DEGENERATE SKLYANIN ALGEBRAS AND GENERALIZED TWISTED HOMOGENEOUS COORDINATE RINGS

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Abstract. In this work, we introduce the point parameter ring B, a generalized twisted homogeneous coordinate ring associated to a degenerate version of the three-dimensional Sklyanin algebra. The surprising geometry of these algebras yields an analogue to a result of Artin-Tate-van den Bergh, namely that B is generated in degree one and thus is a factor of the corresponding degenerate Sklyanin algebra.

1. Introduction

Let k be an algebraically closed field of characteristic 0. We say a k-algebra R is *connected graded (cg)* when $R = \bigoplus_{i \in \mathbb{N}} R_i$ is $\mathbb{N}$-graded with $R_0 = k$.

A vital development in the field of Noncommutative Projective Algebraic Geometry is the investigation of connected graded noncommutative rings with use of geometric data. In particular, a method was introduced by Artin-Tate-van den Bergh in [3] to construct corresponding well-behaved graded rings, namely twisted homogeneous coordinate rings (tcr) [2, 12, 18]. However, there exist noncommutative rings that do not have sufficient geometry to undergo this process [12]. The purpose of this paper is to explore a recipe suggested in [3] for building a generalized analogue of a tcr for *any* connected graded ring. As a result, we provide a geometric approach to examine all degenerations of the Sklyanin algebras studied in [3].

We begin with a few historical remarks. In the mid-1980s, Artin and Schelter [1] began the task of classifying noncommutative analogues of the polynomial ring in three variables, yet the rings of interest were not well

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understood. How close were these noncommutative rings to the commutative counterpart $k[x,y,z]$? Were they Noetherian? Domains? Global dimension 3? These questions were answered later in [3] and the toughest challenge was analyzing the following class of algebras.

Definition 1.1. Let $k\{x,y,z\}$ denote the free algebra on the noncommuting variables x, y, and z. The *three-dimensional Sklyanin algebras* are defined as

$$S(a,b,c) \;=\; \frac{k\{x,y,z\}}{\begin{pmatrix} ayz + bzy + cx^2, \\ azx + bxz + cy^2, \\ axy + byx + cz^2 \end{pmatrix}} \tag{1.1}$$

for $[a:b:c] \in \mathbb{P}^2_k \setminus \mathfrak{D}$ where

$$\mathfrak{D} = \{[0:0:1],[0:1:0],[1:0:0]\} \;\cup\; \{[a:b:c] \mid a^3 = b^3 = c^3 = 1\}.$$

As algebraic techniques were exhausted, two seminal papers [3] and [4] arose introducing algebro-geometric methods to examine noncommutative analogues of the polynomial ring. In fact, a geometric framework was specifically associated to the Sklyanin algebras $S(a,b,c)$ via the following definition and result of [3].

Definition 1.2. A *point module* over a ring R is a cyclic graded left R-module M where $\dim_k M_i = 1$ for all i.

Theorem 1.3. *Point modules for $S = S(a,b,c)$ with $[a:b:c] \notin \mathfrak{D}$ are parameterized by the points of a smooth cubic curve*

$$E = E_{a,b,c} : (a^3+b^3+c^3)xyz - (abc)(x^3+y^3+z^3) = 0 \subset \mathbb{P}^2. \tag{1.2}$$

The curve E is equipped with $\sigma \in Aut(E)$ and the invertible sheaf $i^*\mathcal{O}_{\mathbb{P}^2}(1)$ from which we form the corresponding twisted homogeneous coordinate ring B. There exists a regular normal element $g \in S$, homogeneous of degree 3, so that $B \cong S/gS$ as graded rings. The ring B is a Noetherian domain and thus so is S. Moreover for $d \geq 1$, we get $dim_k B_d = 3d$. Hence S has the same Hilbert series as $k[x,y,z]$, namely $H_S(t) = \frac{1}{(1-t)^3}$. □

In short, the tcr B associated to $S(a,b,c)$ proved useful in determining the Sklyanin algebras' behavior.

Due to the importance of the Sklyanin algebras, it is natural to understand their degenerations to the set $\mathfrak{D}$.

Definition 1.4. The rings $S(a,b,c)$ from (1.1) with $[a:b:c] \in \mathfrak{D}$ are called the *degenerate three-dimensional Sklyanin algebras*. Such a ring is denoted by $S(a,b,c)$ or S_{deg} for short.

In section 2, we study the basic properties of degenerate Sklyanin algebras resulting in the following proposition.

Proposition 1.5. *The degenerate three-dimensional Sklyanin algebras have Hilbert series* $H_{S_{deg}}(t) = \frac{1+t}{1-2t}$, *they have infinite Gelfand Kirillov dimension, and are not left or right Noetherian, nor are they domains. Furthermore, the algebras* S_{deg} *are Koszul and have infinite global dimension.*

The remaining two sections construct a generalized twisted homogeneous coordinate ring $B = B(S_{deg})$ for the degenerate Sklyanin algebras. We are specifically interested in point modules over S_{deg} (Definition 1.2). Unlike their nondegenerate counterparts, the point modules over S_{deg} are not parameterized by a projective scheme so care is required. Nevertheless, the degenerate Sklyanin algebras do have geometric data which is described by the following definition and theorem.

Definition 1.6. A *truncated point module of length* d over a ring R is a cyclic graded left R-module M where $\dim_k M_i = 1$ for $0 \leq i \leq d$ and $\dim_k M_i = 0$ for $i > d$. The d^{th} *truncated point scheme* V_d parameterizes isomorphism classes of length d truncated point modules.

Theorem 1.7. *For* $d \geq 2$, *the truncated point schemes* $V_d \subset (\mathbb{P}^2)^{\times d}$ *corresponding to* S_{deg} *are isomorphic to a union of*

$$\begin{cases} \text{three copies of } (\mathbb{P}^1)^{\times \frac{d-1}{2}} \text{ and three copies of } (\mathbb{P}^1)^{\times \frac{d+1}{2}}, & \text{for } d \text{ odd;} \\ \text{six copies of } (\mathbb{P}^1)^{\times \frac{d}{2}}, & \text{for } d \text{ even.} \end{cases}$$

The precise description of V_d as a subset of $(\mathbb{P}^2)^{\times d}$ is provided in Proposition 3.13. Furthermore, this scheme is not a disjoint union and Remark 4.2 describes the singularity locus of V_d.

In the language of [16], observe that the point scheme data of degenerate Sklyanin algebras does not stabilize to produce a projective scheme (of finite type) and as a consequence we cannot construct a tcr associated to S_{deg}. Instead, we use the truncated point schemes V_d produced in Theorem 1.7 and a method from [3, page 19] to form the $\mathbb{N}$-graded, associative ring B defined below.

Definition 1.8. The *point parameter ring* $B = \bigoplus_{d \geq 0} B_d$ is a ring associated to the sequence of subschemes V_d of $(\mathbb{P}^2)^{\times d}$ (Definition 1.6). We have $B_d = H^0(V_d, \mathcal{L}_d)$ where $\mathcal{L}_d$ is the restriction of invertible sheaf

$$pr_1^* \mathcal{O}_{\mathbb{P}^2}(1) \otimes \ldots \otimes pr_d^* \mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}_{(\mathbb{P}^2)^{\times d}}(1, \ldots, 1)$$

to V_d. The multiplication map $B_i \times B_j \to B_{i+j}$ is defined by applying H^0 to the isomorphism $pr_{1,\ldots,i}(\mathcal{L}_i) \otimes_{\mathcal{O}_{V_{i+j}}} pr_{i+1,\ldots,i+j}(\mathcal{L}_j) \to \mathcal{L}_{i+j}$.

Despite point parameter rings not being well understood in general, the final section of this paper verifies the following properties of $B = B(S_{deg})$.

Theorem 1.9. *The point parameter ring B for a degenerate three-dimensional Sklyanin algebra S_{deg} has Hilbert series $H_B(t) = \frac{(1+t^2)(1+2t)}{(1-2t^2)(1-t)}$ and is generated in degree one.*

Hence we have a surjection of S_{deg} onto B, which is akin to the result involving Sklyanin algebras and corresponding tcrs (Theorem 1.3).

Corollary 1.10. *The ring $B = B(S_{deg})$ has exponential growth and therefore infinite GK dimension. Moreover B is neither right Noetherian, Koszul, nor a domain. Furthermore B is a factor of the corresponding S_{deg} by an ideal K where K has six generators of degree 4 (and possibly more of higher degree).*

Therefore the behavior of $B(S_{deg})$ resembles that of S_{deg}. It is natural to ask if other noncommutative algebras can be analyzed in a similar fashion, though we will not address this here.

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2. Structure of degenerate Sklyanin algebras

In this section, we establish Proposition 1.5. We begin by considering the degenerate Sklyanin algebras $S(a,b,c)_{deg}$ with $a^3 = b^3 = c^3 = 1$ (Definition 1.1) and the following definitions from [10].

Definition 2.1. Let α be an endomorphism of a ring R. An *α-derivation* on R is any additive map $\delta : R \to R$ so that $\delta(rs) = \alpha(r)\delta(s) + \delta(r)s$ for all $r, s \in R$. The set of α-derivations of R is denoted α-Der(R).

We write $S = R[z; \alpha, \delta]$ provided S is isomorphic to the polynomial ring $R[z]$ as a left R-module but with multiplication given by $zr = \alpha(r)z + \delta(r)$ for all $r \in R$. Such a ring S is called an *Ore extension of R*.

By generalizing the work of [7] we see that most degenerate Sklyanin algebras are factors of Ore extensions of the free algebra on two variables.

Proposition 2.2. *In the case of $a^3 = b^3 = c^3 = 1$, assume without loss of generality $a = 1$. Then for $[1 : b : c] \in \mathfrak{D}$ we get the ring isomorphism*

$$S(1,b,c) \cong \frac{k\{x,y\}[z;\alpha,\delta]}{(\Omega)} \tag{2.1}$$

where $\alpha \in$*End*$(k\{x,y\})$ *is defined by* $\alpha(x) = -bx$, $\alpha(y) = -b^2y$ *and the element* $\delta \in \alpha$*-Der*$(k\{x,y\})$ *is given by* $\delta(x) = -cy^2$, $\delta(y) = -b^2cx^2$*. Here* $\Omega = xy + byx + cz^2$ *is a normal element of* $k\{x,y\}[z,\alpha,\delta]$*.*

Proof. By direct computation α and δ are indeed an endomorphism and α-derivation of $k\{x,y\}$ respectively. Moreover $x\cdot\Omega = \Omega\cdot bx$, $y\cdot\Omega = \Omega\cdot by$, $z\cdot\Omega = \Omega\cdot z$ so Ω is a normal element of the Ore extension. Thus both rings of (2.1) have the same generators and relations. $\square$

Remark 2.3. *Some properties of degenerate Sklyanin algebras are easy to verify without use of the Proposition 2.2. Namely one can find a basis of irreducible monomials via Bergman's Diamond lemma* [6, Theorem 1.2] *to imply* $\dim_k S_d = 2^{d-1}3$ *for* $d \geq 1$*. Equivalently* $S(1,b,c)$ *is free with a basis* $\{1,z\}$ *as a left or right module over* $k\{x,y\}$*. Therefore,* $H_{S_{deg}}(t) = \frac{1+t}{1-2t}$*.*

Therefore due to Proposition 2.2 (for $a^3 = b^3 = c^3 = 1$) or Remark 2.3 we have the following immediate consequence.

Corollary 2.4. *The degenerate Sklyanin algebras have exponential growth, infinite GK dimension, and are not right Noetherian. Furthermore* S_{deg} *is not a domain.*

Proof. The growth conditions follow from Remark 2.3 and the non-Noetherian property holds by [20, Theorem 0.1]. Moreover if $[a:b:c] \in \{[1:0:0], [0:1:0], [0:0:1]\}$, then the monomial algebra $S(a,b,c)$ is obviously not a domain. On the other hand if $[a:b:c]$ satisfies $a^3 = b^3 = c^3 = 1$, then assume without loss of generality that $a = 1$. As a result we have

$$f_1 + bf_2 + cf_3 \;=\; (x + by + bc^2z)(cx + cy + b^2z),$$

where $f_1 = yz + bzy + cx^2$, $f_2 = zx + bxz + cy^2$, and $f_3 = xy + byx + cz^2$ are the relations of $S(1,b,c)$. $\square$

Now we verify homological properties of degenerate Sklyanin algebras.

Definition 2.5. Let A be a cg algebra which is locally finite ($\dim_k A_i < \infty$). When provided a minimal resolution of the left A-module $A/\bigoplus_{i\geq 1} A_i \cong k$ determined by matrices M_i, we say A is *Koszul* if the entries of the M_i all belong to A_1.

Proposition 2.6. *The degenerate Sklyanin algebras are Koszul with infinite global dimension.*

Proof. For $S = S(a,b,c)$ with $a^3 = b^3 = c^3 = 1$, consider the description of S in Proposition 2.2. Since $k\{x,y\}$ is Koszul, the Ore extension $k\{x,y\}[z,\alpha,\delta]$ is also Koszul [9, Definition 1.1, Theorem 10.2]. By Proposition 2.2, the element Ω is normal and regular in $k\{x,y\}[z;\alpha,\delta]$. Hence the factor S is Koszul by [17, Theorem 1.2].

