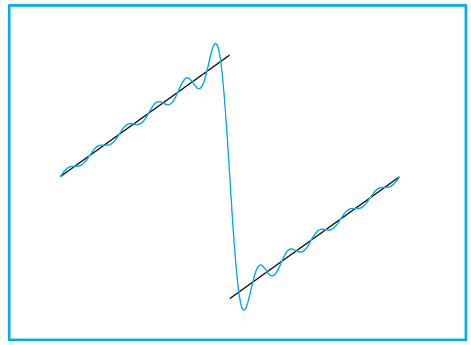
CHAPTER 12



Fourier Series

In 1807, the French mathematician and physicist Joseph Fourier submitted a paper on heat conduction to the Academy of Sciences of Paris. In this paper Fourier made the claim that any function $f(x)$ can be expanded into an infinite sum of trigonometric functions,

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k \cos(kx) + b_k \sin(kx)], \quad \text{for } -\pi \le x \le \pi.$$

The paper was rejected after it was read by some of the leading mathematicians of his day. They objected to the fact that Fourier had not presented much in the way of proof for this statement, and most of them did not believe it.

In spite of its less than glorious start, Fourier's paper was the impetus for major developments in mathematics and in the application of mathematics. His ideas forced mathematicians to come to grips with the definition of a function. This, together with other metamathematical questions, caused nineteenth-century mathematicians to rethink completely the foundations of their subject, and to put it on a more rigorous foundation. Fourier's ideas gave rise to a new part of mathematics, called harmonic analysis or Fourier analysis. This, in turn, fostered the introduction at the end of the nineteenth century of a completely new theory of integration, now called the Lebesgue integral.

The applications of Fourier analysis outside of mathematics continue to multiply. One important application pertains to signal analysis. Here, $f(x)$ could represent the amplitude of a sound wave, such as a musical note, or an electrical signal from a CD player or some other device (in this case x represents time and is usually replaced by t). The Fourier series representation of a signal represents a decomposition of this signal into its various frequency components. The terms $\sin kx$ and $\cos kx$

oscillate with numerical frequency[1] of $k/2\pi$. Signals are often corrupted by noise, which usually involves the high-frequency components (when k is large). Noise can sometimes be filtered out by setting the high-frequency coefficients (the a_k and b_k when k is large) equal to zero.

Data compression is another increasingly important problem. One way to accomplish data compression uses Fourier series. Here the goal is to be able to store or transmit the essential parts of a signal using as few bits of information as possible. The Fourier series approach to the problem is to store (or transmit) only those a_k and b_k that are larger than some specified tolerance and discard the rest. Fortunately, an important theorem (the Riemann–Lebesgue Lemma, which is our Theorem 2.10) assures us that only a small number of Fourier coefficients are significant, and therefore the aforementioned approach can lead to significant data compression.

12.1 Computation of Fourier Series

The problem that we wish to address is the one faced by Fourier. Suppose that $f(x)$ is a given function on the interval $[-\pi, \pi]$. Can we find coefficients, a_n and b_n, so that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx], \quad \text{for } -\pi \le x \le \pi? \tag{1.1}$$

Notice that, except for the term $a_0/2$, the series is an infinite linear combination of the basic terms $\sin nx$ and $\cos nx$ for n a positive integer. These functions are periodic with period $2\pi/n$, so their graphs trace through n periods over the interval $[-\pi, \pi]$. Figure 1 shows the graphs of $\cos x$ and $\cos 5x$, and Figure 2 shows the graphs of $\sin x$ and $\sin 5x$. Notice how the functions become more oscillatory as n increases.

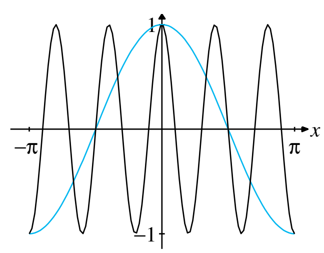


Figure 1 The graphs of $\cos x$ and $\cos 5x$.

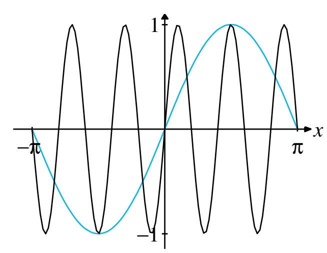


Figure 2 The graphs of $\sin x$ and $\sin 5x$.

The orthogonality relations

Our task of finding the coefficients a_n and b_n for which (1.1) is true is facilitated by the following lemma. These orthogonality relations are one of the keys to the whole theory of Fourier series.

[1] Be sure you know the difference between angular frequency, k in this case, and numerical frequency. It is explained in Section 4.1.

LEMMA 1.2 Let p and q be positive integers. Then we have the following ***orthogonality relations***.

$$\int_{-\pi}^{\pi} \sin px\,dx = \int_{-\pi}^{\pi} \cos px\,dx = 0 \tag{1.3}$$

$$\int_{-\pi}^{\pi} \sin px \cos qx\,dx = 0 \tag{1.4}$$

$$\int_{-\pi}^{\pi} \cos px \cos qx\,dx = \begin{cases} \pi, & \text{if } p = q \\ 0, & \text{if } p \neq q \end{cases} \tag{1.5}$$

$$\int_{-\pi}^{\pi} \sin px \sin qx\,dx = \begin{cases} \pi, & \text{if } p = q \\ 0, & \text{if } p \neq q \end{cases} \tag{1.6}$$

We will leave the proof of these identities for the exercises.

Computation of the coefficients

The orthogonality relations enable us to find the coefficients a_n and b_n in (1.1). Suppose we are given a function f that can be expressed as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx] \tag{1.7}$$

on the interval $[-\pi, \pi]$. To find a_0, we simply integrate the series (1.7) term by term. Using the orthogonality relation (1.3), we see that

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0\pi. \tag{1.8}$$

To find a_n for $n \geq 1$, we multiply both sides of (1.7) by $\cos nx$ and integrate term by term, getting

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos nx\,dx &= \int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos kx + b_k \sin kx] \right) \cos nx\,dx \\ &= \frac{a_0}{2} \int_{-\pi}^{\pi} \cos nx\,dx \\ &\quad + \sum_{k=1}^{\infty} a_k \int_{-\pi}^{\pi} \cos kx \cos nx\,dx \\ &\quad + \sum_{k=1}^{\infty} b_k \int_{-\pi}^{\pi} \sin kx \cos nx\,dx. \end{aligned} \tag{1.9}$$

Using the orthogonality relations in Lemma 1.2, we see that all the terms on the right-hand side of (1.9) are equal to zero, except for

$$a_n \int_{-\pi}^{\pi} \cos nx \cos nx\,dx = a_n \cdot \pi.$$

Hence, equation (1.9) becomes

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx = a_n \cdot \pi, \quad \text{for } n \geq 1,$$

so, including equation (1.8), [2]

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad \text{for } n \geq 0. \tag{1.10}$$

To find b_n, we multiply equation (1.7) by $\sin nx$ and then integrate. By reasoning similar to the computation of a_n, we obtain

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad \text{for } n \geq 1. \tag{1.11}$$

Definition of Fourier series

If f is a piecewise continuous function on the interval $[-\pi, \pi]$, we can compute the coefficients a_n and b_n using (1.10) and (1.11). Thus we can define the Fourier series for any such function.

DEFINITION 1.12 Suppose that f is a piecewise continuous function on the interval $[-\pi, \pi]$. With the coefficients computed using (1.10) and (1.11), we define the ***Fourier series associated to*** f by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]. \tag{1.13}$$

The finite sum

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}[a_n \cos nx + b_n \sin nx] \tag{1.14}$$

is called the ***partial sum of order*** N for the Fourier series in (1.13). We say that the Fourier series converges at x if the sequence of partial sums converges at x as $N \to \infty$. We use the symbol $\sim$ in (1.13) because we cannot be sure that the series converges. We will explore the question of convergence in the next section, and we will see in Theorem 2.3 that for functions that are minimally well behaved, the $\sim$ can be replaced by an equals sign for most values of x.

