: If an object gets deformed, so should its collection of symmetries. (there should some framework for this)

Aside

Ex. $\triangle$ equilateral triangle $\rightsquigarrow$ $\text{Aut}(\triangle) = D_6$

deformation

$\triangle$ scalene triangle $\rightsquigarrow$ $\text{Aut}(\triangle) = \langle e \rangle$

Aside

By altering triangle slightly, the group of symmetries drop dramatically

There should be a framework for symmetries (beyond groups) where the "drop" doesn't occur.

Ex. $A = \mathbb{k}[v_1, v_2] \rightsquigarrow \mathrm{Aut}_{\text{linear}}(A) = GL_2(\mathbb{k}) =: G$

$= \dfrac{\mathbb{k}\langle v_1, v_2\rangle}{(v_2v_1 - v_1v_2)}$ free algebra

$v_1 \mapsto \alpha_{11}v_1 + \alpha_{12}v_2$

$v_2 \mapsto \alpha_{21}v_1 + \alpha_{22}v_2$ $\quad (\alpha_{ij}) \in G.$

deformation

$q \in \mathbb{k}^\times$ $\quad A_q = \dfrac{\mathbb{k}\langle v_1, v_2\rangle}{(v_2v_1 - qv_1v_2)} \rightsquigarrow \mathrm{Aut}_{\text{linear}}(A_q) = \begin{cases} GL_2(\mathbb{k}) & q=1 \\ \text{skew/ diag. matrices} & q=-1 \\ \text{diag. matrices} & q \neq \pm 1. \end{cases}$

(= A as $\mathbb{k}$-vs but multip'n is deformed)

conditions on $(\alpha_{ij}) \in G$.

Namely need relation to be preserved:

$$v_2v_1 - qv_1v_2 \mapsto (1-q)\alpha_{11}\alpha_{21}v_1^2 + (\alpha_{21}\alpha_{12} - q\alpha_{11}\alpha_{22})v_1v_2 + (\alpha_{22}\alpha_{11} - q\alpha_{12}\alpha_{21})\underbrace{v_2v_1}_{=qv_1v_2} + (1-q)\alpha_{12}\alpha_{22}v_2^2$$

need framework that circumvents this "drop" of symmetries phenomenon

WANT $= \lambda(v_2v_1 - qv_1v_2)$ for some $\lambda \in \mathbb{k}$.

Symmetries of algebras in general

Working defn: Given a $\mathbb{k}$-algebra $(A, m_A: A\otimes A \to A, u_A: \mathbb{k} \to A)$, a $\mathbb{k}$-linear structure (*) H captures symmetries of A if

- A = H-module: $\exists$ $\mathbb{k}$-linear map $\mu_A: H\otimes A \to A$ so that

$$\begin{array}{ccc} H\otimes H\otimes A & \xrightarrow{m_H \otimes id_A} & H\otimes A \\ {\scriptstyle id_H\otimes\mu}\downarrow & \circlearrowleft & \downarrow{\scriptstyle\mu} \\ H\otimes A & \xrightarrow{\mu} & A \end{array}$$

(*) need map m_H

• $m_A : A\otimes A \to \Bbbk$ is preserved under μ_A

$$\begin{tikzcd}
H\otimes A\otimes A \arrow[r, "id_H \otimes m_A"] \arrow[d, "\mu_{A\otimes A}"'] \arrow[dr, phantom, "\circlearrowright"] & H\otimes A \arrow[d, "\mu_A"] \\
A\otimes A \arrow[r, "m_A"] & A
\end{tikzcd}$$

(*) need $A\otimes A$ is an H-module

• $u_A : \Bbbk \to A$ is preserved under μ_A

$$\begin{tikzcd}
H\otimes \Bbbk \arrow[r, "id_H \otimes u_A"] \arrow[d, "\mu_{\Bbbk}"'] \arrow[dr, phantom, "\circlearrowright"] & H\otimes A \arrow[d, "\mu_A"] \\
\Bbbk \arrow[r, "u_A"] & A
\end{tikzcd}$$

(*) need $\Bbbk$ is an H-module.

