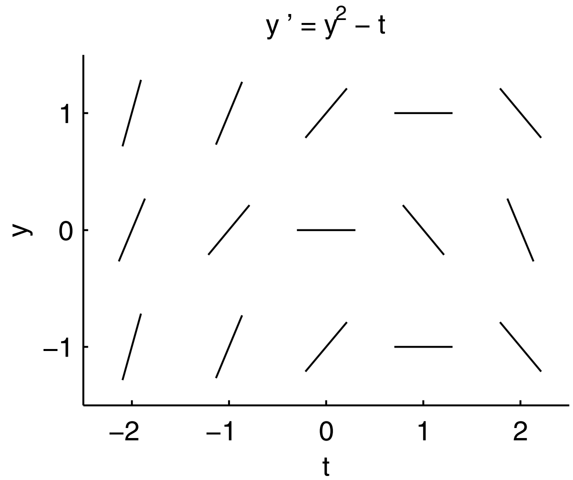
Math 211

Homework #2

January 26, 2001

2.1.18. Plot the direction field for the differential equation $y' = y^2 - t$ by hand. Do this by drawing short lines of the appropriate slope centered at each of the integer valued coordinates (t, y), where $-2 \le t \le 2$ and $-1 \le y \le 1$.

Answer:



2.2.4. $y' = (1 + y^2)e^x$

Answer: Separate the variables and integrate.

$$
\begin{aligned}
\frac{dy}{dx} &= (1 + y^2)e^x \\
\frac{1}{1 + y^2}\,dy &= e^x\,dx \\
\tan^{-1} y &= e^x + C \\
y &= \tan(e^x + C)
\end{aligned}
$$

2.2.10. $x^2y' = y \ln y - y'$

Answer: First a little algebra.

$$
\begin{aligned}
x^2y' &= y \ln y - y' \\
(x^2 + 1)y' &= y \ln y
\end{aligned}
$$

Separate the variables and integrate.

$$\begin{aligned}
\frac{1}{y\ln y}\,dy &= \frac{1}{x^2+1}\,dx \\
\frac{1}{u}\,du &= \frac{1}{x^2+1}\,dx, \qquad \text{where } u = \ln y \text{ and } du = \tfrac{1}{y}\,dy. \\
\ln|u| &= \tan^{-1}x + C
\end{aligned}$$

Solve for u.

$$\begin{aligned}
|u| &= e^{\tan^{-1}x+C} \\
u &= \pm e^C e^{\tan^{-1}x}
\end{aligned}$$

Let $D = \pm e^C$, replace u with $\ln y$, and solve for y.

$$\begin{aligned}
\ln y &= De^{\tan^{-1}x} \\
y &= e^{De^{\tan^{-1}x}}
\end{aligned}$$

2.2.15. $y' = (\sin x)/y$, $y(\pi/2) = 1$

Answer:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\sin x}{y} \\
y\,dy &= \sin x\,dx \\
\frac{1}{2}y^2 &= -\cos x + C_1 \\
y^2 &= -2\cos x + C \quad (C = 2C_1) \\
y(x) &= \pm\sqrt{C - 2\cos x}
\end{aligned}$$

Using the initial condition we notice that we need the plus sign, and $1 = y(\pi/2) = \sqrt{C}$. Thus $C = 1$ and the solution is

$$y(x) = \sqrt{1 - 2\cos x}.$$

The interval of existence will be the interval containing $\pi/2$ where $2\cos x < 1$. This is $\pi/3 < x < 5\pi/3$.

2.2.18. $y' = x/(1+2y)$, $y(-1) = 0$

Answer:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x}{1+2y} \\
(1+2y)\,dy &= x\,dx \\
y + y^2 &= x^2/2 + C
\end{aligned}$$

This last equation can be writen as $y^2 + y - (x^2/2 + C) = 0$. We solve for y using the quadratic formula

$$\begin{aligned} y(x)[-1 \pm \sqrt{1 + 4(x^2/2 + C_1)}]/2 \\ = [-1 \pm \sqrt{2x^2 + C}]/2 \quad (C = 1 + 4C_1) \end{aligned}$$

For the initial condition $y(-1) = 0$ we need to take the plus sign in order to counter the -1. Then the initial condition becomes $0 = [-1 + \sqrt{2 + C}]/2$, which means that $C = -1$. Thus the solution is

$$y(x) = \frac{-1 + \sqrt{2x^2 - 1}}{2}.$$

For the interval of existence we need the interval containing -1 where $2x^2 - 1 > 0$. This is $-\infty < x < -1/\sqrt{2}$.