EXAMPLE 1.15 ◆ Find the Fourier series associated with the function

$$f(x) = \begin{cases} 0, & \text{for } -\pi \leq x < 0, \\ \pi - x, & \text{for } 0 \leq x \leq \pi. \end{cases}$$

[2] We used the expression $a_0/2$ instead of a_0 for the constant term in the Fourier series (1.7) so formulas like equation (1.10) would be true for $n = 0$ as well as for larger n.

We compute the coefficient a_0 using (1.8) or (1.10). We have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} (\pi - x)\,dx = \frac{\pi}{2}.$$

For $n \geq 1$, we use (1.10), and integrate by parts to get

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx \\ &= \frac{1}{n\pi}\int_0^{\pi}(\pi - x)\,d\sin nx \\ &= \frac{1}{n\pi}(\pi - x)\sin nx\bigg|_0^{\pi} + \frac{1}{n\pi}\int_0^{\pi}\sin nx\,dx \\ &= \frac{1}{n^2\pi}(1 - \cos n\pi). \end{aligned}$$

Thus, since $\cos n\pi = (-1)^n$, the even numbered coefficients are $a_{2n} = 0$, and the odd numbered coefficients are $a_{2n+1} = 2/[\pi(2n+1)^2]$ for $n \geq 0$.

We compute b_n using (1.11). Again we integrate by parts to get

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx \\ &= -\frac{1}{n\pi}\int_0^{\pi}(\pi - x)\,d\cos nx \\ &= -\frac{1}{n\pi}(\pi - x)\cos nx\bigg|_0^{\pi} - \frac{1}{n\pi}\int_0^{\pi}\cos nx\,dx \\ &= \frac{1}{n}. \end{aligned}$$

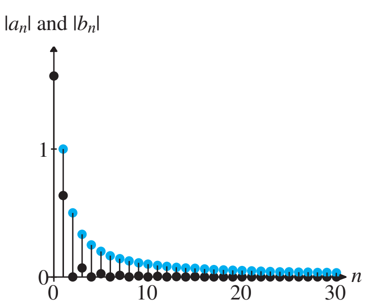


Figure 3 The Fourier coefficients for the function in Example 1.15.

The magnitude of the coefficients is plotted in Figure 3, with $|a_n|$ in black and $|b_n|$ in blue. Notice how the coefficients decay to 0. The Fourier series for f is

$$f(x) \sim \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2} + \sum_{n=1}^{\infty}\frac{\sin nx}{n}. \tag{1.16}$$

◆

Let's examine the experimental evidence for convergence of the Fourier series in Example 1.15. The partial sums of orders 3, 30, and 300 for the Fourier series in Example 1.15 are shown in Figures 4, 5, and 6, respectively. In these figures the function f is plotted in black and the partial sum in blue. The evidence of these figures is that the Fourier series converges to $f(x)$, at least away from the discontinuity of the function at $x = 0$.

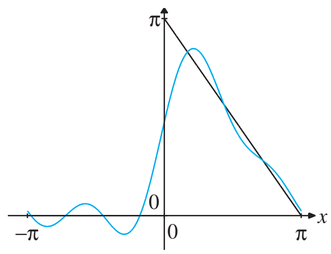


Figure 4 The partial sum of order 3 for the function in Example 1.15.

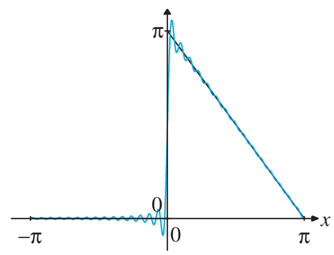


Figure 5 The partial sum of order 30 for the function in Example 1.15.

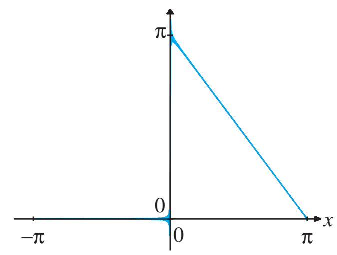


Figure 6 The partial sum of order 300 for the function in Example 1.15.

Fourier series on a more general interval

It is very natural to consider functions defined on $[-\pi, \pi]$ when studying Fourier series because in applications the argument x is frequently an angle. However, in other applications (such as heat transfer and the vibrating string) the argument represents a length. In such a case it is more natural to assume that x is in an interval of the form $[-L, L]$. It is a matter of a simple change of variable to go from $[-\pi, \pi]$ to a more general integral.

Suppose that $f(x)$ is defined for $-L \le x \le L$. Then the function $F(y) = f(Ly/\pi)$ is defined for $-\pi \le y \le \pi$. For F we have the Fourier series defined in Definition 1.20. Using the formula $y = \pi x/L$, the coefficients a_n are given by

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(y)\cos ny\,dy \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{Ly}{\pi}\right)\cos ny\,dy \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx. \end{aligned}$$

The formula for b_n is derived similarly. Thus equations (1.10) and (1.11) are the special case for $L = \pi$ of the following more general result.

THEOREM 1.17 If $f(x) = a_0/2 + \sum_{n=1}^{\infty}[a_n\cos(n\pi x/L) + b_n\sin(n\pi x/L)]$ for $-L \le x \le L$, then

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad \text{for } n \ge 0, \tag{1.18}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad \text{for } n \ge 1. \tag{1.19}$$

Keep in mind that Theorem 1.17 only shows that *if* f can be expressed as a Fourier series, then the coefficients a_n and b_n must be given by the formulas in (1.18) and (1.19). The theorem does not say that an arbitrary function can be expanded into a convergent Fourier series.

The special case when $n = 0$ in (1.18) deserves special attention. Since $\cos 0 = 1$, it says

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx.$$

Thus $a_0/2$ is the average of f over the interval $[-L, L]$.

We will also extend Definition 1.12 to functions defined on the interval $[-L, L]$.

DEFINITION 1.20 Suppose that f is a piecewise continuous function on the interval $[-L, L]$. With the coefficients computed using (1.18) and (1.19), we define the ***Fourier series associated to*** f by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]. \tag{1.21}$$

Even and odd functions

The computation of the Fourier coefficients can often be facilitated by taking note of the symmetries of the function f.

DEFINITION 1.22 A function $f(x)$ defined on an interval $-L \le x \le L$ is said to be ***even*** if $f(-x) = f(x)$ for $-L \le x \le L$, and ***odd*** if $f(-x) = -f(x)$ for $-L \le x \le L$.

The graph of an even function is symmetric about the y-axis as shown in Figure 7. Examples include $f(x) = x^2$ and $f(x) = \cos x$. The graph of an odd function is symmetric about the origin as shown in Figure 8. Examples include $f(x) = x^3$ and $f(x) = \sin x$.

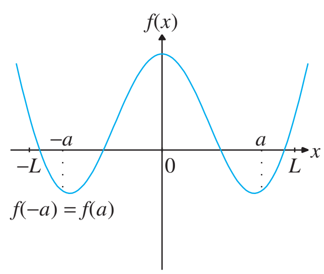


Figure 7 The graph of an even function.

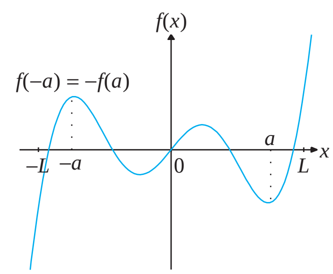


Figure 8 The graph of an odd function.

The following properties follow from the definition.

PROPOSITION 1.23 Suppose that f and g are defined on the interval $-L \le x \le L$.

1. If both f and g are even, then fg is even.

2. If both f and g are odd, then fg is even.
3. If f is even and g is odd, then fg is odd.
4. If f is even, then

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.$$

5. If f is odd, then

$$\int_{-L}^{L} f(x)\,dx = 0.$$

We will leave the proof for the exercises. If we remember that the integral of f computes the algebraic area under the graph of f, parts 4 and 5 of Proposition 1.23 can be seen in Figures 7 and 8.