Say that H acts on A in this case

Example

① Replace group with group algebra $H = \Bbbk G$

$\Bbbk G = \bigoplus_{g\in G} \delta_g$ as a $\Bbbk$-vs, where $\delta_g = \Bbbk g$. 1-dim'l $\Bbbk$-vs

have m_H → with multiplication: $\delta_g \otimes \delta_{g'} \overset{\text{def}}{=} \delta_{gg'}$ $\forall g, g' \in G$.

Here G-modules $\equiv$ $\Bbbk G$-modules

• $A = \Bbbk G$-module via group action $G\otimes A \to A$, $g\otimes a \mapsto g\cdot a$ & by extending linearly to $\mu : \Bbbk G\otimes A \to A$

• $A\otimes A = \Bbbk G$-module via diagonal map $\Delta : G \to G\times G$, $g \mapsto g\otimes g$ & extending linearly to $\Bbbk G$

have $\mu_{A\otimes A}$ →

$$\begin{tikzcd}
\mu_{A\otimes A} : \Bbbk G\otimes A\otimes A \arrow[r] \arrow[d, "\Delta\otimes id\otimes id"'] \arrow[dr, phantom, "\circlearrowright \text{ defn}"] & A\otimes A \\
\Bbbk G\otimes \Bbbk G\otimes A\otimes A \arrow[r, "id\otimes flip\otimes id"'] & \Bbbk G\otimes A\otimes \Bbbk G\otimes A \arrow[u, "\mu_A\otimes\mu_A"']
\end{tikzcd}$$

$$\Leftrightarrow \quad g\cdot(a\otimes b) = (g\cdot a)\otimes(g\cdot b).$$

$\forall g\in G;\ a, b\in A$

• $\Bbbk = \Bbbk G$-module via augmentation map $\varepsilon : \Bbbk G \to \Bbbk$
$\sum_i \alpha_g \delta_g \mapsto \sum \alpha_g$.

Have $\mu_{\Bbbk}$ →

$$\begin{tikzcd} \Bbbk G \otimes \Bbbk \arrow[rr, "\mu_{\Bbbk}"] \arrow[d, "\varepsilon \otimes id"'] & & \Bbbk \\ \Bbbk \otimes \Bbbk \arrow[urr, "\sim"', "2\dim"] & & \end{tikzcd}$$

$\iff \quad g \cdot \lambda = \varepsilon(g)\lambda = \lambda \qquad \forall g \in G,\ \lambda \in \Bbbk.$

Putting this together:

A group algebra $\Bbbk G$ (or equivalently a group G)
acts on an algebra (A, m, u) if
- $A = \Bbbk G$-module
- $m, u = \Bbbk G$-module maps, used $\begin{cases} A \otimes A \\ \Bbbk \end{cases}$ are $\Bbbk G$-modules to get this
  $\left[\text{determined by } \begin{array}{l} g\cdot(ab) = (g\cdot a)(g \cdot b) \\ g \cdot 1_A = \varepsilon(g) 1_A \end{array} \right]$

A clean framework

Fix H with underlying $\Bbbk$-algebra structure : get m_H

Take $\mathcal{C} = H$-mod, the category of H-modules
- objects : H-modules
- morphisms : $\Bbbk$-linear maps $f : M \to N$, for H-mods M, N
  so that
$$\begin{tikzcd} H \otimes M \arrow[r, "id \otimes f"] \arrow[d, "\mu_M"'] \arrow[dr, phantom, "\circlearrowleft"] & H \otimes N \arrow[d, "\mu_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Also have bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
$(M, N) \mapsto M \otimes N$ get $\mu_{A \otimes A}$

Along with distinguished object $\mathbb{1} \in \mathcal{C}$ get $\mu_{\Bbbk}$
satisfying compatibility axioms.