To conclude $\text{gl.dim}(S)=\infty$, note that the Koszul dual of S is

$$S(1,b,c)^! \cong \frac{k\{x,y,z\}}{\left(\begin{array}{ll} z^2-cxy, & yz-c^2x^2, \\ zy-b^2yz, & y^2-bcxz, \\ zx-bxz, & yx-b^2xy \end{array}\right)}.$$

Taking the ordering $x<y<z$, we see that all possible ambiguities of $S^!$ are resolvable in the sense of [6]. Bergman's Diamond lemma [6, Theorem 1.2] implies that $S^!$ has a basis of irreducible monomials $\{x^i, x^jy, x^kz\}_{i,j,k\in\mathbb{N}}$. Hence $S^!$ is not a finite dimensional k-vector space and by [13, Corollary 5], S has infinite global dimension.

For $S=S(a,b,c)$ with $[a:b:c]\in\{[1:0:0],[0:1:0],[0:0:1]\}$, note that S is Koszul as its ideal of relations is generated by quadratic monomials [14, Corollary 4.3]. Denote these monomials m_1, m_2, m_3. The Koszul dual of S in this case is

$$S^! \cong \frac{k\{x,y,z\}}{(\text{the six monomials not equal to } m_i)}.$$

Since $S^!$ is again a monomial algebra, it contains no hidden relations and has a nice basis of irreducible monomials. In particular, $S^!$ contains $\bigoplus_{i\geq 0} kw_i$ where w_i is the length i word:

$$w_i = \begin{cases} \underbrace{xyzxyzx\ldots}_{i}, & \text{if } [a:b:c]=[1:0:0] \\ \underbrace{xzyxzyx\ldots}_{i}, & \text{if } [a:b:c]=[0:1:0] \\ x^i, & \text{if } [a:b:c]=[0:0:1]. \end{cases}$$

Therefore $S^!$ is not a finite dimensional k-vector space. By [13, Corollary 5], the three remaining degenerate Sklyanin algebras are of infinite global dimension. □

3. Truncated point schemes of S_{deg}

The goal of this section is to construct the family of truncated point schemes $\{V_d \subseteq (\mathbb{P}^2)^{\times d}\}$ associated to the degenerate three-dimensional Sklyanin algebras S_{deg} (see Definition 1.4). These schemes will be used in §4 for the construction of a generalized twisted homogeneous coordinate ring, namely the point parameter ring (Definition 1.8). Nevertheless the family $\{V_d\}$ has immediate importance for understanding point modules over $S=S_{deg}$.

Definition 3.1. A graded left S-module M is called a *point module* if M is cyclic and $H_M(t) = \sum_{i=0}^{\infty} t^i = \frac{1}{1-t}$. Moreover a graded left S-module M is called a *truncated point module of length* d if M is again cyclic and $H_M(t) = \sum_{i=0}^{d-1} t^i$.

Note that point modules share the same Hilbert series as a point in projective space in Classical Algebraic Geometry.

Now we proceed to construct schemes V_d that will parameterize length d truncated point modules. This yields information regarding point modules over $S(a,b,c)$ for *any* $[a:b:c] \in \mathbb{P}^2$ due to the following result.

Lemma 3.2. [3, Proposition 3.9, Corollary 3.13] *Let $S = S(a,b,c)$ for any $[a:b:c] \in \mathbb{P}^2$. Denote by Γ the set of isomorphism classes of point modules over S and Γ_d the set of isomorphism classes of truncated point modules of length $d+1$. With respect to the truncation function $\rho_d : \Gamma_d \to \Gamma_{d-1}$ given by $M \mapsto M/M_{d+1}$, we have that Γ is the projective limit of $\{\Gamma_d\}$ as a set.*

The sets Γ_d can be understood by the schemes V_d defined below.

Definition 3.3. [3, §3] The *truncated point scheme of length d*, $V_d \subseteq (\mathbb{P}^2)^{\times d}$, is the scheme defined by the multilinearizations of relations of $S(a,b,c)$ from Definition 1.1. More precisely $V_d = \mathbb{V}(f_i, g_i, h_i)_{0 \le i \le d-2}$ where

$$\begin{aligned} f_i &= ay_{i+1}z_i + bz_{i+1}y_i + cx_{i+1}x_i \\ g_i &= az_{i+1}x_i + bx_{i+1}z_i + cy_{i+1}y_i \\ h_i &= ax_{i+1}y_i + by_{i+1}x_i + cz_{i+1}z_i. \end{aligned} \tag{3.1}$$

For example, $V_1 = \mathbb{V}(0) \subseteq \mathbb{P}^2$ so we have $V_1 = \mathbb{P}^2$. Similarly, $V_2 = \mathbb{V}(f_0, g_0, h_0) \subseteq \mathbb{P}^2 \times \mathbb{P}^2$.

Lemma 3.4. [3] *The set Γ_d is parameterized by the scheme V_d.*

In short, to understand point modules over $S(a,b,c)$ for any $[a:b:c] \in \mathbb{P}^2$, Lemmas 3.2 and 3.4 imply that we can now restrict our attention to truncated point schemes V_d.

On the other hand, we point out another useful result pertaining to V_d associated to $S(a,b,c)$ for any $[a:b:c] \in \mathbb{P}^2$.

Lemma 3.5. *The truncated point scheme V_d lies in d copies of $E \subseteq \mathbb{P}^2$ where E is the cubic curve $E : (a^3+b^3+c^3)xyz - (abc)(x^3+y^3+z^3) = 0$.*

Proof. Let p_i denote the point $[x_i : y_i : z_i] \in \mathbb{P}^2$ and

$$\mathbb{M}_{abc,i} := \mathbb{M}_i := \begin{pmatrix} cx_i & az_i & by_i \\ bz_i & cy_i & ax_i \\ ay_i & bx_i & cz_i \end{pmatrix} \in \mathrm{Mat}_3(kx_i \oplus ky_i \oplus kz_i). \tag{3.2}$$

A d-tuple of points $p = (p_0, p_1, \ldots, p_{d-1}) \in V_d \subseteq (\mathbb{P}^2)^{\times d}$ must satisfy the system $f_i = g_i = h_i = 0$ for $0 \leq i \leq d-2$ by definition of V_d. In other words, one is given $\mathbb{M}_{abc,j} \cdot (x_{j+1}\ y_{j+1}\ z_{j+1})^T = 0$ or equivalently $(x_j\ y_j\ z_j) \cdot \mathbb{M}_{abc,j+1} = 0$ for $0 \leq j \leq d-2$. Therefore for $0 \leq j \leq d-1$, $\det(\mathbb{M}_{abc,j}) = 0$. This implies $p_j \in E$ for each j. Thus $p \in E^{\times d}$. □

3.1. **On the truncated point schemes of some S_{deg}.** We will show that to study the truncated point schemes V_d of degenerate Sklyanin algebras, it suffices to understand the schemes of specific four degenerate Sklyanin algebras. Recall that V_d parameterizes length d truncated point modules (Lemma 3.4). Moreover note that according to [21], two graded algebras A and B have equivalent graded left module categories (A-Gr and B-Gr) if A is a Zhang twist of B. The following is a special case of [21, Theorem 1.2].

Theorem 3.6. *Given a $\mathbb{Z}$-graded k-algebra $S = \bigoplus_{n\in\mathbb{Z}} S_n$ with graded automorphism σ of degree 0 on S, we form a Zhang twist S^σ of S by preserving the same additive structure on S and defining multiplication $*$ as follows: $a * b = ab^{\sigma^n}$ for $a \in S_n$. Furthermore if S and S^σ are cg and generated in degree one, then S-Gr and S^σ-Gr are equivalent categories.* □

Realize $\mathfrak{D}$ from Definition 1.1 as the union of three point sets Z_i:

$$\begin{array}{llll} Z_1 := \{[1:1:1], & [1:\zeta:\zeta^2], & [1:\zeta^2:\zeta]\}, \\ Z_2 := \{[1:1:\zeta], & [1:\zeta:1], & [1:\zeta^2:\zeta^2]\}, \\ Z_3 := \{[1:\zeta:\zeta], & [1:1:\zeta^2], & [1:\zeta^2:1]\}, \\ Z_0 := \{[1:0:0], & [0:1:0], & [0:0:1]\}. \end{array} \tag{3.3}$$

where $\zeta = e^{2\pi i/3}$. Pick respective representatives $[1:1:1]$, $[1:1:\zeta]$, $[1:\zeta:\zeta]$, and $[1:0:0]$ of Z_1, Z_2, Z_3, and Z_0.

Lemma 3.7. *Every degenerate Sklyanin algebra is a Zhang twist of one the following algebras: $S(1,1,1)$, $S(1,1,\zeta)$, $S(1,\zeta,\zeta)$, and $S(1,0,0)$.*

Proof. A routine computation shows that the following graded automorphisms of degenerate $S(a,b,c)$,

$$\sigma : \{x \mapsto \zeta x,\ y \mapsto \zeta^2 y,\ z \mapsto z\} \text{ and } \tau : \{x \mapsto y,\ y \mapsto z,\ z \mapsto x\},$$

yield the Zhang twists:

$$\begin{array}{lll} S(1,1,1)^\sigma = S(1,\zeta,\zeta^2), & S(1,1,1)^{\sigma^{-1}} = S(1,\zeta^2,\zeta) & \text{for } Z_1; \\ S(1,1,\zeta)^\sigma = S(1,\zeta,1), & S(1,1,\zeta)^{\sigma^{-1}} = S(1,\zeta^2,\zeta^2) & \text{for } Z_2; \\ S(1,\zeta,\zeta)^\sigma = S(1,\zeta^2,1), & S(1,\zeta,\zeta)^{\sigma^{-1}} = S(1,1,\zeta^2) & \text{for } Z_3; \\ S(1,0,0)^\tau = S(0,1,0), & S(1,0,0)^{\tau^{-1}} = S(0,0,1) & \text{for } Z_0. \end{array}$$ □

Therefore it suffices to study a representative of each of the four classes of degenerate three-dimensional Sklyanin algebras due to Theorem 3.6.

3.2. **Computation of V_d for $S(1,1,1)$.** We now compute the truncated point schemes of $S(1,1,1)$ in detail. Calculations for the other three representative degenerate Sklyanin algebras, $S(1,1,\zeta)$, $S(1,\zeta,\zeta)$, $S(1,0,0)$, will follow with similar reasoning. To begin we first discuss how to build a truncated point module M' of length d, when provided with a truncated point module M of length $d-1$.

Let us explore the correspondence between truncated point modules and truncated point schemes for a given d; say $d \geq 3$. When given a truncated point module $M = \bigoplus_{i=0}^{d-1} M_i \in \Gamma_{d-1}$, multiplication from $S = S(a,b,c)$ is determined by a point $p = (p_0, \dots, p_{d-2}) \in V_{d-1}$ (Definition 3.3, (3.2)) in the following manner. As M is cyclic, M_i has basis say $\{m_i\}$. Furthermore for $x, y, z \in S$ with $p_i = [x_i : y_i : z_i] \in \mathbb{P}^2$, we get the left S-action on m_i determined by p_i:

$$\begin{aligned} x \cdot m_i &= x_i m_{i+1}, \quad x \cdot m_{d-1} = 0; \\ y \cdot m_i &= y_i m_{i+1}, \quad y \cdot m_{d-1} = 0; \\ z \cdot m_i &= z_i m_{i+1}, \quad z \cdot m_{d-1} = 0. \end{aligned} \tag{3.4}$$

Conversely given a point $p = (p_0, \dots, p_{d-2}) \in V_{d-1}$, one can build a module $M \in \Gamma_{d-1}$ unique up to isomorphism by reversing the above process. We summarize this discussion in the following remark.

Remark 3.8. *Refer to notation from Lemma 3.2. To construct $M' \in \Gamma_d$ from $M \in \Gamma_{d-1}$ associated to $p \in V_{d-1}$, we require $p_{d-1} \in \mathbb{P}^2$ such that $p' = (p, p_{d-1}) \in V_d$.*

Now we begin to study the behavior of truncated point modules over S_{deg} through the examination of truncated point schemes in the next two lemmas.

Lemma 3.9. *Let $p = (p_0, \dots, p_{d-2}) \in V_{d-1}$ with $p_{d-2} \notin Z_i$ (refer to (3.3)). Then there exists a unique $p_{d-1} \in Z_i$ so that $p' := (p, p_{d-1}) \in V_d$.*

Proof of 3.9. For Z_1, we study the representative algebra $S(1,1,1)$. If such a p_{d-1} exists, then $f_{d-2} = g_{d-2} = h_{d-2} = 0$ so we would have

$$\mathbb{M}_{111,d-2} \cdot (x_{d-1} \; y_{d-1} \; z_{d-1})^T = 0$$

(Definition 3.3, Eq. (3.2)). Since $\operatorname{rank}(\mathbb{M}_{111,d-2}) = 2$ when $p_{d-2} \notin \mathfrak{D}$, the tuple $(x_{d-1}, \; y_{d-1}, \; z_{d-1})$ is unique up to scalar multiple and thus the point p_{d-1} is unique.