The Fourier coefficients of even and odd functions

Parts 4 and 5 of Proposition 1.23 simplify the computation of the Fourier coefficients of a function that is either even or odd. For example, if f is even, then, since $\sin(n\pi x/L)$ is odd, $f(x)\sin(n\pi x/L)$ is odd by part 3 of Proposition 1.23, and by part 5,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 0.$$

Consequently, no computations are necessary to find b_n. Using similar reasoning, we see that $f(x)\cos(n\pi x/L)$ is even, and therefore

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx.$$

Frequently integrating from 0 to L is simpler than integrating from $-L$ to L.

Just the opposite occurs for an odd function. In this case the a_n are zero and the b_n can be expressed as an integral from 0 to L. We will leave this as an exercise. We summarize the preceding discussion in the following theorem.

THEOREM 1.24 Suppose that f is piecewise continuous on the interval $[-L, L]$.

1. If $f(x)$ is an even function, then its associated Fourier series will involve only the cosine terms. That is, $f(x) \sim a_0/2 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L)$ with

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad \text{for } n \geq 0.$$

2. If $f(x)$ is an odd function, then its associated Fourier series will involve only the sine terms. That is, $f(x) \sim \sum_{n=1}^{\infty} b_n \sin(n\pi x/L)$ with

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad \text{for } n \geq 1.$$

Let's look at another example of a Fourier series.

EXAMPLE 1.25 ◆ Find the Fourier series associated to the function $f(x) = x$ on $-\pi \le x \le \pi$.

The function f is odd, so according to Theorem 1.24 its Fourier series will involve only the sine terms. The coefficients are

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin(nx)\, dx.$$

Using integration by parts, we obtain

$$b_n = \frac{2}{\pi}\left(\frac{-\pi \cos n\pi}{n} + \frac{1}{n}\int_0^{\pi} \cos n\pi x\, dx\right) = 2\frac{(-1)^{n+1}}{n}.$$

Thus, the Fourier series of $f(x) = x$ on the interval $[-\pi, \pi]$ is

$$f(x) \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx. \tag{1.26}$$

◆

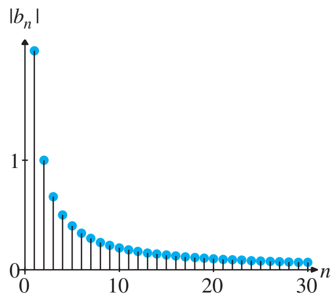


Figure 9 The Fourier coefficients for the function in Example 1.25.

The magnitude of the coefficients is plotted in Figure 9. The a_n are not shown, since they are all equal to 0. The partial sums of orders 5, 11, and 51 for the Fourier series in (1.26) are shown in Figures 10, 11, and 12 respectively. In these figures the function $f(x) = x$ is plotted in black and the partial sum in blue. These figures provide evidence that the Fourier series converges to $f(x)$, at least on the open interval $(-\pi, \pi)$. At $x = \pm\pi$ every term in the series is equal to 0. Therefore the series converges to 0 at $\pm\pi$, but not to $f(\pm\pi) = \pm\pi$.

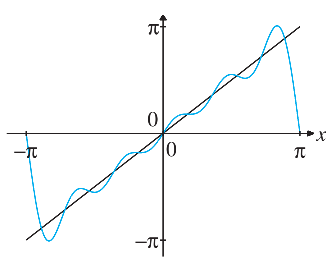


Figure 10 The partial sum of order 5 for $f(x) = x$.

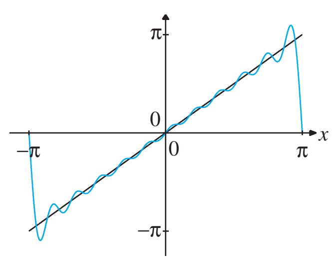


Figure 11 The partial sum of order 11 for $f(x) = x$.

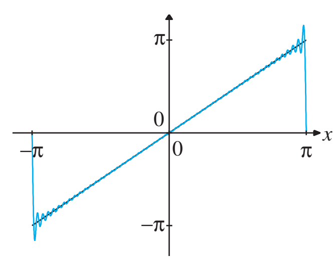


Figure 12 The partial sum of order 51 for $f(x) = x$.

EXAMPLE 1.27 ◆ Compute the Fourier series for the saw-tooth wave f graphed in Figure 13 on the interval $[-1, 1]$.

The graph in Figure 13 on the interval $[-1, 1]$ consists of two lines with slope $+2$ and -2 respectively. The formula for f on the interval $-1 \le x \le 1$ is given by

$$f(x) = \begin{cases} 1 + 2x, & \text{if } -1 \le x \le 0, \\ 1 - 2x, & \text{if } 0 \le x \le 1. \end{cases}$$

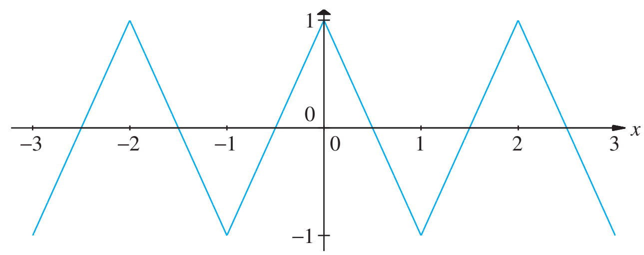


Figure 13 A saw-tooth shaped wave.

The function f is even and is periodic with period 2.

Since f is an even function, we see using Theorem 1.24 that only the cosine terms appear in the Fourier series, and the coefficients are given by

$$a_n = 2\int_0^1 (1-2x)\cos(n\pi x)\,dx, \quad \text{for } n \geq 0.$$

For $n = 0$ we can compute the integral by observation,

$$a_0 = 2\int_0^1 (1-2x)\,dx = 0.$$

For $n > 0$, we use integration by parts to obtain

$$a_n = 2\int_0^1 (1-2x)\cos(n\pi x)\,dx = \frac{4}{n^2\pi^2}(1-\cos n\pi).$$

Since $\cos n\pi = (-1)^n$, we see that

$$a_{2n} = 0 \quad \text{and} \quad a_{2n+1} = \frac{8}{(2n+1)^2\pi^2}, \quad \text{for } n \geq 0.$$

Thus we have

$$f(x) \sim \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos((2n+1)\pi x). \qquad ◆$$

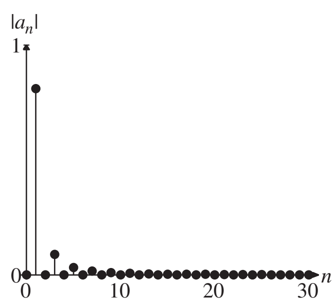


Figure 14 The Fourier coefficients for the function in Example 1.27.

The magnitude of the coefficients is plotted in Figure 14. The b_n are not shown, since they are all equal to 0. Notice how fast the coefficients decay to 0, in comparison to those in Figures 3 and 9. The graph of the partial sum of order 3,

$$S_3(x) = \frac{8}{\pi^2}\left[\cos\pi x + \frac{1}{9}\cos 3\pi x\right],$$

is shown in Figure 15. The sum of these two terms gives a pretty accurate approximation of the saw-tooth wave. This reflects the fact that the coefficients decay rapidly, as shown in Figure 14. The partial sum of order 9 is plotted in Figure 16. Notice that the poorest approximation occurs at the "corners" of the graph of the saw-tooth. This is where the function fails to be differentiable, and these facts are connected.

EXAMPLE 1.28 ◆ Find the Fourier series of the function $f(x) = \sin 3x + 2\cos 4x$ on the interval $[-\pi, \pi]$.