2.2.21. $y' = (y^2 + 1)/y$, $y(1) = 2$

Answer:

$$\begin{aligned} \frac{dy}{dx} &= \frac{y^2 + 1}{y} \\ \frac{y\,dy}{y^2 + 1} &= dx \\ \frac{1}{2}\ln(y^2 + 1) &= x + C \\ y^2 + 1 &= e^{2x+2C} = Ae^{2x} \quad (A = e^{2C}) \\ y^2 &= Ae^{2x} - 1 \\ y(x) &= \pm\sqrt{Ae^{2x} - 1} \end{aligned}$$

This is the general solution. The initial condition becomes

$$\begin{aligned} 2 &= +\sqrt{Ae^2 - 1} \\ 4 &= Ae^2 - 1 \\ A &= 5e^{-2} \end{aligned}$$

The particular solution is

$$y(x) = \sqrt{5e^{2x-2} - 1}.$$

The interval of existence requires that

$$\begin{aligned} 5e^{2x-2} - 1 &> 0 \\ 2x - 2 &> \ln(1/5) \\ x &> 1 - \ln(5)/2 \approx 0.1953. \end{aligned}$$

Thus the interval of existence is $1 - \ln(5)/2 < x < \infty$.

2.2.35. Suppose a cold beer at 40°F is placed into a warm room at 70°F. Suppose 10 minutes later, the temperature of the beer is 48°F. Use Newton's law of cooling to find the temperature 25 minutes after the beer was placed into the room.

Answer: Let $y(t)$ be the temperature of the beer at time t minutes after being placed into the room. From Newton's law of cooling, we obtain

$$y'(t) = k(70 - y(t)) \quad y(0) = 40$$

Note k is positive since $70 > y(t)$ and $y'(t) > 0$ (the beer is warming up). This equation separates as

$$\frac{dy}{70 - y} = k\, dt$$

which has solution $y = 70 - Ce^{-kt}$. From the initial condition, $y(0) = 40$, $C = 30$. Using $y(10) = 48$, we obtain $48 = 70 - 30e^{-10k}$ or $k = (-1/10)\ln(11/15)$ or $k = .0310$. When $t = 25$, we obtain $y(25) = 70 - 30e^{-.598} \approx 56.18°$.

2.2.37. John and Mary were both served hot coffee at the same time and at the same temperature. John immediately pours cream at room temperature and lets it sit for 10 minutes. Mary lets her coffee sit for 10 minutes and then pours in the same amount of cream. Who has the warmest coffee 10 minutes after they were served? Why?

Answer: A complete analysis of the physics here reveals that the two cups will have the same temperature. This is a common problem in ODE texts, and students are supposed to conclude that John will have warmer coffee. This is incorrect, and based on an incomplete analysis of the problem.

2.3.2. A ballon is ascending at a rate of 15 m/s at a height of 100 m above the ground when a package is dropped from the gondola. How long will it take the package to reach the ground? Ignore air resistance.

Answer: We need $0 = -9.8t^2/2 + 15t + 100$. The answer is 6.3 seconds.

2.3.7. A particle moves along a line with x, v, and a representing position, velocity, and acceleration, respectively. The chain rule states that

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}.$$

Assuming constant acceleration a and the fact that $dv/dt = a$, show that

$$v^2 = v_0^2 + 2a(x - x_0),$$

where x_0 and v_0 are the position and velocity of the particle at time $t = 0$, respectively. A car's speed is reduced from 60 mi/h to 30 mi/h in a span covering 500 ft. Calculate the magnitude and direction of the constant deceleration.