To verify the existence of p_{d-1}, say $p_{d-2} = [0 : y_{d-2} : z_{d-2}]$. We require p_{d-2} and p_{d-1} to satisfy the system of equations:

$$\begin{aligned} f_{d-2} = g_{d-2} = h_{d-2} &= 0 \qquad \text{(Eq. (3.1))} \\ y_{d-2}^3 + z_{d-2}^3 = x_{d-1}^3 + y_{d-1}^3 + z_{d-1}^3 &= 0 \qquad (p_{d-2},\, p_{d-1} \in E, \text{ Lemma 3.5}). \end{aligned}$$

However basic algebraic operations imply $y_{d-2} = z_{d-2} = 0$, thus producing a contradiction. Therefore, without loss of generality $p_{d-2} = [1 : y_{d-2} : z_{d-2}]$. With similar reasoning we must examine the system

$$\begin{aligned} y_{d-1}z_{d-2} + z_{d-1}y_{d-2} + x_{d-1} &= 0 \\ z_{d-1} + x_{d-1}z_{d-2} + y_{d-1}y_{d-2} &= 0 \\ x_{d-1}y_{d-2} + y_{d-1} + z_{d-1}z_{d-2} &= 0 \\ 1 + y_{d-2}^3 + z_{d-2}^3 &= 3y_{d-2}z_{d-2} \\ x_{d-1}^3 + y_{d-1}^3 + z_{d-1}^3 &= 3x_{d-1}y_{d-1}z_{d-1}. \end{aligned} \tag{3.5}$$

There are three solutions $(p_{d-2}, p_{d-1}) \in (E \setminus Z_1) \times E$ to (3.5):

$$\left\{ \begin{array}{cc} ([1 : -(1+z_{d-2}) : z_{d-2}], & [1:1:1]), \\ ([1 : -\zeta(1+\zeta z_{d-2}) : z_{d-2}], & [1:\zeta:\zeta^2]), \\ ([1 : -\zeta(\zeta + z_{d-2}) : z_{d-2}], & [1:\zeta^2:\zeta]) \end{array} \right\}.$$

Thus when $p_{d-2} \notin Z_1$, there exists an unique point $p_{d-1} \in Z_1$ so that $(p_0, \dots, p_{d-2}, p_{d-1}) \in V_d$.

Now having studied $S(1,1,1)$ with care, we leave it to the reader to verify the assertion for the algebras $S(1,1,\zeta)$, $S(1,\zeta,\zeta)$, and $S(1,0,0)$ in a similar manner. $\square$

The next result explores the case when $p_{d-2} \in Z_i$.

Lemma 3.10. *Let $p = (p_0, \dots, p_{d-2}) \in V_{d-1}$ with $p_{d-2} \in Z_i$. Then for any $[y_{d-1} : z_{d-1}] \in \mathbb{P}^1$ there exists a function θ of two variables so that*

$$p_{d-1} = [\theta(y_{d-1}, z_{d-1}) : y_{d-1} : z_{d-1}] \notin Z_i$$

which satisfies $(p_0, \dots, p_{d-2}, p_{d-1}) \in V_d$.

Proof. The point $p' = (p, p_{d-1}) \in V_d$ needs to satisfy $f_i = g_i = h_i = 0$ for $0 \leq i \leq d-2$ (Definition 3.3). Since $p \in V_{d-1}$, we need only to consider the equations $f_{d-2} = g_{d-2} = h_{d-2} = 0$ with $p_{d-2} \in Z_i$.

We study $S(1,1,1)$ for Z_1 so the relevant system of equations is

$$\begin{aligned} f_{d-2}:&\ y_{d-1}z_{d-2} + z_{d-1}y_{d-2} + x_{d-1}x_{d-2} = 0 \\ g_{d-2}:&\ z_{d-1}x_{d-2} + x_{d-1}z_{d-2} + y_{d-1}y_{d-2} = 0 \\ h_{d-2}:&\ x_{d-1}y_{d-2} + y_{d-1}x_{d-2} + z_{d-1}z_{d-2} = 0. \end{aligned}$$

If $p_{d-2} = [1:1:1] \in Z_1$, then $x_{d-1} = -(y_{d-1} + y_{d-1})$ is required. On the other hand, if $p_{d-2} = [1 : \zeta : \zeta^2]$ or $[1 : \zeta^2 : \zeta]$, we require $x_{d-1} = -\zeta(y_{d-1} + \zeta z_{d-1})$ or $x_{d-1} = -\zeta(\zeta y_{d-1} + z_{d-1})$ respectively. Thus our function θ is defined as

$$\theta(y_{d-1}, z_{d-1}) = \begin{cases} -(y_{d-1} + z_{d-1}), & \text{if } p_{d-2} = [1:1:1] \\ -(\zeta y_{d-1} + \zeta^2 z_{d-1}), & \text{if } p_{d-2} = [1:\zeta:\zeta^2] \\ -(\zeta^2 y_{d-1} + \zeta z_{d-1}), & \text{if } p_{d-2} = [1:\zeta^2:\zeta]. \end{cases}$$

The arguments for $S(1,1,\zeta)$, $S(1,\zeta,\zeta)$, and $S(1,0,0)$ proceed in a likewise fashion. $\square$

Fix a pair $(S_{deg},\, Z_i(S_{deg}))$. We now know if $p_{d-2} \notin Z_i$, then from every truncated point module of length d over S_{deg} we can produce a unique truncated point module of length $d+1$. Otherwise if $p_{d-2} \in Z_i$, we get a $\mathbb{P}^1$ worth of length $d+1$ modules. We summarize this in the following statement which is made precise in Proposition 3.13.

Proposition 3.11. *The parameter space of Γ_d over S_{deg} is isomorphic to the singular and nondisjoint union of*

$$\begin{cases} \text{three copies of } (\mathbb{P}^1)^{\times \frac{d-1}{2}} \text{ and three copies of } (\mathbb{P}^1)^{\times \frac{d+1}{2}}, & \text{for } d \text{ odd;} \\ \text{six copies of } (\mathbb{P}^1)^{\times \frac{d}{2}}, & \text{for } d \text{ even.} \end{cases}$$

The detailed statement and proof of this proposition will follow from the results below. We restrict our attention to $S(1,1,1)$ for reasoning mentioned in the proofs of Lemmas 3.9 and 3.10.

3.2.1. *Parameterization of* Γ_2. Recall that length 3 truncated point modules of Γ_2 are in bijective correspondence to points on $V_2 \subset \mathbb{P}^2 \times \mathbb{P}^2$ (Lemma 3.4) and it is our goal to depict this truncated point scheme. By Lemma 3.5, we know that $V_2 \subseteq E \times E$. Furthermore note that with $\zeta = e^{2\pi i/3}$, the curve $E = E_{111}$ is the union of three projective lines:

$$\mathbb{P}^1_A : x = -(y+z),\ \mathbb{P}^1_B : x = -(\zeta y + \zeta^2 z),\ \mathbb{P}^1_C : x = -(\zeta^2 y + \zeta z) \tag{3.6}$$

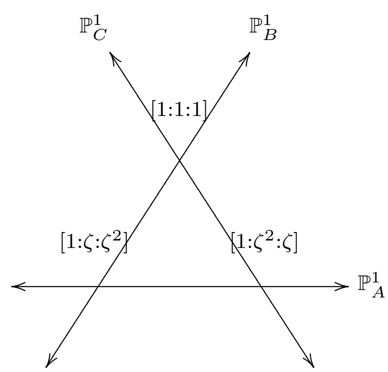

Figure 1: The curve $E = E_{111} \subseteq \mathbb{P}^2 :\ x^3 + y^3 + z^3 - 3xyz = 0$.

Now to calculate V_2, recall that Γ_2 consists of length 3 truncated point modules $M_{(3)} := M_0 \oplus M_1 \oplus M_2$ where M_i is a 1-dimensional k-vector space say with basis m_i. The module $M_{(3)}$ has action determined by $(p_0, p_1) \in V_2$ (Eq. (3.4)). Moreover Lemmas 3.9 and 3.10 provide the precise conditions for (p_0, p_1) to lie in $E \times E$. Namely,

Lemma 3.12. *Refer to (3.6) for notation. The set of length 3 truncated point modules Γ_2 is parametrized by the scheme $V_2 = \mathbb{V}(f_0, g_0, h_0)$ which is the union of the six subsets:*

$$\begin{array}{ll} \mathbb{P}^1_A \times [1:1:1]; & [1:1:1] \times \mathbb{P}^1_A; \\ \mathbb{P}^1_B \times [1:\zeta:\zeta^2]; & [1:\zeta:\zeta^2] \times \mathbb{P}^1_B; \\ \mathbb{P}^1_C \times [1:\zeta^2:\zeta]; & [1:\zeta^2:\zeta] \times \mathbb{P}^1_C. \end{array}$$

of $E \times E$. Thus Γ_2 is isomorphic to 6 copies of $\mathbb{P}^1$. □

3.2.2. *Parameterization of Γ_d for general d.* To illustrate the parametrization of Γ_d, we begin with a truncated point module $M_{(d+1)}$ of length $d+1$ corresponding to $(p_0, p_1, \ldots, p_{d-1}) \in V_d \subseteq (\mathbb{P}^2)^{\times d}$. Due to Lemmas 3.5, 3.9, and 3.10, we know that $(p_0, p_1, \ldots, p_{d-1})$ belongs to either

$$\underbrace{(E \setminus Z_1) \times Z_1 \times (E \setminus Z_1) \times Z_1 \times \ldots}_{d} \text{ or } \underbrace{Z_1 \times (E \setminus Z_1) \times Z_1 \times (E \setminus Z_1) \times \ldots}_{d}$$

where Z_1 is defined in (3.3).

By adapting the notation of Lemma 3.10, we get in the first case that the point $(p_0, p_1, \ldots, p_{d-1})$ is of the form

$$([\theta(y_0, z_0) : y_0 : z_0],\ [1 : \omega : \omega^2],\ [\theta(y_2, z_2) : y_2 : z_2],\ [1 : \omega : \omega^2], \ldots) \in (\mathbb{P}^2)^{\times d}$$

where $\omega^3 = 1$ and $\theta(y, z) = -(\omega y + \omega^2 z)$. Thus in this case, the set of length d truncated point modules is parameterized by three copies of $(\mathbb{P}^1)^{\times \lceil d/2 \rceil}$ with coordinates $([y_0 : z_0], [y_2 : z_2], \ldots, [y_{2\lceil d/2 \rceil - 1} : z_{2\lceil d/2 \rceil - 1}])$.

In the second case $(p_0, p_1, \ldots, p_{d-1})$ takes the form

$$([1 : \omega : \omega^2],\ [\theta(y_1, z_1) : y_1 : z_1],\ [1 : \omega : \omega^2], [\theta(y_3, z_3) : y_3 : z_3], \ldots) \in (\mathbb{P}^2)^{\times d}$$

and the set of truncated point modules is parameterized with three copies of $(\mathbb{P}^1)^{\times \lfloor d/2 \rfloor}$ with coordinates $([y_1 : z_1], [y_3 : z_3], \ldots, [y_{2\lfloor d/2 \rfloor - 1} : z_{2\lfloor d/2 \rfloor - 1}])$.

In other words, we have now proved the next result.

Proposition 3.13. *Refer to (3.6) for notation. For $d \geq 2$ the truncated point scheme V_d for $S(1,1,1)$ is equal to the union of the six subsets $\bigcup_{i=1}^{6} W_{d,i}$ of $(\mathbb{P}^2)^{\times d}$ where*

$$\begin{array}{ll} W_{d,1} & = \mathbb{P}^1_A \times [1:1:1] \times \mathbb{P}^1_A \times [1:1:1] \times \ldots, \\ W_{d,2} & = [1:1:1] \times \mathbb{P}^1_A \times [1:1:1] \times \mathbb{P}^1_A \times \ldots, \\ W_{d,3} & = \mathbb{P}^1_B \times [1:\zeta:\zeta^2] \times \mathbb{P}^1_B \times [1:\zeta:\zeta^2] \times \ldots, \\ W_{d,4} & = [1:\zeta:\zeta^2] \times \mathbb{P}^1_B \times [1:\zeta:\zeta^2] \times \mathbb{P}^1_C \times \ldots, \\ W_{d,5} & = \mathbb{P}^1_C \times [1:\zeta^2:\zeta] \times \mathbb{P}^1_C \times [1:\zeta^2:\zeta] \times \ldots, \\ W_{d,6} & = [1:\zeta^2:\zeta] \times \mathbb{P}^1_C \times [1:\zeta^2:\zeta] \times \mathbb{P}^1_C \times \ldots. \end{array}$$

□

As a consequence, we obtain the proof of Proposition 3.11 for $S(1,1,1)$ and this assertion holds for the remaining degenerate Sklyanin algebras due to Lemma 3.7, and analogous proofs for Lemmas 3.9 and 3.10. □

We thank Karen Smith for suggesting the following elegant way of interpreting the point scheme of $S(1,1,1)$.