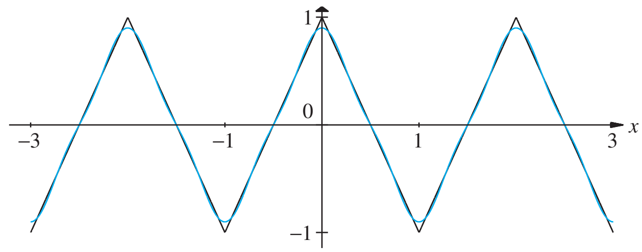


Figure 15 The partial sum of order 3 for the saw-tooth function.

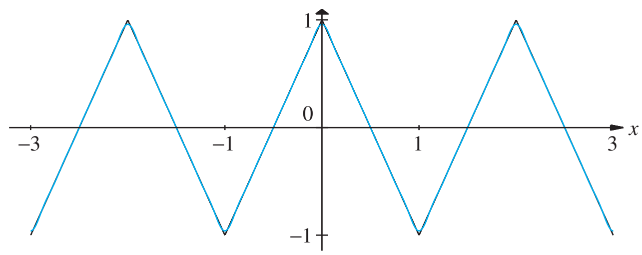


Figure 16 The partial sum of order 9 for the saw-tooth function.

Since f is already given as a sum of sines and cosines, no work is needed. The Fourier series of f is just $\sin 3x + 2\cos 4x$. This example illustrates an important point. According to Theorem 1.17, the Fourier coefficients of a function are uniquely determined by the function. Thus, by inspection, $b_3 = 1$, $a_4 = 2$ and all other coefficients are equal to 0. By uniqueness, these are the same values as would have been obtained by computing the integrals in Theorem 1.17 for the a_n and b_n. ◆

EXAMPLE 1.29 ◆ Find the Fourier series of the function $f(x) = \sin^2 x$ on the interval $[-\pi, \pi]$.

In this example, f is not explicitly given as a linear combination of sines and cosines, so there is some work to do. However, if we use the trigonometric identity

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x),$$

the right side is the desired Fourier series, since it is a finite linear combination of terms of the form $\cos nx$. ◆

EXERCISES

In Exercises 1–6 expand the given function in a Fourier series valid on the interval $-\pi \le x \le \pi$. Plot the function and two partial sums of your choice over the interval $-\pi \le x \le \pi$. Plot the same partial sums over the interval $-3\pi \le x \le 3\pi$.

1. $f(x) = |\sin x|$

2. $f(x) = |x|$

3. $f(x) = \begin{cases} 0, & -\pi \le x < 0, \\ x, & 0 \le x \le \pi. \end{cases}$

4. $f(x) = \begin{cases} 0, & -\pi \le x < 0, \\ \sin x, & 0 \le x \le \pi. \end{cases}$

5. $f(x) = x \cos x$

6. $f(x) = x \sin x$

In Exercises 7–16 find the Fourier series for the indicated function on the indicated interval. Plot the function and two partial sums of your choice over the interval.

7. $f(x) = \begin{cases} 1 + x, & \text{for } -1 \le x \le 0, \\ 1, & \text{for } 0 < x \le 1 \end{cases}$ on $[-1, 1]$

8. $f(x) = 4 - x^2$ on $[-2, 2]$

9. $f(x) = x^3$ on $[-1, 1]$

10. $f(x) = \sin x \cos^2 x$ on $[-\pi, \pi]$

11. $f(x) = \begin{cases} 0, & \text{for } -1 \le x \le 0, \\ x^2, & \text{for } 0 < x \le 1 \end{cases}$ on $[-1, 1]$

12. $f(x) = \begin{cases} \sin \pi x/2, & \text{for } -2 \le x \le 0, \\ 0, & \text{for } 0 < x \le 2 \end{cases}$ on $[-2, 2]$

13. $f(x) = \begin{cases} \cos \pi x, & \text{for } -1 \le x \le 0, \\ 1, & \text{for } 0 < x \le 1 \end{cases}$ on $[-1, 1]$

14. $f(x) = \begin{cases} 1 + x, & \text{for } -1 \le x \le 0, \\ 1 - x, & \text{for } 0 < x \le 1 \end{cases}$ on $[-1, 1]$

15. $f(x) = \begin{cases} 2 + x, & \text{for } -2 \le x \le 0, \\ -2 + x, & \text{for } 0 < x \le 2 \end{cases}$ on $[-2, 2]$

16. $f(x) = \begin{cases} 2, & \text{for } -2 \le x \le 0, \\ 2 - x, & \text{for } 0 < x \le 2 \end{cases}$ on $[-2, 2]$

17. Expand the function $f(x) = x^2$ in a Fourier series valid on the interval $-\pi \le x \le \pi$. Plot both f and the partial sum S_N for $N = 1, 3, 5, 7$. Observe how the graphs of the partial sums approximate the graph of f. Plot the same graphs over the interval $-2\pi \le x \le 2\pi$.

18. Expand the function $f(x) = x^2$ in a Fourier series valid on the interval $-1 \le x \le 1$. Plot both f and the partial sum S_N for $N = 1, 3, 5, 7$. Observe how the graphs of the partial sums approximate the graph of f. Plot the same graphs over the interval $-2 \le x \le 2$.

In Exercises 19–22 determine if the function f is even, odd, or neither.

19. $f(x) = |\sin x|$

20. $f(x) = x + 3x^3$

21. $f(x) = e^x$

22. $f(x) = x + x^2$

23. Use the addition formulas for sin and cos to show that

$$\begin{aligned} \cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)], \\ \sin\alpha\sin\beta &= \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)], \\ \sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]. \end{aligned}$$

24. Prove Lemma 1.2. *Hint:* Use Exercise 23.

25. Complete the derivation of equation 1.11 for the coefficient b_n.

26. Prove parts 1, 2, and 3 of Proposition 1.23.

27. Prove parts 4 and 5 of Proposition 1.23.

28. Prove part 2 of Theorem 1.24.

29. From Theorem 1.24, the Fourier series of an odd function consists only of sine-terms. What additional symmetry conditions on f will imply that the sine coefficients with even indices will be zero? Give an example of a function satisfying this additional condition.

30. Suppose that f is a function which is periodic with period T and differentiable. Show that f' is also periodic with period T.

31. Suppose that f is a function defined on $\mathbf{R}$. Show that there is an odd function f_{odd} and an even function f_{even} such that $f(x) = f_{\text{odd}}(x) + f_{\text{even}}(x)$ for all x.

12.2 Convergence of Fourier Series

Suppose that f is a piecewise continuous function on the interval $[-L, L]$, and that

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right] \tag{2.1}$$

is its associated Fourier series. Two questions arise immediately whenever an infinite series is encountered. The first question is, does the series converge? The second question arises if the series converges. Can we identify the sum of the series? In particular, does the Fourier series of a function f converge at x to $f(x)$ or to something else? These are the questions we will address in this section.[3]

[3] Theorem 1.17 does not answer this question, since it assumes that $f(x)$ equals its Fourier series and then describes what the Fourier coefficients have to be.

Fourier Series and periodic functions

The partial sums of the Fourier series in (2.1) have the form

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]. \tag{2.2}$$

The function $S_N(x)$ is a finite linear combination of the trigonometric functions $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$, each of which is periodic with period $2L$.[4] Hence for every N the partial sum S_N is a function that is periodic with period $2L$. Consequently, if the partial sums converge at each point x, the limit function must also be periodic with period $2L$.

Let's consider again the function $f(x) = x$, which we treated in Example 1.25. $f(x)$ is defined for all real numbers x, and it is not periodic. We found that its Fourier series on the interval $[-\pi, \pi]$ is

$$2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx.$$

The partial sums of this series are all periodic with period 2π. Therefore, if the Fourier series converges, the limit function will be periodic with period 2π. Thus the limit cannot be equal to $f(x) = x$ everywhere. The evidence from the graphs of the partial sums in Figures 10, 11, and 12 of the previous section indicates that the series does converge to $f(x) = x$ on the interval $(-\pi, \pi)$. Since the limit must be 2π-periodic, we expect that the limit is closely related to the periodic extension of $f(x) = x$ from the interval $(-\pi, \pi)$. The situation is illustrated in Figure 1, which shows the partial sum of order 5 over 3 periods. The periodic extension of $f(x) = x$ is shown plotted in black.