$\mathcal{C}$ is called a **monoidal category** (of representations or modules of H)

Example

② H = bialgebra / $\Bbbk$. ($\Bbbk G$ is a bialgebra)

$= (H, m, u, \Delta, \varepsilon)$

$(H, m: H\otimes H \longrightarrow H, u: \Bbbk \longrightarrow H)$ "$\Bbbk$-algebra" , $(H, \Delta: H \to H\otimes H, \varepsilon: H \longrightarrow \Bbbk)$ comultip. counit

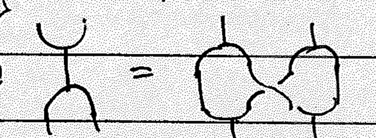
satisfying coassociativity & counit conditions

"$\Bbbk$-coalgebra"

m & Δ are compatible like so

$$\begin{tikzcd} H\otimes H \arrow[rr, "m"] \arrow[d, "\Delta\otimes\Delta"'] & & H \arrow[d, "\Delta"] \\ H^{\otimes 4} \arrow[r, "id\otimes flip\otimes id"] & H^{\otimes 4} \arrow[r, "m\otimes m"] & H\otimes H \end{tikzcd}$$

Graphical calculus



Get $\operatorname{Rep} H \equiv H\text{-mod}$ is a monoidal category with $\mathbb{1} = \Bbbk$.

(so is $\operatorname{Rep}_{\Bbbk} G \equiv \Bbbk G\text{-mod}$ with $\mathbb{1} = \Bbbk$)

Explicit example

$H_q' = \Bbbk\langle g^{\pm 1}, x\rangle$ $\quad q \in \Bbbk^\times$

$(gg^{-1} = g^{-1}g = 1,\ gx = q^2 x g)$ — acts on — $A_q = \dfrac{\Bbbk\langle v_1, v_2\rangle}{(v_2 v_1 - q v_1 v_2)}$

with

$\Delta(g^{\pm 1}) = g^{\pm 1}\otimes g^{\pm 1}$

$\Delta(x) = 1\otimes x + x\otimes g$

$\varepsilon(g^{\pm 1}) = 1$

$\varepsilon(x) = 0$

is a $\Bbbk$-bialgebra

via

$g^{\pm 1}\cdot v_1 = q^{\pm 1} v_1$, $g^{\pm 1}\cdot v_2 = q^{\mp 1} v_2$.

$x \cdot v_1 = 0$, $x\cdot v_2 = v_1$

as $q \to 1$ get group action on $\Bbbk[v_1, v_2]$

so as the multiplication of A_q is deformed by changing q, so is its collection of bialgebra "quantum" syms via H_q

Next talk: we'll broaden the context of quantum symmetries by using $H = \begin{cases} \text{Hopf algebra} \\ \text{weak bialgebra / weak Hopf algebra} \end{cases}$ (structures of indep. interest)

Get that $\mathcal{C} = H\text{-mod} \equiv \text{Rep } H$ is monoidal

- for Hopf algs in a completely analogous way
- for WHAs, not so straightforward.

Results will be focused on H-comods $\equiv \text{Corep } H$, in fact (still monoidal).

(Research talk, so will go a bit faster)

Weak bialgebras & Weak Hopf algebras

—HANDOUT—

Ref: Böhm–Nill–Szlachányi (1998)
Nikshych, Nikshych–Vainerman (~200

Definition A weak bialgebra H is an associative $\Bbbk$-algebra (H, m, u) & a coassociative $\Bbbk$-coalgebra (H, Δ, ε) with compatibility conditions:

(i) m with Δ (as in bialgebras)

$$\begin{tikzcd}
H\otimes H \arrow[r, "m"] \arrow[d, "\Delta\otimes\Delta"'] \arrow[dr, phantom, "\circlearrowleft"] & H \arrow[d, "\Delta"] \\
H^{\otimes 4} \arrow[r, "\mathrm{id}\otimes\mathrm{flip}\otimes\mathrm{id}"'] & H^{\otimes 4} \arrow[r, "m\otimes m"'] & H\otimes H
\end{tikzcd}$$

$$\begin{tikzcd}
h\otimes h' \arrow[r, mapsto] \arrow[d, mapsto] & hh' \arrow[d, mapsto] \\
\Delta(h)\otimes\Delta(h') \arrow[r, mapsto] & \boxed{\begin{matrix}\Delta(hh') \\ \shortparallel \\ \Delta(h)\Delta(h')\end{matrix}}
\end{tikzcd}$$