Remark 3.14. *We can provide an alternate geometric description of the point scheme of the* Γ *of* $S(1,1,1)$*. Let* $G := \mathbb{Z}_3 \rtimes \mathbb{Z}_2 = <\zeta, \sigma>$ *where* $\zeta = e^{2\pi i/3}$ *and* $\sigma^2 = 1$*. We define a* G*-action on* $\mathbb{P}^2 \times \mathbb{P}^2$ *as follows:*

$$\zeta([x:y:z],[u:v:w]) = ([x:\zeta^2 y:\zeta z],[u:\zeta v:\zeta^2 w])$$
$$\sigma([x:y:z],[u:v:w]) = ([u:v:w],[x:y:z])$$

Note that G *stabilizes* $E \times E$ *and acts transitively on the* $W_{2,i}$*. We extend the action of* G *to* $(\mathbb{P}^2 \times \mathbb{P}^2)^{\times\infty}$ *diagonally. Now we interpret* Γ *as*

$$\Gamma \;=\; \lim_{\longleftarrow} V_d \;=\; \lim_{\longleftarrow} V_{2d} \;=\; \lim_{\longleftarrow} \bigcup_i W_{2d,i} \;=\; G \cdot (\mathbb{P}^1_A \times [1:1:1])^{\times\infty},$$

as sets.

4. Point parameter ring of $S(1,1,1)$

We now construct a graded associative algebra B from truncated point schemes of the degenerate Sklyanin algebra $S = S(1,1,1)$. The analogous result for the other degenerate Sklyanin algebras will follow in a similar fashion and we leave the details to the reader. As is true for the Sklyanin algebras themselves, it will be shown that this algebra B is a proper factor of $S(1,1,1)$ and its properties closely reflect those of $S(1,1,1)$. We will for example show that B is not right Noetherian, nor a domain.

The definition of the algebra B initially appears in [3, §3]. Recall that we have projection maps $pr_{1,\ldots,d-1}$ and $pr_{2,\ldots,d}$ from $(\mathbb{P}^2)^{\times d}$ to $(\mathbb{P}^2)^{\times d-1}$. Restrictions of these maps to the truncated point schemes $V_d \subseteq (\mathbb{P}^2)^{\times d}$ (Definition 3.3) yield

$$pr_{1,\ldots,d-1}(V_d) \subset V_{d-1} \text{ and } pr_{2,\ldots,d}(V_d) \subset V_{d-1} \text{ for all } d.$$

Definition 4.1. Given the above data, the *point parameter ring* $B = B(S)$ is an associative $\mathbb{N}$-graded ring defined as follows. First $B_d = H^0(V_d, \mathcal{L}_d)$ where $\mathcal{L}_d$ is the restriction of invertible sheaf

$$pr_1^*\mathcal{O}_{\mathbb{P}^2}(1) \otimes \ldots \otimes pr_d^*\mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}_{(\mathbb{P}^2)^{\times d}}(1,\ldots,1)$$

to V_d. The multiplication map $\mu_{i,j} : B_i \times B_j \to B_{i+j}$ is then defined by applying H^0 to the isomorphism

$$pr^*_{1,\ldots,i}(\mathcal{L}_i) \otimes_{\mathcal{O}_{V_{i+j}}} pr^*_{i+1,\ldots,i+j}(\mathcal{L}_j) \to \mathcal{L}_{i+j}.$$

We declare $B_0 = k$.

We will later see in Theorem 4.6 that B is generated in degree one; thus S surjects onto B.

To begin the analysis of B for $S(1,1,1)$, recall that $V_1 = \mathbb{P}^2$ so

$$B_1 = H^0(V_1, pr_1^*\mathcal{O}_{\mathbb{P}^2}(1)) = kx \oplus ky \oplus kz$$

where $[x:y:z]$ are the coordinates of $\mathbb{P}^2$. For $d \geq 2$ we will compute $\dim_k B_d$ and then proceed to the more difficult task of identifying the multiplication maps $\mu_{i,j} : B_i \times B_j \to B_{i+j}$. Before we get to specific calculations for $d \geq 2$, let us recall that the schemes V_d are realized as the union of six subsets $\{W_{d,i}\}_{i=1}^6$ of $(\mathbb{P}^2)^{\times d}$ described in Proposition 3.13 and Eq. (3.6). These subsets intersect nontrivially so that each V_d for $d \geq 2$ is singular. More precisely,

Remark 4.2. *A routine computation shows that the singular subset, Sing(V_d), consists of six points:*

$$\begin{array}{lll}
v_{d,1} := & ([1:1:1],\ [1:\zeta:\zeta^2],\ [1:1:1],\ [1:\zeta:\zeta^2], \dots) & \in W_{d,2} \cap W_{d,3}, \\
v_{d,2} := & ([1:1:1],\ [1:\zeta^2:\zeta],\ [1:1:1],\ [1:\zeta^2:\zeta], \dots) & \in W_{d,2} \cap W_{d,5}, \\
v_{d,3} := & ([1:\zeta:\zeta^2],\ [1:1:1],\ [1:\zeta:\zeta^2],\ [1:1:1], \dots) & \in W_{d,1} \cap W_{d,4}, \\
v_{d,4} := & ([1:\zeta:\zeta^2],\ [1:\zeta:\zeta^2],\ [1:\zeta:\zeta^2],\ [1:\zeta:\zeta^2], \dots) & \in W_{d,3} \cap W_{d,4}, \\
v_{d,5} := & ([1:\zeta^2:\zeta],\ [1:1:1],\ [1:\zeta^2:\zeta],\ [1:1:1], \dots) & \in W_{d,1} \cap W_{d,6}, \\
v_{d,6} := & ([1:\zeta^2:\zeta],\ [1:\zeta^2:\zeta],\ [1:\zeta^2:\zeta],\ [1:\zeta^2:\zeta], \dots) & \in W_{d,5} \cap W_{d,6}.
\end{array}$$

where $\zeta = e^{2\pi i/3}$.

4.1. **Computing the dimension of B_d.** Our objective in this section is to prove

Proposition 4.3. *For* $d \geq 1$, $\dim_k B_d = 3\left(2^{\lfloor \frac{d+1}{2} \rfloor} + 2^{\lceil \frac{d-1}{2} \rceil}\right) - 6$.

For the rest of the section, let $\mathbf{1}$ denote a sequence of 1s of appropriate length. Now consider the normalization morphism $\pi : V_d' \to V_d$ where V_d' is the disjoint union of the six subsets $\{W_{d,i}\}_{i=1}^6$ mentioned in Proposition 3.13. This map induces the following short exact sequence of sheaves on V_d:

$$0 \to \mathcal{O}_{V_d}(\mathbf{1}) \to (\pi_*\mathcal{O}_{V_d'})(\mathbf{1}) \to \mathcal{S}(\mathbf{1}) \to 0, \tag{4.1}$$

where $\mathcal{S}$ is the skyscraper sheaf whose support is Sing(V_d), that is $\mathcal{S} = \bigoplus_{k=1}^6 \mathcal{O}_{\{v_{d,k}\}}$.

Note that we have

$$H^0(V_d, (\pi_*\mathcal{O}_{V_d'})(\mathbf{1})) \underset{k-\text{v.s.}}{\cong} H^0(V_d', \mathcal{O}_{V_d'}(\mathbf{1})) \tag{4.2}$$

since the normalization morphism is a finite map, which in turn is an affine map [11, Exercises II.5.17(b), III.4.1]. To complete the proof of the proposition, we make the following assertion:

Claim: $H^1(V_d, \mathcal{O}_{V_d}(\mathbf{1})) = 0$.

Assuming that the claim holds, we get from (4.1) the following long exact sequence of cohomology:

$$\begin{aligned} 0 \to H^0(V_d, \mathcal{O}_{V_d}(\mathbf{1})) & \to H^0(V_d, (\pi_*\mathcal{O}_{V_d'})(\mathbf{1})) \\ & \to H^0(V_d, \mathcal{S}(\mathbf{1})) \;\to\; H^1(V_d, \mathcal{O}_{V_d}(\mathbf{1})) = 0. \end{aligned}$$

Thus, with writing $h^0(X, \mathcal{L}) = \dim_k H^0(X, \mathcal{L})$, (4.2) implies that

$$\begin{aligned} \dim_k B_d = h^0(\mathcal{O}_{V_d}(\mathbf{1})) & = h^0((\pi_*\mathcal{O}_{V_d'})(\mathbf{1})) - h^0(\mathcal{S}(\mathbf{1})) \\ & = h^0(\mathcal{O}_{V_d'}(\mathbf{1})) - h^0(\mathcal{S}(\mathbf{1})) \\ & = \textstyle\sum_{i=1}^{6} h^0(\mathcal{O}_{W_{d,i}}(\mathbf{1})) - 6. \end{aligned}$$

Therefore applying Proposition 3.11 and Künneth's Formula [8, A.10.37] completes the proof of Proposition 4.3. It now remains to verify the claim.

Proof of Claim: By the discussion above, it suffices to show that

$$\delta_d : H^0(V_d', \mathcal{O}_{V_d'}(\mathbf{1})) \to H^0\left(\bigcup_{k=1}^{6}\{v_{d,k}\},\ \mathcal{S}(\mathbf{1})\right)$$

is surjective. Referring to the notation of Proposition 3.13 and Remark 4.2, we choose $v_{d,i} \in \mathrm{Supp}(\mathcal{S}(\mathbf{1}))$ and W_{d,k_i} containing $v_{d,i}$. This W_{d,k_i} contains precisely two points of $\mathrm{Supp}(\mathcal{S}(\mathbf{1}))$ and say the other is $v_{d,j}$ for $j \neq i$. After choosing a basis $\{t_i\}_{i=1}^{6}$ for the six-dimensional vector space $H^0(\mathcal{S}(\mathbf{1}))$ where $t_i(v_{d,j}) = \delta_{ij}$, we construct a preimage of each t_i. Since $\mathcal{O}_{W_{d,k_i}}(\mathbf{1})$ is a very ample sheaf, it separates points. In other words there exists $\tilde{s}_i \in H^0(\mathcal{O}_{W_{d,k_i}}(\mathbf{1}))$ such that $\tilde{s}_i(v_{d,j}) = \delta_{ij}$. Extend this section $\tilde{s}_i$ to $s_i \in H^0(\mathcal{O}_{V_d'}(\mathbf{1}))$ by declaring $s_i = \tilde{s}_i$ on W_{d,k_i} and $s_i = 0$ elsewhere. Thus $\delta_d(s_i) = t_i$ for all i and the map δ_d is surjective as desired. □

This concludes the proof of Proposition 4.3.

Corollary 4.4. *We have* $\lim_{d\to\infty}(\dim_k B_d)^{1/d} = \sqrt{2} > 1$ *so* B *has exponential growth hence infinite GK dimension. By* [20, Theorem 0.1]*,* B *is not left or right Noetherian.* □

On the other hand, we can also determine the Hilbert series of B.

Proposition 4.5. $H_B(t) = \dfrac{(1+t^2)(1+2t)}{(1-2t^2)(1-t)}$.

Proof. Recall from Proposition 4.3 that $\dim_k B_d = 3\left(2^{\lceil \frac{d-1}{2} \rceil} + 2^{\lfloor \frac{d+1}{2} \rfloor}\right) - 6$ for $d \geq 1$ and that $\dim_k B_0 = 1$. Thus

$$\begin{aligned} H_B(t) & = 1 + 3\left(\sum_{d\geq 1} 2^{\lceil \frac{d-1}{2} \rceil} t^d + \sum_{d\geq 1} 2^{\lfloor \frac{d+1}{2} \rfloor} t^d - 2\sum_{d\geq 1} t^d\right) \\ & = 1 + 3\left(t\sum_{d\geq 0} 2^{\lceil \frac{d}{2} \rceil} t^d + 2t\sum_{d\geq 0} 2^{\lfloor \frac{d}{2} \rfloor} t^d - 2t\sum_{d\geq 0} t^d\right). \end{aligned}$$

Consider generating functions $a(t) = \sum_{d\geq 0} a_d t^d$ and $b(t) = \sum_{d\geq 0} b_d t^d$ for the respective sequences $a_d = 2^{\lceil d/2 \rceil}$ and $b_d = 2^{\lfloor d/2 \rfloor}$. Elementary operations result in $a(t) = \frac{1+2t}{1-2t^2}$ and $b(t) = \frac{1+t}{1-2t^2}$. Hence

$$H_B(t) = 1 + 3\left[t\left(\frac{1+2t}{1-2t^2}\right) + 2t\left(\frac{1+t}{1-2t^2}\right) - 2t\left(\frac{1}{1-t}\right)\right] = \frac{(1+t^2)(1+2t)}{(1-2t^2)(1-t)}.$$

□

4.2. **The multiplication maps $\mu_{ij} : B_i \times B_j \to B_{i+j}$.** In this section we examine the multiplication of the point parameter ring B of $S(1,1,1)$. In particular, we show that the multiplication maps are surjective which results in the following theorem.