Since it will appear repeatedly, let's denote the ***periodic extension*** of a function $f(x)$ defined on an interval $[-L, L]$ by $f_p(x)$. Usually it is easier to understand the periodic extension of a function graphically than it is to give an understandable formula for it. This is illustrated for $f(x) = x$ in Figure 1. However, for the record, the formula for the periodic extension[5] is

$$f_p(x) = f(x - 2kL) \quad \text{for } (2k-1)L < x \le (2k+1)L.$$

You will notice that f_p is periodic with period $2L$.

To get a feeling for whether or not the Fourier series of $f(x) = x$ converges to its periodic extension f_p, we graph both f_p and the partial sum of order 21 in Figure 2. Note that the graph of S_{21} (the blue curve) is close to the graph of f_p except at the points of discontinuity of f_p, which occur at the odd multiples of π. The accuracy of the approximation of $f_p(x)$ by $S_{21}(x)$ gets worse as x gets closer

---

[4] We studied periodic functions in Section 5.5, but let's refresh our memory. A function $g(x)$ is ***periodic*** with period T if $g(x+T) = g(x)$ for all x. Notice that every integral multiple of a period is also a period. The smallest period of $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$ is $2L/n$, so $n\cdot(2L/n) = 2L$ is also a period. Thus each function in the partial sum S_N in (2.2) is periodic with period $2L$.

[5] Notice we use less than ($<$) at the lower endpoint of each interval and less than or equal ($\le$) at the upper endpoint. A choice is necessary to avoid having two values at the endpoints. This is not the only possible choice, but it is as good as any.

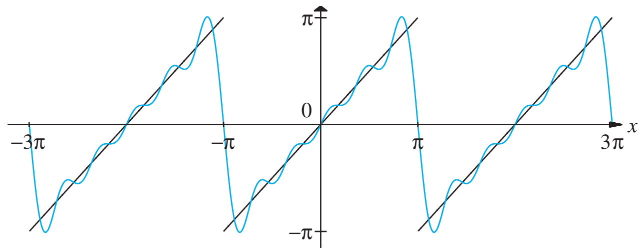


Figure 1 The partial sum of order 5 for the series in Example 1.25 over three periods.

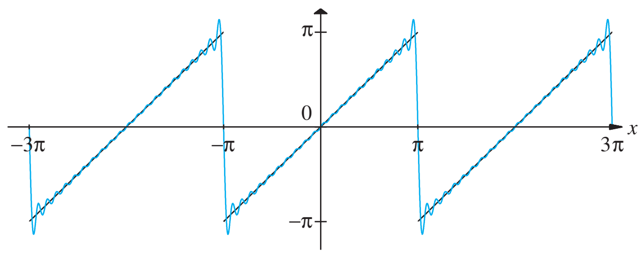


Figure 2 The partial sum of order 21 for the series in Example 1.25 over three periods.

to a point of discontinuity. This is necessary, simply because each partial sum is a continuous function, while f_p is not. Furthermore, we see that $S_N(\pi) = 0$ for all N. Hence the Fourier series converges to 0 at π, and not to $f_p(\pi) = \pi$. The same phenomenon occurs at $x = (2k+1)\pi$ for any integer k, and these are the points of discontinuity of f_p.

We will see these considerations reflected in our convergence theorem.

Piecewise continuous functions

Suppose that f is a function defined in an interval I. We define the ***right-hand and left-hand limits*** of f at a point x_0 to be

$$f(x_0^+) = \lim_{x \to x_0^+} f(x) \quad \text{and} \quad f(x_0^-) = \lim_{x \to x_0^-} f(x).$$

The function f is continuous at x_0 if and only if both limits exist and $f(x_0^+) = f(x_0^-) = f(x_0)$.

In Section 5.1, we defined a function f to be ***piecewise continuous*** if it has only finitely many points of discontinuity in any finite interval, and if both the left- and right-hand limits exist at every point of discontinuity. Thus, for a piecewise continuous function the left- and right-hand limits exist everywhere.

You will notice that the periodic extension f_p of $f(x) = x$ on the interval $[-\pi, \pi]$ that we saw in Example 1.25 is piecewise continuous. The points of discontinuity for f_p are at $(2k+1)\pi$, the odd integral multiples of π. We have $f_p([(2k+1)\pi]^+) = -\pi$ and $f_p([(2k+1)\pi]^-) = \pi$. In fact, if f is any function that is continuous on the interval $[-L, L]$, then the periodic extension f_p is piecewise continuous on all of **R**, and its only possible points of discontinuity are the odd multiples of L.

We will say that the function f has ***left-hand derivative*** at x_0 if the left-hand limit $f(x_0^-)$ exists and the limit

$$\lim_{x \to x_0^-} \frac{f(x) - f(x_0^-)}{x - x_0})$$

exists. Similarly, we will say that f has ***right-hand derivative*** at x_0 if $f(x_0^+)$ exists and the limit

$$\lim_{x \to x_0^+} \frac{f(x) - f(x_0^+)}{x - x_0}$$

exists. If f is differentiable at x_0, then f has left- and right-hand derivatives there and both are equal to $f'(x_0)$.

For an example, we consider again the periodic extension f_p of $f(x) = x$ on the interval $[-\pi, \pi]$. The left- and right-hand derivatives exist at every point and are equal to 1, even at the points of discontinuity. Another example is the saw-tooth wave in Example 1.27. Notice that the saw-tooth function is continuous everywhere, but fails to be differentiable where x is an integer. However, at these points the left- and right-hand derivatives of the saw-tooth wave exist.

Convergence

Since a Fourier series converges to a periodic function, we may as well assume from the beginning that the function f is already periodic. If, as was the case in Example 1.25, we are given a function that is not periodic, then it is necessary to look at the periodic extension of the function.

We are now in a position to state our main theorem on the convergence of Fourier series. A proof is beyond the scope of this text, but one can be found in any advanced book on Fourier series.

THEOREM 2.3 Suppose $f(x)$ is a piecewise continuous function that is periodic with period $2L$. If the left- and right-hand derivatives of f exist at x_0 then the Fourier series for f converges at x_0 to

$$\frac{f(x_0^+) + f(x_0^-)}{2}.$$

If f is continuous at x_0, then $f(x_0^+) = f(x_0^-) = f(x_0)$, so assuming that the left- and right-hand derivatives of f exist at x_0, Theorem 2.3 concludes that the Fourier series converges at x_0 to $f(x_0)$.

Theorem 2.3 assumes the existence of the left- and right-hand derivatives of f *only at* the point x_0, and concludes that the Fourier series converges *only at* x_0. This indicates that it is only the smoothness of f near x_0 that affects the convergence there. However, in most of the examples that we will consider, the left- and right-hand derivatives will exist everywhere. We will consider this special case in the next corollary.

COROLLARY 2.4 Suppose $f(x)$ is a piecewise continuous function that is periodic with period $2L$. Suppose in addition that f has left- and right-hand derivatives at every point.

1. At every point x_0 where f is continuous the Fourier series for f converges to $f(x_0)$.
2. At every point x_0 where f is not continuous the Fourier series for f converges to

$$\frac{f(x_0^+) + f(x_0^-)}{2}. \tag{2.5}$$

EXAMPLE 2.6 ◆ We have verified that the hypothesis of Corollary 2.4 holds for the periodic extension f_p of $f(x) = x$ on $-\pi \le x \le \pi$. Show that the conclusion of Corollary 2.4 holds at any point of discontinuity.