(ii) Δ with u (weaker than bialg. compatibility
$$\begin{tikzcd}
& \Bbbk \arrow[dl, "u"'] \arrow[dr, "u\otimes u"] & \\
H \arrow[rr, "\Delta"'] & & H\otimes H
\end{tikzcd}, \quad \Bbbk \cong \Bbbk\otimes\Bbbk \Big)$$

$$\begin{tikzcd}
& \Bbbk \arrow[dl, "u"'] \arrow[dr, "u\otimes u"] & \\
H \arrow[d, "\Delta"'] & & H\otimes H \arrow[dd, "\Delta\otimes\Delta"] \\
H\otimes H \arrow[d, "\Delta\otimes\mathrm{id}"'] & & \\
H^{\otimes 3} & & H^{\otimes 4} \arrow[ll, "\mathrm{id}\otimes m\otimes\mathrm{id}"'] \arrow[dl, "\mathrm{id}\otimes\mathrm{flip}\otimes\mathrm{id}"] \\
& H^{\otimes 4} \arrow[ul, "\mathrm{id}\otimes m\otimes\mathrm{id}"] &
\end{tikzcd}$$

commutes

$$\boxed{\begin{matrix}\Delta^2(1) \\ \shortparallel \\ 1_1\otimes 1_2 1'_1\otimes 1'_2 \\ \shortparallel \\ 1_1\otimes 1'_1 1_2\otimes 1'_2\end{matrix}}$$

$1_H =: 1 = 1'$ another copy

(iii) m with ε (weaker than bialg. compatibility
$$\begin{tikzcd}
H\otimes H \arrow[rr, "m"] \arrow[dr, "\varepsilon\otimes\varepsilon"'] & & H \arrow[dl, "\varepsilon"] \\
& \Bbbk &
\end{tikzcd}, \quad \Bbbk \cong \Bbbk\otimes\Bbbk \Big)$$

$$\begin{tikzcd}
& H^{\otimes 3} \arrow[dl, "\mathrm{id}\otimes\Delta^{op}\otimes\mathrm{id}"'] \arrow[d, "\mathrm{id}\otimes\Delta\otimes\mathrm{id}"] \arrow[dr, "m\otimes\mathrm{id}"] & \\
H^{\otimes 4} \arrow[d, "m\otimes m"'] & H^{\otimes 4} \arrow[dl, "m\otimes m"] & H\otimes H \arrow[d, "m"] \\
H\otimes H \arrow[dr, "\varepsilon\otimes\varepsilon"'] & & H \arrow[dl, "\varepsilon"] \\
& \Bbbk &
\end{tikzcd}$$

$$\boxed{\begin{matrix}\varepsilon(hh'h'') \\ \shortparallel \\ \varepsilon(hh'_1)\,\varepsilon(h'_2h'') \\ \shortparallel \\ \varepsilon(hh'_2)\,\varepsilon(h'_1h'')\end{matrix}}$$

A weak Hopf algebra H is a weak bialgebra along with $\Bbbk$-linear map $S: H \to H$ "weak antipode"

so that
$$\boxed{\begin{aligned} S(h_1)h_2 &= 1_1\,\varepsilon(h1_2) \\ h_1S(h_2) &= \varepsilon(1_1h)\,1_2 \\ S(h) &= S(h_1)h_2S(h_3)\end{aligned}}$$

(weaker than Hopf compatibility:
$$S(h_1)h_2 = \varepsilon(h)1_H = h_1S(h_2)\Big)$$

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special maps $H \longrightarrow H$ that measure how far H is from being a Hopf algebra

$$\boxed{\varepsilon_t} : H \longrightarrow H, \quad h \longmapsto \varepsilon(1_1 h) 1_2 \qquad \& \qquad \boxed{\varepsilon_s} : H \longrightarrow H, \quad h \longmapsto 1_1 \varepsilon(h 1_2)$$

<u>source</u> & <u>target</u> <u>counital maps</u>

$\boxed{H_s}$ = image (ε_s)

$\boxed{H_t}$ = image ε_t

<u>source & target</u>
<u>counital subalgebras</u>

these are separable $\Bbbk$-algebras

A weak Hopf alg is a Hopf alg.
$\iff \Delta(1_H) = 1_H \otimes 1_H$
$\iff \varepsilon$ is an algebra map
$\iff H_s = H_t = \Bbbk 1_H$

<u>Example</u>: Groupoid algebra (categorical def'n)

<u>Definition</u> A <u>groupoid</u> is a small category $\mathcal{G}$ in which every morphism is an isom. $\longrightarrow$ ob($\mathcal{G}$), $\mathrm{Hom}_{\mathcal{G}}(X,Y)$ are sets.