Theorem 4.6. *The point parameter ring B of $S(1,1,1)$ is generated in degree one.*

With similar reasoning, $B = B(S_{deg})$ is generated in degree one for all S_{deg}.

Proof. It suffices to prove that the multiplication maps $\mu_{d,1} : B_d \times B_1 \to B_{d+1}$ are surjective for $d \geq 1$. Recall from Definition 4.1 that $\mu_{d,1} = H^0(m_d)$ where m_d is the isomorphism

$$m_d : \mathcal{O}_{V_d \times \mathbb{P}^2}(1,\ldots,1,0) \otimes_{\mathcal{O}_{V_{d+1}}} \mathcal{O}_{(\mathbb{P}^2)^{\times d}}(0,\ldots,0,1) \to \mathcal{O}_{V_{d+1}}(1,\ldots,1).$$

To use the isomorphism m_d, we employ the following commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_{V_d \times \mathbb{P}^2}(1,\ldots,1,0) \otimes_{\mathcal{O}_{(\mathbb{P}^2)^{\times d+1}}} \mathcal{O}_{(\mathbb{P}^2)^{\times d+1}}(0,\ldots,0,1) \arrow[d] \arrow[dr, "t_d"] & \\
\mathcal{O}_{V_d \times \mathbb{P}^2}(1,\ldots,1,0) \otimes_{\mathcal{O}_{V_{d+1}}} \mathcal{O}_{(\mathbb{P}^2)^{\times d+1}}(0,\ldots,0,1) \arrow[r, "m_d"] & \mathcal{O}_{V_{d+1}}(1,\ldots,1).
\end{tikzcd} \tag{4.3}$$

The source of t_d is isomorphic to $\mathcal{O}_{V_d \times \mathbb{P}^2}(1,\ldots,1)$ and the map t_d is given by restriction to V_{d+1}. Hence we have the short exact sequence

$$0 \longrightarrow \mathcal{I}_{\frac{V_{d+1}}{V_d \times \mathbb{P}^2}}(\mathbf{1}) \longrightarrow \mathcal{O}_{V_d \times \mathbb{P}^2}(\mathbf{1}) \xrightarrow{t_d} \mathcal{O}_{V_{d+1}}(\mathbf{1}) \longrightarrow 0, \tag{4.4}$$

where $\mathcal{I}_{\frac{V_{d+1}}{V_d \times \mathbb{P}^2}}$ is the ideal sheaf of V_{d+1} defined in $V_d \times \mathbb{P}^2$. Since the Künneth formula and the claim from §4.1 implies that $H^1(\mathcal{O}_{V_d \times \mathbb{P}^2}(\mathbf{1})) = 0$, the cokernel of $H^0(t_d)$ is $H^1\Big(\mathcal{I}_{\frac{V_{d+1}}{V_d \times \mathbb{P}^2}}(\mathbf{1})\Big)$. Now we assert

Proposition 4.7. $H^1\Big(\mathcal{I}_{\frac{V_{d+1}}{V_d \times \mathbb{P}^2}}(\mathbf{1})\Big) = 0$ *for* $d \geq 1$.

By assuming that Proposition 4.7 holds, we get the surjectivity of $H^0(t_d)$ for $d \geq 1$. Now by applying the global section functor to Diagram (4.3), we have that $H^0(m_d) = \mu_{d,1}$ is surjective for $d \geq 1$. This concludes the proof of Theorem 4.6. $\square$

Proof of 4.7. Consider the case $d = 1$. We study the ideal sheaf $\mathcal{I}_{\frac{V_2}{\mathbb{P}^2\times\mathbb{P}^2}} := \mathcal{I}_{V_2}$ by using the resolution of the ideal of defining relations (f_0, g_0, h_0) for V_2 (Eqs. (3.1)) in the $\mathbb{N}^2$-graded ring $R = k[x_0, y_0, z_0, x_1, y_1, z_1]$. Note that each of the defining equations have bidegree (1,1) in R and we get the following resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-3,-3) \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-2,-2)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-1,-1)^{\oplus 3} \to \mathcal{I}_{V_2} \to 0.$$

Twisting the above sequence with $\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1)$ we get

$$0 \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-2,-2) \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-1,-1)^{\oplus 3} \to \mathcal{O}^{\oplus 3}_{\mathbb{P}^2\times\mathbb{P}^2} \xrightarrow{f} \mathcal{I}_{V_2}(1,1) \to 0.$$

Let $\mathcal{K} = \ker(f)$. Then $h^0(\mathcal{I}_{V_2}(1,1)) = 3 - h^0(\mathcal{K}) + h^1(\mathcal{K})$. On the other hand, $H^1(\mathcal{O}_{\mathbb{P}^2}(j)) = H^2(\mathcal{O}_{\mathbb{P}^2}(j) = 0$ for $j = -1, -2$. Thus the Künneth formula applied the cohomology of the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-2,-2) \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(-1,-1)^{\oplus 3} \to \mathcal{K} \to 0$$

results in $h^0(\mathcal{K}) = h^1(\mathcal{K}) = 0$. Hence $h^0(\mathcal{I}_{V_2}(1,1)) = 3$.

Now using the long exact sequence of cohomology arising from the short exact sequence

$$0 \to \mathcal{I}_{V_2}(1,1) \to \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1) \to \mathcal{O}_{V_2}(1,1) \to 0,$$

and the facts:

$$\begin{aligned}
&h^0(\mathcal{I}_{V_2}(1,1)) = 3,\\
&h^0(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1)) = 9\\
&h^0(\mathcal{O}_{V_2}(1,1)) = \dim_k B_2 = 6,\\
&h^1(\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1)) = 0,
\end{aligned}$$

we conclude that $H^1(\mathcal{I}_{V_2}(1,1)) = 0$.

For $d \geq 2$ we will construct a commutative diagram to assist with the study of the cohomology of the ideal sheaf $\mathcal{I}_{\frac{V_{d+1}}{V_d\times\mathbb{P}^2}}(\mathbf{1})$. Recall from (4.1) that we have the following normalization sequence for V_d:

$$0 \longrightarrow \mathcal{O}_{V_d} \longrightarrow \bigoplus_{i=1}^{6} \mathcal{O}_{W_{d,i}} \longrightarrow \bigoplus_{k=1}^{6} \mathcal{O}_{\{v_{d,k}\}} \longrightarrow 0. \qquad (\dagger_d)$$

Consider the sequence

$$pr^*_{1,\dots,d}\Big((\dagger_d) \otimes \mathcal{O}_{(\mathbb{P}^2)^{\times d}}(\mathbf{1})\Big) \otimes_{\mathcal{O}_{(\mathbb{P}^2)^{\times d+1}}} pr^*_{d+1}\mathcal{O}_{\mathbb{P}^2}(1)$$

and its induced sequence of restrictions to V_{d+1}, namely

$$0 \to \mathcal{O}_{V_d\times\mathbb{P}^2}(\mathbf{1})\big|_{V_{d+1}} \to \bigoplus_{i=1}^{6} \mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(\mathbf{1})\big|_{V_{d+1}} \to \bigoplus_{k=1}^{6} \mathcal{O}_{\{v_{d,k}\}\times\mathbb{P}^2}(\mathbf{1})\big|_{V_{d+1}} \to 0. \tag{4.5}$$

Now $V_{d+1} \subseteq V_d \times \mathbb{P}^2$ and $(W_{d,i} \times \mathbb{P}^2) \cap V_{d+1} = W_{d+1,i}$ due to Proposition 3.13 and Remark 4.2. We also have that $(\{v_{d,k}\} \times \mathbb{P}^2) \cap V_{d+1} = \{v_{d+1,k}\}$ for all i,k. Therefore the sequence (4.5) is equal to $(\dagger_{d+1}) \otimes \mathcal{O}_{(\mathbb{P}^2)^{\times d+1}}(\mathbf{1})$. In other words, we are given the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{V_d\times\mathbb{P}^2}(\mathbf{1}) \arrow[r] \arrow[d] & \bigoplus_{i=1}^{6} \mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(\mathbf{1}) \arrow[r] \arrow[d] & \bigoplus_{k=1}^{6} \mathcal{O}_{\{v_{d,k}\}\times\mathbb{P}^2}(\mathbf{1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{V_{d+1}}(\mathbf{1}) \arrow[r] & \bigoplus_{i=1}^{6} \mathcal{O}_{W_{d+1,i}}(\mathbf{1}) \arrow[r] & \bigoplus_{k=1}^{6} \mathcal{O}_{\{v_{d+1,k}\}}(\mathbf{1}) \arrow[r] & 0.
\end{tikzcd}$$

Diagram 1: Understanding $\mathcal{I}_{\frac{V_{d+1}}{V_d\times\mathbb{P}^2}}(1,\dots,1)$.

where the vertical maps are given by restriction to V_{d+1}. Observe that the kernels of the vertical maps (from left to right) are respectively $\mathcal{I}_{\frac{V_{d+1}}{V_d\times\mathbb{P}^2}}(\mathbf{1})$, $\bigoplus_i \mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(\mathbf{1})$, and $\bigoplus_k \mathcal{I}_{\frac{\{v_{d+1,k}\}}{\{v_{d,k}\}\times\mathbb{P}^2}}(\mathbf{1})$, and the cokernels are all 0.

By the claim in §4.1 and the Künneth formula, we have that

$$H^1(\mathcal{O}_{V_d\times\mathbb{P}^2}(\mathbf{1})) = H^1(\mathcal{O}_{V_{d+1}}(\mathbf{1})) = 0.$$

Hence the application of the global section functor to Diagram 1 yields Diagram 2 below. Now by the Snake Lemma, we get the following sequence:

$$\dots \longrightarrow \bigoplus_{i=1}^{6} H^0\left(\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(\mathbf{1})\right) \xrightarrow{\psi} \bigoplus_{k=1}^{6} H^0\left(\mathcal{I}_{\frac{\{v_{d+1,k}\}}{\{v_{d,k}\}\times\mathbb{P}^2}}(\mathbf{1})\right)$$

$$\longrightarrow H^1\left(\mathcal{I}_{\frac{V_{d+1}}{V_d\times\mathbb{P}^2}}(\mathbf{1})\right) \longrightarrow \bigoplus_{i=1}^{6} H^1\left(\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(\mathbf{1})\right) \to \dots.$$

In Lemma 4.8, we will show that $\bigoplus_i H^1\Big(\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(\mathbf{1})\Big) = 0$ for $d \geq 2$. Furthermore the surjectivity of the map ψ will follow from Lemma 4.9. This will complete the proof of Proposition 4.7.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & H^0\left(V_d \times \mathbb{P}^2, \mathcal{I}_{\frac{V_{d+1}}{V_d \times \mathbb{P}^2}}(\mathbf{1})\right) \arrow[d] & \bigoplus_{i=1}^{6} H^0\left(W_{d,i} \times \mathbb{P}^2, \mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i} \times \mathbb{P}^2}}(\mathbf{1})\right) \arrow[r, dashed, "\psi"] \arrow[d, "\alpha"'] & \bigoplus_{k=1}^{6} H^0\left(\{v_{d,k}\} \times \mathbb{P}^2, \mathcal{I}_{\frac{\{v_{d+1,k}\}}{\{v_{d,k}\} \times \mathbb{P}^2}}(\mathbf{1})\right) \arrow[d, "\lambda"] & \\
0 \arrow[r] & H^0\left(V_d \times \mathbb{P}^2, \mathcal{O}_{V_d \times \mathbb{P}^2}(\mathbf{1})\right) \arrow[r] \arrow[d] & \bigoplus_{i=1}^{6} H^0\left(W_{d,i} \times \mathbb{P}^2, \mathcal{O}_{W_{d,i} \times \mathbb{P}^2}(\mathbf{1})\right) \arrow[r, "\nu"] \arrow[d, "\beta"'] & \bigoplus_{k=1}^{6} H^0\left(\{v_{d,k}\} \times \mathbb{P}^2, \mathcal{O}_{\{v_{d,k}\} \times \mathbb{P}^2}(\mathbf{1})\right) \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & H^0\left(V_{d+1}, \mathcal{O}_{V_{d+1}}(\mathbf{1})\right) \arrow[r] \arrow[d] & \bigoplus_{i=1}^{6} H^0\left(W_{d+1,i}, \mathcal{O}_{W_{d+1,i}}(\mathbf{1})\right) \arrow[r] \arrow[d] & \bigoplus_{k=1}^{6} H^0\left(\{v_{d+1,k}\}, \mathcal{O}_{\{v_{d+1,k}\}}(\mathbf{1})\right) \arrow[r] \arrow[d] & 0 \\
 & H^1\left(V_d \times \mathbb{P}^2, \mathcal{I}_{\frac{V_{d+1}}{V_d \times \mathbb{P}^2}}(\mathbf{1})\right) & \bigoplus_{i=1}^{6} H^1\left(W_{d,i} \times \mathbb{P}^2, \mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i} \times \mathbb{P}^2}}(\mathbf{1})\right) & \bigoplus_{k=1}^{6} H^1\left(\{v_{d,k}\} \times \mathbb{P}^2, \mathcal{I}_{\frac{\{v_{d+1,k}\}}{\{v_{d,k}\} \times \mathbb{P}^2}}(\mathbf{1})\right) &
\end{tikzcd}$$

Diagram 2: Induced Cohomology from Diagram 1

Lemma 4.8. $\displaystyle\bigoplus_{i=1}^{6} H^1\Big(W_{d,i}\times\mathbb{P}^2,\ \mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(\mathbf{1})\Big) = 0$ *for* $d \geq 2$.