Answer: The velocities must be changed to ft/s, so $v_0 = 60$ mi/h $= 60 \times 5280/3600 = 88$ft/s, and $v = 30$ mi/h $= 44$ft/s. Then $a = (v^2 - v_0^2)/2(x - x_0) = -5.8\text{ft/s}^2$.

2.3.9. A ball having mass $m = 0.1$ kg falls from rest under the influence of gravity in a medium that provides a resistance that is proportional to its velocity. For a velocity of 0.2 m/s, the force due to the resistance of the medium is -1 N. [One Newton [N] is the force required to accelerate a 1kg mass at a rate of 1 m/s^2. Hence, 1 N = 1 kg m/s^2.] Find the limiting velocity of the ball.

Answer: The resistance force has the form $R = -rv$. When $v = 0.2$, $R = -1$ so $r = 5$. The terminal velocity is $v_{\text{term}} = -mg/r = 0.196$m/s.

2.3.14. Suppose that an object with mass m is launched from the earth's surface with initial velocity v_0. Let y represent its position above the earth's surface, as shown in ...

(a) If air resistance is ignored, show that

$$v\,\frac{dv}{dy} = -\frac{GM}{(R+y)^2}.$$

Answer: Follows from $a = \dfrac{dv}{dt} = \dfrac{dv}{dy}\dfrac{dy}{dt} = v\dfrac{dv}{dy}$.

(b) Assuming that $y(0) = 0$ (the object is launched from earth's surface) and $v(0) = v_0$, solve the above equation to show that

$$v^2 = v_0^2 - 2GM\left(\frac{1}{R} - \frac{1}{R+y}\right).$$

Answer:

$$\begin{aligned} v\,dv &= -\frac{GM}{(R+y)^2}\,dy \\ \int_{v_0}^{v} v\,dv &= -\int_0^y \frac{GM}{(R+s)^2}\,ds \\ \frac{1}{2}(v^2 - v_0^2) &= -GM\left(\frac{1}{R} - \frac{1}{R+y}\right) \\ v^2 &= v_0^2 - 2GM\left(\frac{1}{R} - \frac{1}{R+y}\right) \end{aligned}$$

(c) Show that the maximum height reached by the object is given by

$$y = \frac{v_0^2 R}{2GM/R - v_0^2}.$$

Answer: If y is the maximum height, the corresponding velocity is $v = 0$, so from (3.16)

$$0 = v_0^2 - 2GM\left(\frac{1}{R} - \frac{1}{R+y}\right).$$

This equation can be solved for y,

$$\begin{aligned}\frac{2GM}{R+y} &= \frac{2GM}{R} - v_0^2 \\ \frac{R+y}{2GM} &= \frac{1}{2GM/R - v_0^2} \\ y &= \frac{2GM}{2GM/R - v_0^2} - R \\ &= \frac{Rv_0^2}{2GM/R - v_0^2}.\end{aligned}$$

(d) Show that the initial velocity

$$v_0 = \sqrt{\frac{2GM}{R}}$$

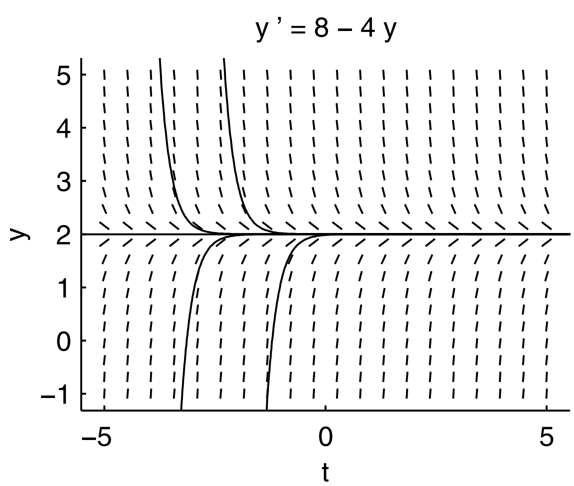
is the minimum required for the object to "escape" earth's gravitational field. *Hint:* If an object "escapes" earth's gravitational field, then the maximum height acquired by the object is potentially infinite.