More precisely, $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ where $\mathcal{G}_0$ = set of objects, $\mathcal{G}_1$ = set of maps / arrows:

- $\forall X, Y \in \mathcal{G}_0$, $\exists$ set of morphisms $\mathcal{G}(X,Y)$ of arrows $f : X \longrightarrow Y$ ($X = s(f)$, $Y = t(f)$)
- $\forall X \in \mathcal{G}_0$, $\exists\, id_X \in \mathcal{G}(X,X)$
- $\forall X, Y, Z \in \mathcal{G}_0$, $\exists$ function $\mathcal{G}(X,Y) \times \mathcal{G}(Y,Z) \longrightarrow \mathcal{G}(X,Z)$, $(g, f) \longmapsto gf$ (composition)
- $\forall X, Y \in \mathcal{G}_0$, $\exists$ function $\mathcal{G}(X,Y) \longrightarrow \mathcal{G}(Y,X)$, $f \longmapsto f^{-1}$ (inverse)

so that $f\, id_X = f$, $id_Y f = f$, $(hg)f = h(gf)$, $ff^{-1} = id_Y$, $f^{-1}f = id_X$

Ex. $\mathcal{G}_0 = \{X\}$ one object, get $\mathcal{G}$ = group : $\mathcal{G}(X,X) \equiv \{g : X \longrightarrow X\}$ elts of group

composition $\equiv$ operation of group
inverse $\equiv$ inverse of group

<u>Definition</u>: A <u>groupoid algebra</u> $\Bbbk\mathcal{G}$ is $\Bbbk$-vs $\bigoplus_{g \in \mathcal{G}_1} \Bbbk g$ with

$$m(g \otimes h) = \begin{cases} gh & \text{if } t(h) = s(g) \\ 0 & \text{else} \end{cases} \qquad \Delta(g) = g \otimes g$$

$$u(1_\Bbbk) = \textstyle\sum_{X \in \mathcal{G}_0} id_X \qquad \varepsilon(g) = 1$$

$$S(g) = g^{-1} \quad \forall g \in \mathcal{G}_1$$

is a weak Hopf alg

$H_s = H_t = \bigoplus_{X \in \mathcal{G}_0} \Bbbk\, id_X$

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Part II. Research Talk

joint with Elizabeth Wicks & Robert Won, in preparation/progress

Fix $\Bbbk$-field, $\otimes = \otimes_{\Bbbk}$

Goal: To study symmetries of $\Bbbk$-algebras ... using weak bi/Hopf alg. actions.

- $\Bbbk$-algebra: $A = (\underbrace{A}_{\Bbbk\text{-vs}}, \underbrace{m: A\otimes A \to A}_{\text{multip'l map}}, \underbrace{u: \Bbbk \to A}_{\text{unit}})$ satisfying assoc & unit constraints

- Ways to study symmetries of A:
  - group (G)-actions: $A|_{\Bbbk\text{-vs}}$ is a G-module; m, u are G-maps:
    $\forall g \in G,\ a, b \in A$: $g\cdot(ab) = (g\cdot a)(g\cdot b)$ $\quad g\cdot 1_A = 1_A$.
  - bialgebra actions: $A|_{\Bbbk\text{-vs}}$ is an H-module; m, u are H-maps:
    $(H, m, u, \Delta: H \to H\otimes H, \varepsilon: H \to \Bbbk)$ $\qquad h\cdot(ab) = (h_1\cdot a)(h_2\cdot b)$
    $h \mapsto h_1 \otimes h_2$ sumless Sweedler notation $\qquad h\cdot 1_A = \varepsilon(h)1_A \quad \forall h \in H, a, b \in A$.