Proof. We consider the different parities of d and i separately. For d even and i odd,

$$\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}} \;\cong\; \mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(0,\dots,0,-1)$$

because $W_{d+1,i}$ is defined in $W_{d,i}\times\mathbb{P}^2$ by one equation of degree $(0,\dots,0,1)$ (Proposition 3.13). Twisting by $\mathcal{O}_{(\mathbb{P}^2)^{\times d+1}}(1,\dots,1)$ results in

$$H^1\left(\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(1,\dots,1)\right) \cong H^1\left(\mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(1,\dots,1,0)\right). \tag{4.6}$$

Since $W_{d,i}$ is the product of $\mathbb{P}^1$ and points lying in $\mathbb{P}^2$ and $H^1(\mathcal{O}_{\mathbb{P}^1}(1)) = H^1(\mathcal{O}_{\{pt\}}(1)) = H^1(\mathcal{O}_{\mathbb{P}^2}) = 0$, the Künneth formula implies that the right hand side of (4.6) is equal to zero.

Consider the case of d and i even. As $pr_{1,\dots,d}(W_{d+1,i}) = W_{d,i}$ and $pr_{d+1}(W_{d+1,i}) = [1:\omega:\omega^2]$ for $\omega = \omega_{d,i}$ a third of unity, we have that $W_{d+1,i}$ is defined in $W_{d,i}\times\mathbb{P}^2$ by two equations of degree $(0,\dots,0,1)$. The defining equations (in variables x,y,z) of $[1:\omega:\omega^2]$ form a $k[x,y,z]$-regular sequence and so we have the Koszul resolution of $\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}} \otimes \mathcal{O}_{(\mathbb{P}^2)^{\times d+1}}(1,\dots,1)$:

$$\begin{aligned} 0 \to \mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(1,\dots,1,-1) &\to \mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(1,\dots,1,0)^{\oplus 2} \\ &\to \mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(1,\dots,1) \;\to\; 0. \end{aligned} \tag{4.7}$$

Now apply the global section functor to sequence (4.7) and note that

$$H^j(\mathcal{O}_{W_{d,i}}(1,\dots,1)) = H^j(\mathcal{O}_{\mathbb{P}^2}) = H^j(\mathcal{O}_{\mathbb{P}^2}(-1)) = 0 \text{ for } j = 1,2.$$

Hence the Künneth formula yields

$$H^1\left(\mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(1,\dots,1,0)\right)^{\oplus 2} = H^2\left(\mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(1,\dots,1,-1)\right) = 0.$$

Therefore $H^1\Big(\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(\mathbf{1})\Big) = 0$ for d and i even.

We conclude that for d even, we know $\displaystyle\bigoplus_{i=1}^{6} H^1\Big(\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(\mathbf{1})\Big) = 0$. For d odd, the same conclusion is drawn by swapping the arguments for the i even and i odd subcases. $\square$

Lemma 4.9. *The map* ψ *is surjective for* $d \geq 2$.

Proof. Refer to the notation from Diagram 2. To show ψ is onto, here is our plan of attack.

(1) Choose a basis of $\bigoplus_k H^0\Big(\mathcal{I}_{\frac{\{v_{d+1,k}\}}{\{v_{d,k}\}\times\mathbb{P}^2}}(\mathbf{1})\Big)$ so that each basis element t lies in $H^0\Big(\mathcal{I}_{\frac{\{v_{d+1,k_0}\}}{\{v_{d,k_0}\}\times\mathbb{P}^2}}(\mathbf{1})\Big)$ for some $k = k_0$. For such a basis element t, identify its image under λ in $\bigoplus_k H^0\left(\mathcal{O}_{\{v_{d,k}\}\times\mathbb{P}^2}(\mathbf{1})\right)$.

(2) Construct for $\lambda(t)$ a suitable preimage $s \in \nu^{-1}(\lambda(t))$.

(3) Prove $s \in \ker(\beta)$.

As a consequence, s lies in $\bigoplus_i H^0\Big(\mathcal{I}_{\frac{W_{d+1,i}}{W_{d,i}\times\mathbb{P}^2}}(\mathbf{1})\Big)$ and serves as a preimage to t under ψ. In other words, ψ is surjective. To begin, fix such a basis element t and integer k_0.

Step 1: Observe that $pr_{1,\dots,d}(\{v_{d+1,k_0}\}) = \{v_{d,k_0}\}$ and $pr_{d+1}(\{v_{d+1,k_0}\}) = [1 : \omega : \omega^2]$ for some ω, a third root of unity (Remark 4.2). Thus our basis element $t \in \bigoplus_k H^0\Big(\mathcal{I}_{\frac{\{v_{d+1,k}\}}{\{v_{d,k}\}\times\mathbb{P}^2}}(\mathbf{1})\Big)$ is of the form

$$t = a(\omega x_d - y_d) + b(\omega^2 x_d - z_d) \tag{4.8}$$

for some $a, b \in k$, with $\{\omega x_d - y_d,\ \omega^2 x_d - z_d\}$ defining $[1 : \omega : \omega^2]$ in the $(d+1)^{st}$ copy of $\mathbb{P}^2$. Note that λ is the inclusion map so we may refer to $\lambda(t)$ as t. This concludes Step 1. $\square$

Step 2: Next we construct a suitable preimage $s \in \nu^{-1}(\lambda(t))$. Referring to Remark 4.2, let us observe that for all k, there is an unique even integer $:= i''_k$ and unique odd integer $:= i'_k$ so that $v_{d,k} \in W_{d,i''_k} \cap W_{d,i'_k}$ for all $k = 1, \dots, 6$. For instance with $k_0 = 1$, we consider the membership $v_{d,1} \in W_{d,2} \cap W_{d,3}$; hence $i''_1 = 2$ and $i'_1 = 3$.

As a consequence, $\lambda(t)$ has preimages under ν in

$$H^0\left(W_{d,i''_{k_0}} \times \mathbb{P}^2, \mathcal{O}_{W_{d,i''_{k_0}}\times\mathbb{P}^2}(\mathbf{1})\right) \oplus\ H^0\left(W_{d,i'_{k_0}} \times \mathbb{P}^2, \mathcal{O}_{W_{d,i'_{k_0}}\times\mathbb{P}^2}(\mathbf{1})\right).$$

For d even (respectively odd) we write $i_{k_0} := i''_{k_0}$ (respectively $i_{k_0} := i'_{k_0}$). Therefore we intend to construct $s \in \nu^{-1}(t)$ belonging to $H^0\Big(\mathcal{O}_{W_{d,i_{k_0}}\times\mathbb{P}^2}(\mathbf{1})\Big)$. However this $W_{d,i_{k_0}}$ will also contain another point $v_{d,j}$ for some $j \neq k_0$. Let us define the global section $\tilde{s} \in H^0\Big(\mathcal{O}_{W_{d,i_{k_0}}\times\mathbb{P}^2}(\mathbf{1})\Big)$ as follows. Since $\mathcal{O}_{W_{d,i_{k_0}}}(\mathbf{1})$ is a very ample sheaf, we have a global section $\tilde{s}_{k_0}$ separating the points v_{d,k_0} and $v_{d,j}$; say $\tilde{s}_{k_0}(v_{d,k}) = \delta_{k_0,k}$. We then use (4.8) to define $\tilde{s}$ by

$$\tilde{s} = \tilde{s}_{k_0} \cdot [a(\omega x_d - y_d) + b(\omega^2 x_d - z_d)].$$

where $[1 : \omega : \omega^2] = pr_{d+1}(\{v_{d+1,k_0}\})$. We now extend this section $\tilde{s}$ to

$$s \in \bigoplus_{i=1}^{6} H^0\left(\mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(\mathbf{1})\right) \cong \left(\bigoplus_{i=1}^{6} H^0\left(\mathcal{O}_{W_{d,i}}(\mathbf{1})\right)\right) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)).$$

This is achieved by setting $s = \tilde{s}$ on $W_{d,i_{k_0}} \times \mathbb{P}^2$ and 0 elsewhere. To check that $\nu(s) = t$, note

$$s = \bigoplus_{i=1}^{6} s_i \text{ where } s_i \in H^0\left(\mathcal{O}_{W_{d,i}\times\mathbb{P}^2}(\mathbf{1})\right), \; s_i = \begin{cases} \tilde{s}, & i = i_{k_0}, \\ 0, & i \neq i_{k_0}; \end{cases} \tag{4.9}$$

Therefore by the construction of $\tilde{s}$, we have $\nu(\tilde{s}) = t\big|_{\{v_{d,k_0}\}\times\mathbb{P}^2}$. Hence we have built our desired preimage $s \in \nu^{-1}(t)$ and this concludes Step 2. $\square$

Step 3: Recall the structure of s from (4.9). By definition of β, we have that $\beta(s) = \beta\left(\bigoplus_{i=1}^6 s_i\right)$ is equal to $\bigoplus_{i=1}^6 \left(s_i\big|_{W_{d+1,i}}\right)$.

For $i \neq i_{k_0}$, we clearly get that $s_i\big|_{W_{d+1,i}} = 0$. On the other hand, the key point of our construction is that $W_{d+1,i_{k_0}} = W_{d,i_{k_0}} \times [1 : \epsilon : \epsilon^2]$ for some $\epsilon^3 = 1$ as i_{k_0} is chosen to be even (respectively odd) when d is even (respectively odd) (Proposition 3.13). Moreover $v_{d+1,k_0} \in W_{d+1,i_{k_0}}$ and

$$pr_{d+1}(W_{d+1,i_{k_0}}) = pr_{d+1}(\{v_{d+1,k_0}\}) = [1 : \omega : \omega^2]$$

where ω is defined by Step 1 and Remark 4.2. Thus $\epsilon = \omega$. Now we have

$$s_{i_{k_0}}\big|_{W_{d+1,i_{k_0}}} = \tilde{s}_{k_0} \cdot [a(\omega x_d - y_d) + b(\omega^2 x_d - z_d)]\Big|_{[1:\omega:\omega^2]} = 0.$$

Therefore $s_i\big|_{W_{d+1,i}} = 0$ for all $i = 1, \ldots, 6$. Hence $\beta(s) = 0$. $\square$

Hence Steps 1-3 are complete which concludes the proof of Lemma 4.9. $\square$

Consequently, we have verified Proposition 4.7. $\square$

One of the main results why twisted homogeneous coordinate rings are so useful for studying Sklyanin algebras is that tcrs are factors of their corresponding Sklyanin algebra (by some homogeneous element; refer to Theorem 1.3). The following corollaries to Theorem 4.6 illustrate an analogous result for S_{deg}.

Corollary 4.10. *Let B be the point parameter ring of a degenerate Sklyanin algebra S_{deg}. Then $B \cong S_{deg}/K$ for some ideal K of S_{deg} that has six generators of degree 4 and possibly higher degree generators.*

Proof. By Theorem 4.6, S_{deg} surjects onto B say with kernel K. By Remark 2.3 we have that $\dim_k S_4 = 57$, yet we know $\dim_k B_4 = 63$ by Proposition 4.3. Hence $\dim_k K_4 = 6$. The same results also imply that $\dim_k S_d = \dim_k B_d$ for $d \leq 3$. $\square$

Corollary 4.11. *The ring $B = B(S_{deg})$ is neither a domain or Koszul.*

Proof. By Corollary 2.4, there exist linear nonzero elements $u, v \in S$ with $uv = 0$. The image of u and v are nonzero, hence B is not a domain due to Corollary 4.10. Since B has degree 4 relations, it does not possess the Koszul property. $\square$

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CORRIGENDUM TO "DEGENERATE SKLYANIN ALGEBRAS AND GENERALIZED TWISTED HOMOGENEOUS COORDINATE RINGS", J. ALGEBRA 322 (2009) 2508-2527

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There is an error in the computation of the truncated point schemes V_d of the degenerate Sklyanin algebra $S(1,1,1)$. We are grateful to S. Paul Smith for pointing out that V_d is larger than was claimed in Proposition 3.13. All 2 or 3 digit references are to the above paper, while 1 digit references are to the results in this corrigendum. We provide a description of the correct V_d in Proposition 5 below. Results about the corresponding point parameter ring B associated to the schemes $\{V_d\}_{d\geq 1}$ are given afterward.

Acknowledgments. I thank Sue Sierra for pointing out a typographical error in Lemma 3.9, and for providing several insightful suggestions. I also thank Paul Smith for suggesting that a quiver could be used for the bookkeeping required in Proposition 5. Moreover, I am grateful to Paul Smith and Toby Stafford for providing detailed remarks, which improved the exposition of this manuscript.