The points of discontinuity are $(2k+1)\pi$, the odd integral multiples of π. We have $f_p([(2k+1)\pi]^+) = -\pi$ and $f_p([(2k+1)\pi]^-) = \pi$. Therefore,

$$\frac{f_p([(2k+1)\pi]^+) + f_p([(2k+1)\pi]^-)}{2} = 0.$$

We have also seen that the Fourier series of f is

$$2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

When $x = (2k+1)\pi$ every term in the series is equal to 0. Hence the series converges to 0, so the conclusion of Theorem 2.5 is valid at $x = (2k+1)\pi$. ◆

EXAMPLE 2.7 ◆ Let $f(x) = x^2$ on the interval $-1 \le x \le 1$. Without computing the Fourier coefficients, explicitly describe the sum of the Fourier series of f for all x.[6]

Of course, $f(x) = x^2$ is not periodic. Therefore, we must consider its periodic extension, f_p, graphed in black in Figure 3. Note that f_p is continuous everywhere, and it is differentiable except at the odd integers, $x = 2k+1$. At these points the left- and right-hand derivatives exist. Thus the left- and right-hand derivatives exist everywhere, and Corollary 2.4 implies that its Fourier series converges to $f_p(x)$ for all x. ◆

EXAMPLE 2.8 ◆ Consider the function

$$f(x) = \begin{cases} -1, & \text{for } -1 \le x \le 0, \\ 0, & \text{for } x = 0, \\ 1, & \text{for } 0 < x \le 1. \end{cases}$$

Find the Fourier series for f, and describe the sum of its Fourier series.

[6] The computation of this series is Exercise 18 in Section 1.

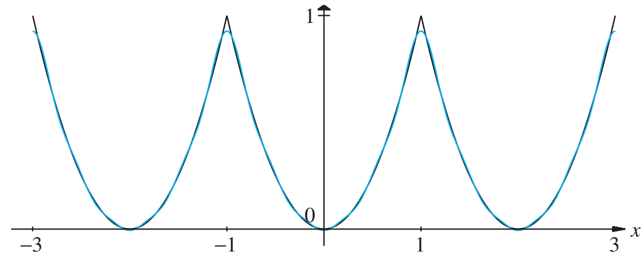


Figure 3 The partial sum $S_5(x)$ for the function in Example 2.7.

Since f is only defined over the interval $[-1, 1]$, we are really looking at the Fourier series of its periodic extension f_p, shown plotted in black in Figure 4. For obvious reasons, f_p is called the ***square wave***. Since f (and f_p) is an odd function, Theorem 1.24 says that only the sine terms are present in the Fourier series, and that the coefficients are

$$b_n = 2\int_0^1 f(x)\sin n\pi x\,dx = 2\int_0^1 \sin n\pi x\,dx = -\frac{2}{n\pi}[(-1)^n - 1].$$

Hence

$$b_{2n} = 0 \quad \text{and} \quad b_{2n+1} = \frac{4}{(2n+1)\pi}.$$

The Fourier series associated to f_p (and to f) is

$$f_p(x) \sim \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(2n+1)\pi x. \tag{2.9}$$

The function f_p is piecewise continuous, with discontinuities at all of the integers. In addition, its left- and right-hand derivatives exist everywhere. Thus, the Fourier series converges to $f_p(x)$ if x is not an integer. If $x = k$ is an integer, the series converges to

$$\frac{f_p(k^+) + f_p(k^-)}{2} = \frac{1 + (-1)}{2} = 0.$$

In fact, each term of the Fourier series in (2.9) is equal to 0 when $x = k$ is an integer. The partial sum of the Fourier series of order 11 is shown plotted in Figure 4. ◆

Gibb's phenomenon

Suppose that the piecewise continuous function f has a discontinuity at x_0, but that the left- and right-hand derivatives of f exist at x_0. As Theorem 2.3 points out, the Fourier series of f converges at x_0 to $[f(x_0^+) + f(x_0^-)]/2$. However, if you look closely at the graphs of the partial sums near the points of discontinuity in Figures 1, 2, and 4, we see that the graph of the partial sum overreaches the graph of the function on each side of the discontinuity. This effect is called ***Gibb's phenomenon***.

To examine Gibb's phenomenon a little more deeply, let's look at the graphs of some high order partial sums for the square wave. The partial sums S_{301} and S_{601}

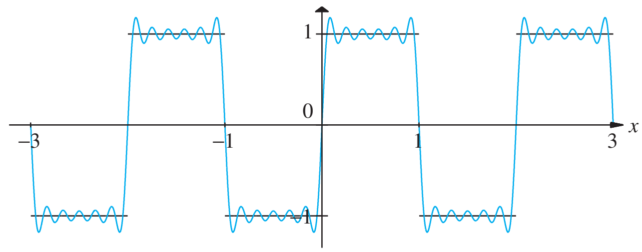


Figure 4 The partial sum $S_{11}(x)$ for the square wave in Example 2.8.

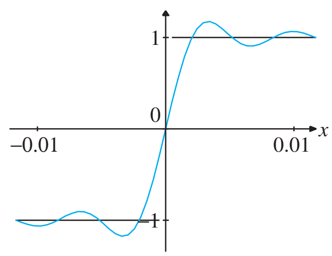


Figure 5 The partial sum $S_{301}(x)$ for the square wave.

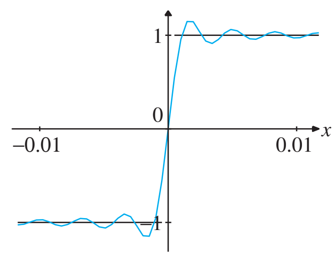


Figure 6 The partial sum $S_{601}(x)$ for the square wave.

are plotted in Figures 5 and 6, but only for $-0.01 \le x \le 0.01$, so that we may see the overrun clearly. Notice that both partial sums display the overrun that is characteristic of Gibb's phenomenon. In the two cases the amount of the overrun is approximately the same, but for the higher order sum the duration is smaller.

It can be proved that whenever a function f satisfies the hypotheses of Theorem 2.3, but has a discontinuity at x_0, the graphs of the partial sums of the Fourier series display Gibb's phenomenon near x_0. Furthermore, the ratio of the length of the interval between the upper peak and the lower peak of the partial sum to $|f(x_0^+) - f(x_0^-)|$ is approximately 1.179 in every case.

The Riemann–Lebesgue Lemma

Notice that in every example we have considered, the Fourier coefficients approach 0 as the frequency gets large. This is demonstrated, in particular, in Figures 3, 9, and 14 in Section 1. These examples are typical of the behavior of Fourier coefficients, as the next theorem, known as the Riemann-Lebesgue lemma, shows.

THEOREM 2.10 Suppose f is a piecewise continuous function on the interval $a \le x \le b$. Then

$$\lim_{k\to\infty} \int_a^b f(x)\cos kx\, dx = \lim_{k\to\infty} \int_a^b f(x)\sin kx\, dx = 0.$$

As we will see in Section 5, this theorem has important applications. Basically, this theorem states that any given signal can be approximated very well by a few dominant Fourier coefficients because most of the Fourier coefficients are near zero.

The intuitive reason behind this theorem is that as k gets very large, $\sin kx$ and $\cos kx$ oscillate much more rapidly than does f (see Figures 1 and 2 in Section 1). If k is large, $f(x)$ is nearly constant on two adjacent periods of $\sin kx$ or $\cos kx$. The integral over each period is almost zero, since the areas above and below the x-axis almost cancel.

Interpretation of the Fourier coefficients

Suppose that f is a function that satisfies the hypotheses of Corollary 2.4. Then $f(x)$ is equal to its Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right], \tag{2.11}$$

except at those points where f is not continuous. Let's look more closely at the nth summand, which we can rewrite in terms of its amplitude and phase[7] as

$$f_n(x) = a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) = A_n \cos\left(\frac{n\pi x}{L} - \phi_n\right), \tag{2.12}$$

where $A_n = \sqrt{a_n^2 + b_n^2}$ and $\tan\phi_n = b_n/a_n$. We see that $f_n(x)$, defined in (2.12), is an oscillation with amplitude A_n and frequency[8] $\omega_n = n\pi/L$. We will call f_n the ***component of*** f ***at frequency*** $\omega_n = n\pi/L$. Notice that $\omega_n = n\omega_1$, where $\omega_1 = \pi/L$, so all of the frequencies are integer multiples of the ***fundamental frequency*** ω_1.