  - cleanest way: A is an algebra in a monoidal category $\mathcal{C}$
    $\mathcal{C} = (\underbrace{\mathcal{C}}_{\text{catg.}}, \underbrace{\otimes: \mathcal{C}\times\mathcal{C} \to \mathcal{C}}_{\text{bifunctor}}, \underbrace{\mathbb{1}}_{\text{unit}}, \text{associativity \& unit isomorphism})$
    satisfying coherence relations

$A = (A, m, u)$ is an algebra in $\mathcal{C}$ ($A \in \mathrm{Alg}(\mathcal{C})$) if
$A|_{\Bbbk\text{-vs}} \in \mathcal{C}$ (object), $\quad m, u \in \mathcal{C}$ (morphisms)

A right H-comod. M is a $\Bbbk$-vs with str. map $\rho: M \to M\otimes H$ satisfying comp. condition

Ex. $\mathcal{C} = \mathrm{Rep}\,H \equiv$ (left) H-modules $=: {}_H\mathcal{M}$
or $\mathcal{C} = \mathrm{Corep}\,H \equiv$ (right) H-comodules $=: \mathcal{M}^H$
for $H = \Bbbk G$ group algebra, or general bialgebra

(action) • If $A \in \mathrm{Alg}(H\text{-mod})$, say that A is a (left) H-module algebra

(coaction) • If $A \in \mathrm{Alg}(\text{comod-}H)$, say that A is a (right) H-comodule algebra

Can use any H for which H-mod or comod-H is <u>monoidal</u>.

Ex. H = Hopf algebra$/\Bbbk$ = bialgebra$/\Bbbk$ equipped with anti-homomorphism $S: H \longrightarrow H$ "antipode" with compatibility cond.

Eg. $H = U(\mathfrak{g})$, $U_q(\mathfrak{g})$ are Hopf algebras that typically act on algs.

$H = \mathcal{O}(GL_2)$, $\mathcal{O}_q(GL_2)$ " " " " " coact on algs

$\mathcal{O}(SL_2)$, $\mathcal{O}_q(SL_2)$

$H = \mathcal{O}(Mat_2(\Bbbk))$ ($= \Bbbk[a,b,c,d]$ as algebras) are just bialgebras

$\mathcal{O}_q(Mat_2(\Bbbk))$ that typically coact on algs

Advantages of "Hopf" over "bialgebra":

- <u>duality</u>: If H is a finite dim'l Hopf algebra, so is
$$H^* = \mathrm{Hom}_{\Bbbk}(H, \Bbbk).$$
<u>Facts</u>: ${}_H\mathcal{M} \sim \mathcal{M}^{H^*}$ & $V \in {}_H\mathcal{M} \Rightarrow V^* \in {}_H\mathcal{M}$ via S.
& $H^{**} \cong H$ as Hopf algs via an evaluation map.

Hopf algebras originally appeared in algebraic topology & algebraic group theory as early as the 1940s (arXiv/0901.2460) & have since appeared in various aspects (quantum) alg/geom/top/physics and functional analysis.

Why go beyond "Hopf symmetries"? Why "weak bialgs/weak Hopf"? (arXiv: 0703441)

- still have monoidal category of co/representations
- still have duality in finite-dim'l case
- still have presence in various fields of math & physics, including quantum groups & rep thy, subfactor theory, Poisson geometry &

*mostly bi/Hopf algebroids = generalization of wbas/whas

My suspicion: Just like one needs to go beyond group actions & use bialgebras / Hopf algebras to properly capture symmetries of noncommutative algebras (esp. deformations of com. algs),

$\mathfrak{g}$ Lie alg
G alg. group

$$\left[\begin{array}{lll} \text{deform } \Big\{ \; \Bbbk[v_1, v_2] & \text{in } \mathrm{Alg}({}_{U(\mathfrak{g})}\mathcal{M}), & \text{in } \mathrm{Alg}(\mathcal{M}^{\mathcal{O}(G)}) \\ \quad q \in \Bbbk^\times \;\; \Bbbk_q[v_1, v_2] & \text{in } \mathrm{Alg}({}_{U_q(\mathfrak{g})}\mathcal{M}), & \text{in } \mathrm{Alg}(\mathcal{M}^{\mathcal{O}_q(G)}) \end{array}\right]$$

I suspect that one needs to use weak bi/Hopf algebras to properly understand symmetries of certain types of algebras.
(just a suspicion for now, projects in progress towards understanding.)