1. Corrections

The main error in the above paper is to the statement of Lemma 3.10. Before stating the correct version, we need some notation.

Notation. Given $\zeta = e^{2\pi i/3}$, let $p_a := [1:1:1]$, $p_b := [1:\zeta:\zeta^2]$, and $p_c := [1:\zeta^2:\zeta]$. Also, let $\check{\mathbb{P}}^1_A := \mathbb{P}^1_A \setminus \{p_b, p_c\}$, $\check{\mathbb{P}}^1_B := \mathbb{P}^1_B \setminus \{p_a, p_c\}$, and $\check{\mathbb{P}}^1_C := \mathbb{P}^1_C \setminus \{p_a, p_b\}$.

We also require the following more precise version of Lemma 3.9; the original result is correct though there is a slight change in the proof as given below.

Lemma 1. *(Correction of Lemma 3.9) Let $p = (p_0, \dots, p_{d-2}) \in V_{d-1}$ with $p_{d-2} \in \check{\mathbb{P}}^1_A$, $\check{\mathbb{P}}^1_B$, or $\check{\mathbb{P}}^1_C$. If $p' = (p, p_{d-1}) \in V_d$, then $p_{d-1} = p_a$, p_b, or p_c respectively.*

Proof. The proof follows from that of Lemma 3.9, except that there is a typographical error in the case when $p_{d-2} = [0 : y_{d-2} : z_{d-2}]$. Here, we require that (p_{d-2}, p_{d-1}) satisfies the system of equations:

$$\begin{aligned} f_{d-2} = g_{d-2} = h_{d-2} &= 0, \\ y_{d-2}^3 + z_{d-2}^3 &= 0, \\ x_{d-1}^3 + y_{d-1}^3 + z_{d-1}^3 - 3x_{d-1}y_{d-1}z_{d-1} &= 0. \end{aligned}$$

This implies that either $y_{d-2} = z_{d-2} = 0$ or $x_{d-1} = y_{d-1} = z_{d-1} = 0$, which produces a contradiction. $\square$

Now the correct version of Lemma 3.10 is provided below. The present version is slightly weaker the original result, where it was claimed that $p_{d-1} \in \check{\mathbb{P}}^1_*$ instead of $p_{d-1} \in \mathbb{P}^1_*$. Here, $\mathbb{P}^1_*$ denotes either $\mathbb{P}^1_A$, $\mathbb{P}^1_B$, or $\mathbb{P}^1_C$.

Lemma 2. *(Correction of Lemma 3.10) Let* $p = (p_0, \dots, p_{d-2}) \in V_{d-1}$ *with* $p_{d-2} = p_a$*,* p_b*, or* p_c*. If* $p' = (p, p_{d-1}) \in V_d$*, then* $p_{d-1} \in \mathbb{P}^1_A$*,* $\mathbb{P}^1_B$*, or* $\mathbb{P}^1_C$ *respectively.*

Proof. The proof from follows that of Lemma 3.10 with the exception that there is a typographical error in the definition of the function θ; it should be defined as:

$$\theta(y_{d-1}, z_{d-1}) = \begin{cases} -(y_{d-1} + z_{d-1}) & \text{if } p_{d-2} = p_a, \\ -(\zeta^2 y_{d-1} + \zeta z_{d-1}) & \text{if } p_{d-2} = p_b, \\ -(\zeta y_{d-1} + \zeta^2 z_{d-1}) & \text{if } p_{d-2} = p_c. \end{cases}$$

□

Remark 3. There are two further minor typographical corrections to the paper.

(1) (Correction of Figure 3.1) The definition of the projective lines $\mathbb{P}^1_B$ and $\mathbb{P}^1_C$ should be interchanged. More precisely, the curve E_{111} is the union of three projective lines:

$$\begin{aligned} &\mathbb{P}^1_A : x + y + z = 0, \\ &\mathbb{P}^1_B : x + \zeta^2 y + \zeta z = 0, \\ &\mathbb{P}^1_C : x + \zeta y + \zeta^2 z = 0. \end{aligned}$$

(2) (Correction to Corollary 4.10) The numbers 57 and 63 should be replaced by 24 and 18 respectively.

2. Consequences

The main consequence of weakening Lemma 3.10 to Lemma 3 is that the truncated point schemes $\{V_d\}_{d\geq 1}$ of $S = S(1,1,1)$ are strictly larger than the truncated point schemes computed in Proposition 3.13 for $d \geq 4$. We discuss such results in §2.1 below. Furthermore, the corresponding point parameter ring associated to the correct point scheme data of S is studied in §2.2.

Notation. (i) Let $W_d := \bigcup_{i=1}^6 W_{d,i}$ with $W_{d,i}$ defined in Proposition 3.13.
(ii) Let $B := \bigoplus_{d\geq 0} H^0(V_d, \mathcal{O}_{V_d}(\mathbf{1}))$ be the point parameter ring of $S(1,1,1)$ as in Definition 1.8.
(iii) Likewise let $P := \bigoplus_{d\geq 0} H^0(W_d, \mathcal{O}_{W_d}(\mathbf{1}))$ be the point parameter ring associated to the schemes $\{W_d\}_{d\geq 1}$.

The results of §4 of the paper are still correct; we describe the ring P, and we show that it is a factor of $S(1,1,1)$. Unfortunately, the ring P is not equal to the point parameter ring B of $S(1,1,1)$. More precisely, the following corrections should be made.

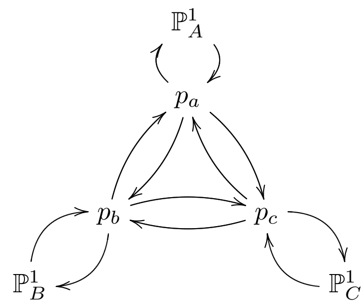
Remark 4. (1) The scheme V_d should be replaced by W_d in Theorem 1.7, in Proposition 3.13, in Remark 3.14, and in all of §4 after Definition 4.1.

(2) The ring B should be replaced by P in §1 after Definition 1.8, and in all of §4 with the exception of the second paragraph.

2.1. **On the truncated point schemes $\{V_d\}_{d\geq 1}$.** We provide a description of the truncated point schemes $\{V_d\}_{d\geq 1}$ as follows.

Notation. Let $\{V_{d,i}\}_{i\in I_d}$ denote the $|I_d|$ irreducible components of the d^{th} truncated point scheme V_d.

Proposition 5. *(Description of V_d) For $d \geq 2$, the length d truncated point scheme V_d is realized as the union of length d paths of the quiver Q below. With $d = 2$, for example, the path $\mathbb{P}^1_A \longrightarrow p_a$ corresponds to the component $\mathbb{P}^1_A \times p_a$ of V_2.*



The quiver Q

Proof. We proceed by induction. Considering the $d = 2$ case, Lemma 3.12 still holds so $V_2 = W_2$, the union of the irreducible components:

$$\begin{array}{lll} \mathbb{P}^1_A \times p_a, & \mathbb{P}^1_B \times p_b, & \mathbb{P}^1_C \times p_c \\ p_a \times \mathbb{P}^1_A, & p_b \times \mathbb{P}^1_B, & p_c \times \mathbb{P}^1_C. \end{array}$$

One can see these components correspond to length 2 paths of the quiver Q. Conversely, any length 2 path of Q corresponds to a component that lies in V_2.

We assume the proposition holds for V_{d-1}, and recall that Lemmas 2 and 3 provide the recipe to build V_d from V_{d-1}. Take a point $(p_0, \ldots, p_{d-2}) \in V_{d-1,i}$, where the irreducible component $V_{d-1,i}$ of V_{d-1} corresponds to a length $d-1$ path of Q. Let $\{V_{d,ij}\}_{j\in J}$ be the set of $|J|$ irreducible components of V_d with

$$(p_0, \ldots, p_{d-2}, p_{d-1}) \in V_{d,ij} \subseteq V_d$$

for some $p_{d-1} \in \mathbb{P}^2$. There are two cases to consider.

<u>Case 1</u>: We have that (p_{d-3}, p_{d-2}) lies in one of the following products:

$$\begin{array}{lll} \mathbb{P}^1_A \times p_a, & \mathbb{P}^1_B \times p_b, & \mathbb{P}^1_C \times p_c, \\ p_a \times \check{\mathbb{P}}^1_A, & p_b \times \check{\mathbb{P}}^1_B, & p_c \times \check{\mathbb{P}}^1_C. \end{array}$$

For the first three choices, Lemma 2 implies that $pr_d(V_{d,ij}) = \mathbb{P}^1_A$, $\mathbb{P}^1_B$, or $\mathbb{P}^1_C$, respectively. For the second three choices, p_{d-2} belongs to $\check{\mathbb{P}}^1_A$, $\check{\mathbb{P}}^1_B$, or $\check{\mathbb{P}}^1_C$, and Lemma 1 implies that $pr_d(V_{d,ij}) = p_a$, p_b, or p_c, respectively. We conclude by induction that the component $V_{d,ij}$ yields a length d path of Q.

Case 2: We have that (p_{d-3}, p_{d-2}) is equal to one of the following points:

$$\begin{array}{ll} p_a \times p_b, & p_a \times p_c, \\ p_b \times p_a, & p_b \times p_c, \\ p_c \times p_a, & p_c \times p_b. \end{array}$$

Now Lemma 2 implies that:

$$pr_d(V_{d,ij}) = \begin{cases} \mathbb{P}^1_A & \text{if } p_{d-2} = p_a, \\ \mathbb{P}^1_B & \text{if } p_{d-2} = p_b, \\ \mathbb{P}^1_C & \text{if } p_{d-2} = p_c. \end{cases}$$

Again we have that in this case, the component $V_{d,ij}$ yields a length d path of Q.

Conversely (in either case), let $\mathcal{P}$ be a length d path of Q. Then, by induction, the embedded length $d-1$ path $\mathcal{P}'$ ending at the $d-1^{\text{st}}$ vertex v' of $\mathcal{P}$ yields a component X' of V_{d-1}. Say v is the d^{th} vertex of $\mathcal{P}$. If v' is equal to $\mathbb{P}^1_A$, $\mathbb{P}^1_B$, or $\mathbb{P}^1_C$, then v must be p_a, p_b, or p_c by the definition of Q, respectively. Lemma 2 then ensures that $\mathcal{P}$ yields a component X of V_d so that $pr_{1\ldots d-1}(X) = X'$. On the other hand, if v' is equal to p_a, p_b, or p_c, then v lies in $\mathbb{P}^1_A$, $\mathbb{P}^1_B$, or $\mathbb{P}^1_C$, respectively. Likewise, Lemma 3 implies that $\mathcal{P}$ yields a component X of V_d so that $pr_{1\ldots d-1}(X) = X'$. $\square$

Corollary 6. *We have that $V_d = W_d$ for $d = 1, 2, 3$, and that $V_d \supsetneq W_d$ for $d \geq 4$.*

Proof. First, $V_1 = \mathbb{P}^2 = W_1$. Next, as mentioned in proof of Proposition 5, $V_2 = W_2$ is the union of the irreducible components:

$$\begin{array}{lll} \mathbb{P}^1_A \times p_a, & \mathbb{P}^1_B \times p_b, & \mathbb{P}^1_C \times p_c \\ p_a \times \mathbb{P}^1_A, & p_b \times \mathbb{P}^1_B, & p_c \times \mathbb{P}^1_C. \end{array}$$

By Proposition 5, we have that $V_3 = X_{3,1} \cup X_{3,2}$ where $X_{3,1}$ consists of the irreducible components:

$$\begin{array}{lll} \mathbb{P}^1_A \times p_a \times \mathbb{P}^1_A, & \mathbb{P}^1_B \times p_b \times \mathbb{P}^1_B, & \mathbb{P}^1_C \times p_c \times \mathbb{P}^1_C, \\ p_a \times \mathbb{P}^1_A \times p_a, & p_b \times \mathbb{P}^1_B \times p_b, & p_c \times \mathbb{P}^1_C \times p_c, \end{array}$$

and $X_{3,2}$ is the union of:

$$\begin{array}{llll} \mathbb{P}^1_A \times p_a \times p_b, & \mathbb{P}^1_A \times p_a \times p_c, & p_a \times p_b \times \mathbb{P}^1_B, & p_a \times p_c \times \mathbb{P}^1_C, \\ p_a \times p_b \times p_a, & p_a \times p_b \times p_c, & p_a \times p_c \times p_a, & p_a \times p_c \times p_b, \\ \mathbb{P}^1_B \times p_b \times p_c, & \mathbb{P}^1_B \times p_b \times p_a, & p_b \times p_c \times \mathbb{P}^1_C, & p_b \times p_a \times \mathbb{P}^1_A, \\ p_b \times p_c \times p_b, & p_b \times p_c \times p_a, & p_b \times p_a \times p_b, & p_b \times p_a \times p_c, \\ \mathbb{P}^1_C \times p_c \times p_a, & \mathbb{P}^1_C \times p_c \times p_b, & p_c \times p_a \times \mathbb{P}^1_A, & p_c \times p_b \times \mathbb{P}^1_B, \\ p_c \times p_a \times p_c, & p_c \times p_a \times p_b, & p_c \times p_b \times p_c, & p_c \times p_b \times p_a. \end{array}$$

Note that $X_{3,2}$ is contained in $X_{3,1}$; hence $V_3 = X_{3,1} = W_3$. Furthermore, one sees that $W_d \subsetneq V_d$ for $d \geq 4$ as follows. The components of W_d are read off the subquiver Q' of Q below.

$$\begin{tikzcd}
& & \mathbb{P}^1_A \arrow[d, bend left] & & \\
& & p_a \arrow[u, bend left] & & \\
& p_b \arrow[dl, bend left] & & p_c \arrow[dr, bend left] & \\
\mathbb{P}^1_B \arrow[ur, bend left] & & & & \mathbb{P}^1_C \arrow[ul, bend left]
\end{tikzcd}$$

The quiver Q'

On the other hand, for $d \geq 4$, the length d path containing

$$\mathbb{P}^1_A \longrightarrow p_a \longrightarrow p_b \longrightarrow \mathbb{P}^1_B$$

corresponds to a component of V_d not contained in W_d. □

2.2. **On the point parameter ring** $B(\{V_d\})$**.** The result that there exists a ring surjection from $S = S(1,1,1)$ onto the ring $P(\{W_d\})$ remains true. However, by Lemma 7 below, B is a larger ring than P, and whether there is a ring surjection from S onto B is unknown. We know that there is a ring homomorphism from S to B with $S_1 \cong B_1$ by [1, Proposition 3.20], and computational evidence suggests that $S \cong B$. The details are given as follows.

