Brief Introduction to Knot Theory

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What are knots and links?

Projections of knots and links.

How can we distinguish two knots one from the other?

Knot invariants
Some projections of knots and links

- the unknot
- the trefoil
- the figure eight
- the unlink
- the Hopf link
Four different projections of the unknot.
How can we recognize if two projections represent the same knot or different knots?

In 1927 Alexander and Briggs, and independently Reidemeister, proved that two knot diagrams belonging to the same knot can be related by a sequence of three kinds of moves on the diagram. These operations are now called the Reidemeister moves.

Move I

Move II

Move III
Sequence of Reidemeister moves for the unknot.
We want to be able to calculate numbers (or bits of algebra such as polynomials) from given knot diagrams, in such a way that these numbers do not change when the diagrams are changed by Reidemeister moves. Numbers or polynomials of this kind are called **invariants** of the knot.

Example: The **crossing number**, $c(K)$ is a knot invariant.

$c(K)$ = the minimal number of crossings over all projections of $K$.

- $c(\text{unknot}) = 0$
- $c(\text{trefoil}) = 3$
- $c(\text{figure eight}) = 4$

There are 7 distinct knots with 7 crossings, 21 distinct knots with 8 crossings, 49 distinct knots with 9 crossings, and 166 distinct knots with 10 crossings.
Example: The **unknotting number**, \( u(K) \) is a knot invariant.

\( u(K) \) = the minimal number of crossing changes which unknots \( K \).

\[
\begin{align*}
 u(\text{unknot}) &= 0 \\
 u(\text{trefoil}) &= 1 \\
 u(\text{figure eight}) &= 1
\end{align*}
\]

Among all knots up to ten crossings, for nine of them the unknotting number is not yet known. For the rest the unknotting number is either 1, 2 or 3.
An **oriented diagram** is a knot or link diagram with a chosen orientation, marked by an arrow.

Given an oriented diagram, we label the crossings as positive or negative according to:

- **positive**
- **negative**
- **smoothing**
A knot polynomial is a knot invariant that is a polynomial. One knot polynomial is the Alexander-Conway polynomial, a polynomial in the variable $z$ with integer coefficients.

Suppose we are given a link diagram which is oriented and suppose $L_+, L_-$ and $L_0$ are oriented link diagrams resulting from changing the diagram at a specified crossing of the diagram according to the above rules. Then the Alexander-Conway polynomial, $C(z)$, is recursively defined according to the rules:

1. $C(O) = 1$ (where $O$ is any diagram representing the unknot)
2. $C(L_+) = C(L_-) + zC(L_0)$

The second rule is what is often referred to as a skein relation. To prove that this is indeed an invariant one needs to check that the polynomial remains unchanged under the three Reidemeister moves.
Intuitively, the trefoil is a different knot than the unknot. Can we prove this? We can try and see if they have different Alexander-Conway polynomials.
\[ C\left( \begin{array}{c} \text{TREFOIL} \\ \end{array} \right) = C\left( \begin{array}{c} \text{UNKNOT} \\ \end{array} \right) + z \cdot C\left( \begin{array}{c} \text{HOPF LINK} \\ \end{array} \right) \]

\[ C\left( \begin{array}{c} \text{HOPF LINK} \\ \end{array} \right) = C\left( \begin{array}{c} \text{UNLINK} \\ \end{array} \right) + z \cdot C\left( \begin{array}{c} \text{UNKNOT} \\ \end{array} \right) \]

\[ C\left( \begin{array}{c} \text{UNKNOT} \\ \end{array} \right) = C\left( \begin{array}{c} \text{UNKNOT} \\ \end{array} \right) + z \cdot C\left( \begin{array}{c} \text{UNLINK} \\ \end{array} \right) \]
We find the following:

\[ C( \text{trefoil} ) = 1 + z \cdot C( \text{Hopf link}) \]
\[ C( \text{Hopf link} ) = C(\text{unlink}) + z \]
\[ 1 = 1 + z \cdot C(\text{unlink}) \]

We find : \( C( \text{trefoil} ) = 1 + z^2 \)

As \( 1 \neq 1 + z^2 \) we can conclude that trefoil \( \neq \) unknot.

We can similarly compute \( C( \text{figure eight} ) = 1 - x^2 \)

How strong is this invariant?

It as shown that trefoil \( \neq \) mirror image of the trefoil ( the mirror image of a knot is obtained by reversing all it’s crossings ).

But, \( C( \text{trefoil} ) = C( \text{mirror image of the trefoil} ) \)
There are other polynomial invariants for knots and links, such as the Jones polynomial and its generalization, the HOMFLY polynomial.

The Jones polynomial $V_K(t, t^{-1})$ is a Laurent polynomial and can generally distinguish mirror images. For example:

$V(\text{trefoil}) \neq V(\text{mirror image of the trefoil})$