Answer: If $v_0 < \sqrt{2GM/R}$, then $v_0^2 < 2GM/R$, and $2GM/R - v_0^2 > 0$. Hence by (c) the object has a finite maximum height and does not escape. However, when $v_0 = \sqrt{2GM/R}$, $2GM/R - v_0^2 = 0$, and there is no maximum height.

M2.6.

Answer:

(a)



(b) As $t \to \infty$ all solutions tend to 2.

(c) The equation is both separable and linear. Separating the variables we have

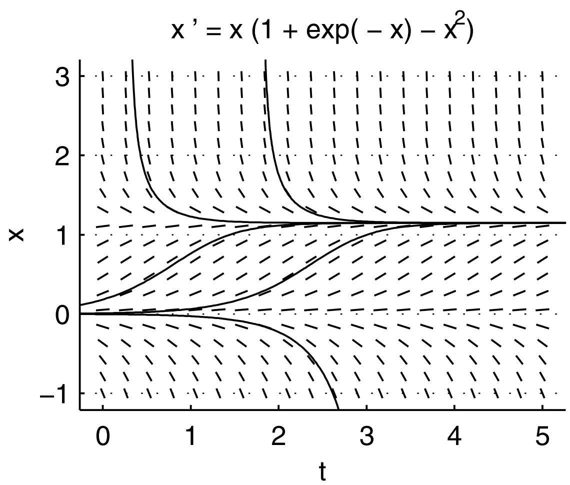
$$\begin{aligned} \frac{dy}{dt} &= 8 - 4y \\ \int \frac{dy}{y-2} &= \int -2\,dt \\ \ln(|y-2|) &= -2t + C \\ y(t) &= 2 + Ae^{-2t} \end{aligned}$$

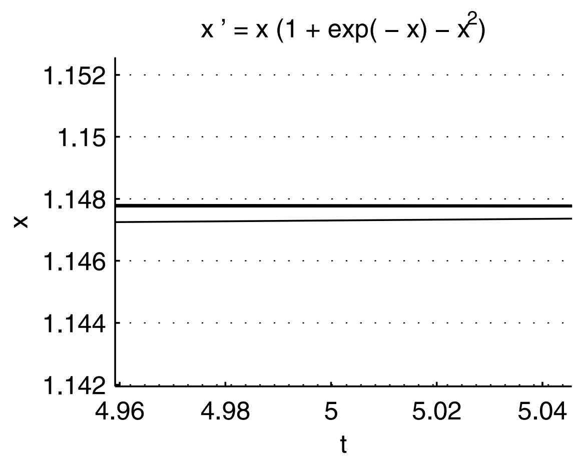
(d) Since the exponential tends to 0, the general solution tends to 2.

M2.17.

Answer:

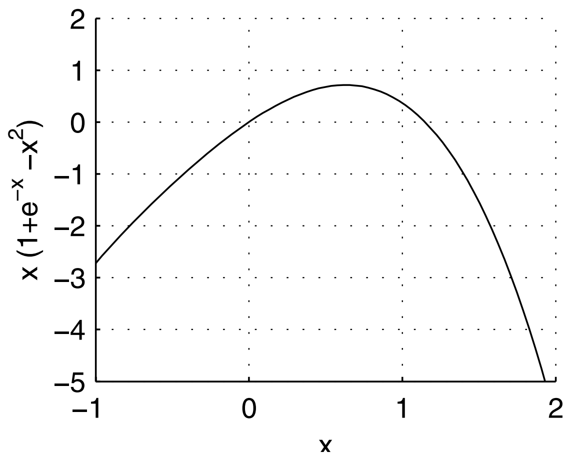
(i & iv)





The left hand side is 0 when $x = 0$, and also when $x = x_1 \approx 1.1476$, as we see from the second `dfield5` figure on the right, which is a "zoom in" from the one on the left. Using `fzero` as indicated we find that $x_1 \approx 1.1478$.

(ii)



The graph of $f(x) = x(1 + e^{-x} - x^2)$ is shown above. $f(x) > 0$ for $0 < x < x_1$, so any solution $x(t)$ is increasing in that range, and any solution outside of this interval is decreasing.

(v) The equilibrium point at 0 is unstable, while x_1 is stable.