We can interpret Corollary 2.4 as saying that any function that satisfies its hypotheses is an infinite linear combination of oscillatory components at frequencies that are integer multiples of the fundamental frequency. The component of f at frequency ω_n has amplitude $A_n = \sqrt{a_n^2 + b_n^2}$. The amplitude is a numerical measure of the importance of the component in the Fourier expansion. By the Riemann–Lebesgue lemma, the Fourier coefficients decay to 0 as n increases, so the amplitudes A_n do as well. As a result, the components at the smaller frequencies dominate the Fourier series in (2.11). This fact is illustrated by the plots of the magnitudes of the coefficients in Figures 3, 9, and 14 in Section 1.

Fourier coefficients for periodic functions

In this section we have been looking at periodic functions, since it is only such functions that can be the sums of Fourier series. It is worth pointing out that for a periodic function with period $2L$, the coefficients can be computed by an integral over any interval of length $2L$. More precisely, we have the following:

[7] See Section 4.4.

[8] This is an angular frequency. Remember that we are using angular frequencies instead of numerical frequencies unless otherwise stated.

PROPOSITION 2.13 Suppose that f is a piecewise continuous function that is periodic with period $2L$. Then for any c the Fourier coefficients for f are given by

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad \text{for } n \geq 0,$$
$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx, \quad \text{for } n \geq 1.$$

We will leave the proof to the exercises.

EXERCISES

In Exercises 1–6 determine if the function f is periodic or not. If it is periodic, find the smallest positive period.

1. $f(x) = |\sin x|$

2. $f(x) = \cos 3\pi x$

3. $f(x) = x$

4. $f(x) = \sin(x) + \cos(x/2)$

5. $f(x) = x^2$

6. $f(x) = e^x$

In Exercises 7–14 find the sum of the Fourier series for indicated function at every point in **R** without computing the series. Each of these is an exercise in Section 1. Although that is not very important, the reference is included in parentheses.

7. $f(x) = \begin{cases} 0, & -\pi \leq x < 0, \\ x, & 0 \leq x \leq \pi \end{cases}$ on $[-\pi, \pi]$ (See Exercise 1.3)

8. $f(x) = \begin{cases} 0, & -\pi \leq x < 0, \\ \sin x, & 0 \leq x \leq \pi \end{cases}$ on $[-\pi, \pi]$ (See Exercise 1.4)

9. $f(x) = \begin{cases} 1 + x, & \text{for } -1 \leq x \leq 0 \\ 1, & \text{for } 0 < x \leq 1 \end{cases}$ on $[-1, 1]$ (See Exercise 1.7)

10. $f(x) = 4 - x^2$ on $[-2, 2]$ (See Exercise 1.8)

11. $f(x) = x^3$ on $[-1, 1]$ (See Exercise 1.9)

12. $f(x) = \begin{cases} 0, & \text{for } -1 \leq x \leq 0, \\ x^2, & \text{for } 0 < x \leq 1 \end{cases}$ on $[-1, 1]$ (See Exercise 1.11)

13. $f(x) = \begin{cases} \sin \pi x/2, & \text{for } -2 \leq x \leq 0, \\ 0, & \text{for } 0 < x \leq 2 \end{cases}$ on $[-2, 2]$ (See Exercise 1.12)

14. $f(x) = \begin{cases} 2, & \text{for } -2 \le x \le 0, \\ 2 - x, & \text{for } 0 < x \le 2 \end{cases}$ on $[-2, 2]$ (See Exercise 1.16)

15. Compute the Fourier series for the function $f(x) = |x|$ on the interval $[-\pi, \pi]$. (See Exercise 1.2.) Use the result and Theorem 2.3 to show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

16. Compute the Fourier series for the function $f(x) = x^2$ on the interval $[-\pi, \pi]$. (See Example 2.7.) Use the result and Theorem 2.3 to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

17. Compute the Fourier series for the function $f(x) = x^4$ on the interval $[-\pi, \pi]$. Use the result, Theorem 2.3, and Exercise 16 to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{120} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

18. Expand the function

$$f(x) = \begin{cases} 0, & -1 < x \le -1/2, \\ 1, & -1/2 < x \le 1/2, \\ 0, & 1/2 < x \le 1, \end{cases}$$

in a Fourier series valid on the interval $-1 \le x \le 1$. Plot the graph of f and the partial sums of order N for $N = 5,\ 10,\ 20,$ and 40, as in Exercise 17 in Section 1. Notice how much slower the series converges to f in this example than in Exercise 17 in Section 1. What accounts for the slow rate of convergence in this example?

19. Expand the function $f(x) = e^{rx}$ in a Fourier series valid for $-\pi \le x \le \pi$. For the case $r = 1/2$, plot the partial sums of orders $N = 10$, 20, and 30 of the Fourier series along with the graph of f_p over the intervals $-\pi \le x \le \pi$ and $-2\pi \le x \le 2\pi$.

20. Use the previous exercise to compute the Fourier coefficients for the function $f(x) = \sinh x = (e^x - e^{-x})/2$ and $f(x) = \cosh(x) = (e^x + e^{-x})/2$ over the interval $-\pi \le x \le \pi$.

21. Use Theorem 2.3 to determine the sum of the Fourier series of the function f defined in Exercise 18 for each x in the interval $-1 \le x \le 1$.

22. Suppose that f is periodic with period T and differntiable. Show that f' is also periodic with period T.

23. Suppose that f is periodic with period T. Show that

$$\int_b^{b+T} f(x)\,dx = \int_a^{a+T} f(x)\,dx$$

for any a and b. Use this result to prove Proposition 2.13.

24. Suppose that f is periodic with period T. Define

$$F(x) = \int_0^x f(y)\,dy.$$

Show that if $\int_0^T f(y)\,dy = 0$, then F is periodic with period T. (*Hint:* Use Exercise 23.)

12.3 Fourier Cosine and Sine Series

In this section we will examine the possibility of finding Fourier series of the forms

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad \text{for } 0 \le x \le L,$$

and

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad \text{for } 0 \le x \le L.$$

The basic idea behind our method comes from Theorem 1.24.

Fourier cosine series

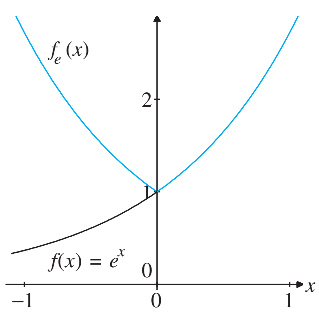


Figure 1 The even extension of $f(x) = e^x$.

According to Theorem 1.24, the Fourier series of an even function contains only cosine terms. If the function $f(x)$ is defined only for $0 \le x \le L$, we extend it to $-L \le x \le 0$ as an even function. The ***even extension*** of f is defined by

$$f_e(x) = \begin{cases} f(x), & \text{if } 0 \le x \le L, \\ f(-x), & \text{if } -L \le x < 0. \end{cases}$$

For the function $f(x) = e^x$ on the interval $[0, 1]$, the even extension f_e is plotted in blue in Figure 1.

Since the function f_e is an even function defined on $[-L, L]$, Theorem 1.24 tells us that its Fourier series has the form

$$f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad \text{for } -L \le x \le L, \tag{3.1}$$

where

$$a_n = \frac{2}{L}\int_0^L f_e(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad \text{for } n \ge 0.$$

Since $f_e(x) = f(x)$ for $0 \le x \le L$, this formula becomes

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad \text{for } n \ge 0. \tag{3.2}$$

Furthermore, if we restrict ourselves to the interval $[0, L]$, where $f_e(x) = f(x)$, we can write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \quad \text{for } 0 \le x \le L, \tag{3.3}$$

with the coefficients given by (3.2). The series in (3.3), with the coefficients given in (3.2), is called the ***Fourier cosine series*** for f on the interval $[0, L]$.