Lemma 7. *The k-vector space dimension of B_d is equal to* $\dim_k S(1,1,1)_d$ *for* $d = 0, 1, \ldots, 4$*. In particular,* $\dim_k B_4 \neq \dim_k P_4$*.*

It is believed that analogous computations will show that $\dim_k B_d = \dim_k S(1,1,1)_d = 3 \cdot 2^{d-1}$ for $d = 5, 6$.

Proof. By Corollary 6, we know that $V_d = W_d$ for $d = 1, 2, 3$; hence

$$\dim_k B_d = 3 \cdot 2^{d-1} = \dim_k S(1,1,1)_d \text{ for } d = 0, 1, 2, 3.$$

To compute $\dim_k B_4$, note that by Proposition 5, V_4 equals the union $X_{4,1} \cup X_{4,2} \subseteq (\mathbb{P}^2)^{\times 4}$ as follows. Here, $X_{4,1}$ consists of the following irreducible components

$$\begin{array}{ll}
\mathbb{P}^1_A \times p_a \times \mathbb{P}^1_A \times p_a, & p_a \times \mathbb{P}^1_A \times p_a \times \mathbb{P}^1_A, \\
\mathbb{P}^1_B \times p_b \times \mathbb{P}^1_B \times p_b, & p_b \times \mathbb{P}^1_B \times p_b \times \mathbb{P}^1_B, \\
\mathbb{P}^1_C \times p_c \times \mathbb{P}^1_C \times p_c, & p_c \times \mathbb{P}^1_C \times p_c \times \mathbb{P}^1_C;
\end{array}$$

and $X_{4,2}$ is the union of

$$\begin{array}{ll}
\mathbb{P}^1_A \times p_a \times p_b \times \mathbb{P}^1_B, & \mathbb{P}^1_A \times p_a \times p_c \times \mathbb{P}^1_C, \\
\mathbb{P}^1_B \times p_b \times p_a \times \mathbb{P}^1_A, & \mathbb{P}^1_B \times p_b \times p_c \times \mathbb{P}^1_C, \\
\mathbb{P}^1_C \times p_c \times p_a \times \mathbb{P}^1_A, & \mathbb{P}^1_C \times p_c \times p_b \times \mathbb{P}^1_B.
\end{array}$$

We consider a component such as $\mathbb{P}^1_A \times p_a \times p_b \times p_a$ contained in $\mathbb{P}^1_A \times p_a \times p_b \times \mathbb{P}^1_B$ to be included as part of $X_{4,2}$.

Since $X_{4,1} = W_4$ we get that $h^0(\mathcal{O}_{X_{4,1}}(1,1,1,1)) = 6 \cdot 4 - 6 = 18$ by Proposition 4.3. Moreover, $h^0(\mathcal{O}_{X_{4,2}}(1,1,1,1)) = 6 \cdot 4 = 24$ as $X_{4,2}$ is a disjoint union of its irreducible components.

Consider the finite morphism

$$\pi_1 : X_{4,1} \uplus X_{4,2} \longrightarrow V_4 = X_{4,1} \cup X_{4,2},$$

which by twisting by $\mathcal{O}_{(\mathbb{P}^2)^{\times 4}}(1,1,1,1)$, we get the exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{V_4}(1,1,1,1) &\longrightarrow [(\pi_1)_*\mathcal{O}_{X_{4,1}\uplus X_{4,2}}](1,1,1,1) \\ &\longrightarrow \mathcal{O}_{X_{4,1}\cap X_{4,2}}(1,1,1,1) \\ &\longrightarrow 0. \end{aligned} \tag{$\dagger$}$$

Here, $X_{4,1} \cap X_{4,2}$ is the union of the following irreducible components:

$$\begin{array}{ll} \mathbb{P}^1_A \times p_a \times p_b \times p_a, & p_b \times p_a \times p_b \times \mathbb{P}^1_B, \\ \mathbb{P}^1_A \times p_a \times p_c \times p_a, & p_c \times p_a \times p_c \times \mathbb{P}^1_C, \\ \mathbb{P}^1_B \times p_b \times p_a \times p_b, & p_a \times p_b \times p_a \times \mathbb{P}^1_A, \\ \mathbb{P}^1_B \times p_b \times p_c \times p_b, & p_c \times p_b \times p_c \times \mathbb{P}^1_C, \\ \mathbb{P}^1_C \times p_c \times p_a \times p_c, & p_a \times p_c \times p_a \times \mathbb{P}^1_A, \\ \mathbb{P}^1_C \times p_c \times p_b \times p_c, & p_b \times p_c \times p_b \times \mathbb{P}^1_B, \end{array}$$

a union that is not disjoint. Let $(X_{4,1} \cap X_{4,2})'$ be the disjoint union of these twelve components and consider the finite morphism

$$\pi_2 : (X_{4,1} \cap X_{4,2})' \to X_{4,1} \cap X_{4,2}.$$

Again by twisting by $\mathcal{O}_{\mathbb{P}^2}(1,1,1,1)$, we get the exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{X_{4,1}\cap X_{4,2}}(1,1,1,1) &\longrightarrow [(\pi_2)_*\mathcal{O}_{(X_{4,1}\cap X_{4,2})'}](1,1,1,1) \\ &\longrightarrow \mathcal{O}_{\mathcal{S}}(1,1,1,1) \\ &\longrightarrow 0, \end{aligned} \tag{$\ddagger$}$$

where $\mathcal{S}$ is the union of the following six points:

$$\begin{array}{lll} p_a \times p_b \times p_a \times p_b, & p_b \times p_a \times p_b \times p_a, & p_a \times p_c \times p_a \times p_c, \\ p_c \times p_a \times p_c \times p_a, & p_b \times p_c \times p_b \times p_c, & p_c \times p_b \times p_c \times p_b. \end{array}$$

Claim 1. $H^1(\mathcal{O}_{X_{4,1}\cap X_{4,2}}(1,1,1,1)) = 0.$

Note that $H^0([(\pi_2)_*\mathcal{O}_{(X_{4,1}\cap X_{4,2})'}](1,1,1,1)) \cong H^0(\mathcal{O}_{(X_{4,1}\cap X_{4,2})'}(1,1,1,1))$ as k-vector spaces since π_2 is an affine map [2, Exercise III 4.1]. Hence, if Claim 1 holds, then by ($\ddagger$):

$$\begin{aligned} h^0(\mathcal{O}_{X_{4,1}\cap X_{4,2}}(1,1,1,1)) &= h^0(\mathcal{O}_{(X_{4,1}\cap X_{4,2})'}(1,1,1,1)) - h^0(\mathcal{O}_{\mathcal{S}}(1,1,1,1)) \\ &= 12 \cdot 2 - 6 = 18. \end{aligned}$$

Claim 2. $H^1(\mathcal{O}_{V_4}(1,1,1,1)) = 0.$

Note that $H^0([(\pi_1)_*\mathcal{O}_{X_{4,1}\uplus X_{4,2}}](1,1,1,1)) \cong H^0(\mathcal{O}_{X_{4,1}\uplus X_{4,2}}(1,1,1,1))$ as k-vector spaces since π_1 is an affine map [2, Exercise III 4.1]. Hence, if Claim 2 is also true, then by ($\dagger$) and the computation above, we note that:

$$\begin{aligned}
\dim_k B_4 &= h^0(\mathcal{O}_{V_4}(1,1,1,1)) \\
&= h^0(\mathcal{O}_{X_{4,1}\uplus X_{4,2}}(1,1,1,1)) - h^0(\mathcal{O}_{X_{4,1}\cap X_{4,2}}(1,1,1,1)) \\
&= h^0(\mathcal{O}_{X_{4,1}}(1,1,1,1)) + h^0(\mathcal{O}_{X_{4,2}}(1,1,1,1)) - h^0(\mathcal{O}_{X_{4,1}\cap X_{4,2}}(1,1,1,1)) \\
&= 18 + 24 - 18 = 24.
\end{aligned}$$

Therefore,

$$\dim_k B_4 = \dim_k S(1,1,1)_4 = 24 \neq 18 = \dim_k P_4.$$

Now we prove Claims 1 and 2 above. Here, we refer to the linear components of $(\mathbb{P}^2)^{\times 4}$ of dimensions 1 or 2 by "lines" or "planes", respectively.

Proof of Claim 1. It suffices to show that

$$\theta : H^0\left(\mathcal{O}_{(X_{4,1}\cap X_{4,2})'}(1,1,1,1)\right) \longrightarrow H^0(\mathcal{O}_{\mathcal{S}}(1,1,1,1))$$

is surjective. Say $\mathcal{S} = \{v_i\}_{i=1}^6$, the union of points v_i. Each point v_i is contained in two lines of $(X_1 \cap X_2)'$, and each of the twelve lines of $(X_1 \cap X_2)'$ contains a unique point of $\mathcal{S}$.

Choose a basis $\{t_i\}_{i=1}^6$ for $H^0(\mathcal{S}(1,1,1,1))$, where $t_i(v_j) = \delta_{ij}$. For each i, there exists a unique line L_i of $(X_{4,1} \cap X_{4,2})'$ containing v_i so that $pr_{234}(L_i) = pr_{234}(v_i)$. Now we define a preimage of t_i by first extending t_i to a global section s_i of $\mathcal{O}_{L_i}(1,1,1,1)$. Moreover, extend s_i to a global section $\tilde{s_i}$ on $\mathcal{O}_{(X_{4,1}\cap X_{4,2})'}(1,1,1,1)$ by declaring that $\tilde{s_i} = s_i$ on L_i and zero elsewhere. Now $\theta(\tilde{s_i}) = t_i$ for all i, and θ is surjective. □

Proof of Claim 2. It suffices to show that

$$\tau : H^0(\mathcal{O}_{X_{4,1}\uplus X_{4,2}}(1,1,1,1)) \longrightarrow H^0(\mathcal{O}_{X_{4,1}\cap X_{4,2}}(1,1,1,1))$$

is surjective.

Recall that $X_{4,1} \cap X_{4,2}$ is the union of twelve lines $\{L_i\}$, and $X_{4,1} \uplus X_{4,2}$ is the union of twelve planes $\{P_i\}$. Here, each line L_i of $X_{4,1} \cap X_{4,2}$ is contained in precisely two planes of $X_{4,1} \uplus X_{4,2}$, and each plane P_i of $X_{4,1} \uplus X_{4,2}$ contains precisely two lines of $X_{4,1} \cap X_{4,2}$.

Choose a basis $\{t_i\}_{i=1}^{12}$ of $H^0\left(\mathcal{O}_{X_{4,1}\cap X_{4,2}}(1,1,1,1)\right)$ so that $t_i(L_j) = \delta_{ij}$. For each i, we want a preimage of t_i in $H^0\left(\mathcal{O}_{X_{4,1}\uplus X_{4,2}}(1,1,1,1)\right)$.

Say P_i is a plane of $X_{4,1} \uplus X_{4,2}$ that contains L_i, and L_j is the other line that is contained in P_i. Since $\mathcal{O}_{P_i}(1,1,1,1)$ is very ample, its global sections separate the lines L_i and L_j. In other words, there exists $s_i \in H^0(\mathcal{O}_{P_i}(1,1,1,1))$ so that $s_i(L_k) = \delta_{ik}$. Extend s_i to $\tilde{s_i} \in H^0\left(\mathcal{O}_{X_{4,1}\cap X_{4,2}}(1,1,1,1)\right)$ by declaring that $\tilde{s_i} = s_i$ on L_i, and zero elsewhere. Now $\tau(\tilde{s_i}) = t_i$ for all i, and τ is surjective. □

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