What is a weak Hopf algebra? [See handout for defns & example]

Theorem [Nill 1998, Böhm - Caenepeel - Janssen 2011] For H wba, H-mod and $\underbrace{\text{comod-}H}$ can be given structure of monoidal category.
In particular, $\mathcal{M}^H = (\text{comod-}H, \underset{\substack{\text{skip details,} \\ \text{not } \otimes}}{\bar{\otimes}}, \underset{!}{\mathbb{1} = H_s}, \underset{\text{skip details}}{\text{assoc, unit}})$

Recall a $\Bbbk$-algebra $A = (\underset{\substack{\Uparrow \\ \text{need comod-}H \checkmark}}{A}, \; m: \underbrace{A \otimes A} \longrightarrow A, \; u: \underbrace{\Bbbk} \longrightarrow A)$
$\notin \mathcal{M}^H$: isn't straightforward
In particular, a wba is not guaranteed to have $\Bbbk$ as co/mod.

Still there are definitions of "H-co/mod-algebra" in the literature, where one requires A to be a H-co/mod & impose certain equations for preservation of m, u generalizing those for H bialg / Hopf algebra $[h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b), \; h \cdot 1_A = \varepsilon(h) 1_A]$

(*) just comod in paper

Result [WWW] : We consider the category of such "H-comodule algebras"(*) & show that its isomorphic to $Alg(\mathcal{M}^H)$.
Same for H-comod coalgs & H-comod Frobenius algebras
(as one can also build coalgs & Frob. algs in monoidal catys)

[The algebra result also appeared in work of (language of bialgebroids)
Brzezinski - Caenepeel - Militaru 2002, include proof anyway
in our work as we build on the result & proof in subsequent results]

Definition [WWW] Take H a wba.

(1) $\boxed{\mathcal{A}^H}$ is the category of $\Bbbk$-algebras (A, m, u) so that
- $A \in$ comod-H via $\rho(a) = a_{[0]} \otimes a_{[1]}$
- $(ab)_{[0]} \otimes (ab)_{[1]} = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]}$
- $\rho(1_A) \in A \otimes H_t$

morphisms = morphisms in $Vec_{\Bbbk}$ that are H-comod maps

(2) $\boxed{\mathcal{C}^H}$ is the category of $\Bbbk$-Coalgs (C, Δ, ε) so that
- $C \in$ comod-H via $\rho(c) = c_{[0]} \otimes c_{[1]}$
- $c_{1,[0]} \otimes c_{2,[0]} \otimes c_{1,[1]} c_{2,[1]} = c_{[0],1} \otimes c_{[0],2} \otimes c_{[1]}$
- $\varepsilon_C(c_{[0]}) c_{[1]} = \varepsilon_C(c_{[0]}) \varepsilon_s(c_{[1]})$

(morphisms same as (1))

(3) $\boxed{\mathcal{F}^H}$ is the category of $\Bbbk$-Frob. algebras $(A, m, u, \Delta, \varepsilon)$ so that
$(A, m, u) \in \mathcal{A}^H$ and $(A, \Delta, \varepsilon) \in \mathcal{C}^H$.
(morphisms same)

Theorem [WWW] We construct explicit category isomorphisms:
$$\mathcal{A}^H \cong Alg(\mathcal{M}^H), \quad \mathcal{C}^H \cong Coalg(\mathcal{M}^H), \quad \mathcal{F}^H \cong FrobAlg(\mathcal{M}^H).$$

(Pf is quite technical - lots of wba identities used, no graphical calc available)

Building weak symmetries:

Take L a Hopf algebra, B a strongly separable L-module algebra,
can build a quantum transformation groupoid (QTG):
a Weak Hopf algebra $H = H(L,B)$ ($\cong B^{op} \otimes L \otimes B$ as algs)
(appearing in survey of Nikshych-Vainerman 2002)
but we had to correct some bits, full proof in appendix of our work

Theorem [WWW] $\exists$ explicit monoidal functor ${}^L\mathcal{M}^L \xrightarrow{T} \mathcal{M}^H$.
So, if A is an algebra in the bi-comodule category ${}^L\mathcal{M}^L$
then $T(A)$ is an H-comodule algebra.

In progress, putting comonoidal, Frobenius monoidal
structure on T so that one can get objects in
$\mathrm{Alg}(\mathcal{M}^H)$
$\mathrm{Coalg}(\mathcal{M}^H)$ from Hopf algebra actions
$\mathrm{FrobAlg}(\mathcal{M}^H)$

Building examples as well.