EXAMPLE 3.4 ◆ Find the Fourier cosine series for $f(x) = e^x$ on the interval $[0, 1]$.

The coefficients in (3.2) become

$$a_n = 2\int_0^1 e^x \cos n\pi x\, dx = \frac{2}{1+n^2\pi^2}\left[(-1)^n e - 1\right].$$

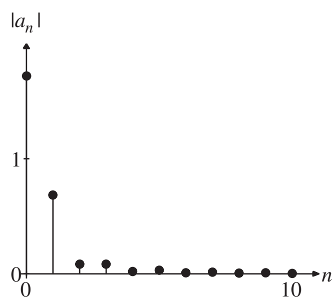


Figure 2 The Fourier cosine coefficients for $f(x) = e^x$.

This evaluation can be done by direct computation, by looking the integral up in an integral table, or by using a computer and a symbolic algebra program. The magnitude of these coefficients is plotted in Figure 2. Notice how quickly the coefficients decay to 0. The Fourier series is

$$\begin{aligned} e^x &\sim (e-1) + 2\sum_{n=1}^{\infty}\frac{(-1)^n e - 1}{1+n^2\pi^2}\cos n\pi x \\ &\sim (e-1) - \frac{2(e+1)}{1+\pi^2}\cos \pi x + \frac{2(e-1)}{1+4\pi^2}\cos 2\pi x + \dots \end{aligned} \tag{3.5}$$

on the interval $[0, 1]$.

The partial sum $S_3(x)$ is plotted in blue in Figure 3. The black curve in Figure 3 is the periodic extension of the even extension of the function $f(x) = e^x$. We will call this the ***even periodic extension*** of f, and we will denote it by f_{ep}. Since, in this case, f_{ep} is continuous and satisfies the hypotheses of Corollary 2.4, the Fourier series converges everywhere to $f_{ep}(x)$. ◆

Fourier sine series

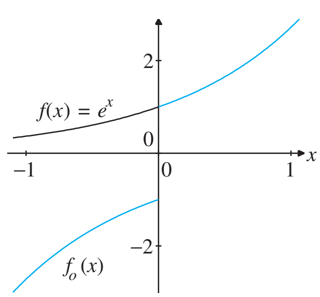


Figure 4 The odd extension of $f(x) = e^x$.

In a similar manner, a function f can be expanded in a series which involves only sine terms. Again motivated by Theorem 1.24, we consider the ***odd extension of*** f, which is defined by

$$f_o(x) = \begin{cases} f(x), & \text{if } 0 < x \le L, \\ 0, & \text{if } x = 0, \\ -f(-x), & \text{if } -L \le x < 0. \end{cases}$$

The odd extension of $f(x) = e^x$ is plotted in blue in Figure 4. Since the function f_o

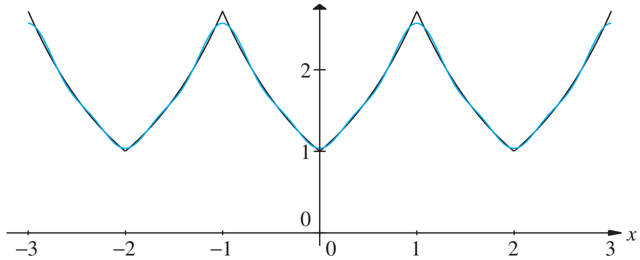


Figure 3 The partial sum S_3 of the Fourier cosine series for $f(x) = e^x$ plotted over three periods.

is an odd function defined on $[-L, L]$, Theorem 1.24 tells us that its Fourier series has only sine terms. Proceeding as before, we find that

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx, \quad \text{for } 0 < x \le L, \tag{3.6}$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad \text{for } n \ge 1. \tag{3.7}$$

The series in (3.6), with the coefficients given in (3.7), is called the ***Fourier sine series*** for f on the interval $[0, L]$.

EXAMPLE 3.8 ◆ Find the Fourier sine series for $f(x) = e^x$ on the interval $[0, 1]$.

The coefficients in (3.7) become

$$b_n = 2\int_0^1 e^x \sin n\pi x \, dx = \frac{2n\pi[1 - (-1)^n e]}{1 + n^2\pi^2}.$$

This evaluation can be done by direct computation, by looking the integral up in an integral table, or by using a computer and a symbolic algebra program. Thus we have

$$\begin{aligned} e^x &\sim \sum_{n=1}^{\infty} \frac{2n\pi[1-(-1)^n e]}{1+n^2\pi^2} \sin n\pi x \\ &\sim \frac{2\pi(e+1)}{1+\pi^2} \sin \pi x - \frac{4\pi(e-1)}{1+4\pi^2} \sin 2\pi x + \frac{6\pi(e+1)}{1+9\pi^2} \sin 3\pi x + \dots \end{aligned} \tag{3.9}$$

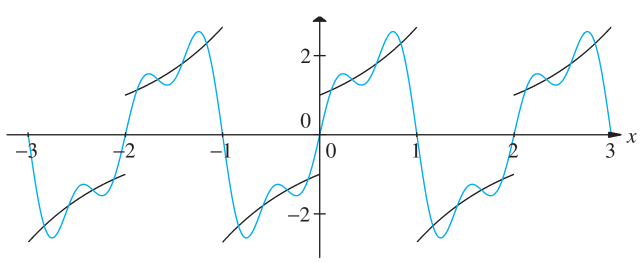


Figure 6 The partial sum S_3 of the Fourier sine series for $f(x) = e^x$ plotted over three periods.

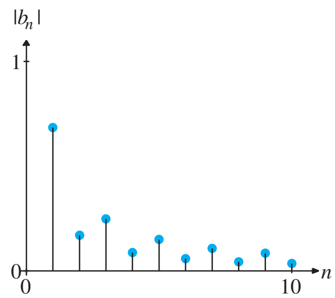


Figure 5 The Fourier sine coefficients for $f(x) = e^x$.

on the interval [0, 1]. The magnitude of the coefficients is plotted in Figure 5. Notice that the sine coefficients do not decay nearly as rapidly as do the cosine coefficients in Figure 2. The partial sum of order 3 is plotted in blue in Figure 6. It is interesting to compare this figure with Figure 3. The black curve in Figure 6 is the ***odd periodic extension*** of $f(x) = e^x$, which we denote by f_{op}. In this case f_{op} satisfies the hypotheses of Corollary 2.4 and fails to be continuous only at the integers. Consequently, the Fourier sine series converges to $f_{op}(x)$ everywhere except at the integers. ◆

Note that while we used the even and odd extensions of f (f_e and f_o) to help derive the cosine and sine expansions, the formulas for a_n and b_n involve only the function f on the interval $[0, L]$. This is reflected by the fact that the cosine and sine expansions converge to f only on $(0, L)$. Outside this interval the cosine and sine expansions converge to f_{ep} and f_{op}, respectively. Examples 3.4 and 3.8 illustrate these facts, but another example might help to put things into perspective.

EXAMPLE 3.10 ◆ Find the complete Fourier series for $f(x) = e^x$ on the interval $[-1, 1]$.

From (1.10) we have

$$a_n = \int_{-1}^{1} e^x \cos n\pi x \, dx = (-1)^n \frac{e - 1/e}{1 + n^2\pi^2}, \quad \text{for } n \geq 0,$$

while from (1.11) we have

$$b_n = \int_{-1}^{1} e^x \sin n\pi x \, dx = (-1)^{n+1} \frac{n\pi(e - 1/e)}{1 + n^2\pi^2}, \quad \text{for } n \geq 1.$$

Again there are several ways to verify this. Hence the complete Fourier series for e^x on $[-1, 1]$ is

$$e^x = \left(e - \frac{1}{e}\right) \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1 + n^2\pi^2} [\cos n\pi x - n\pi \sin n\pi x] \right\}. \tag{3.11}$$

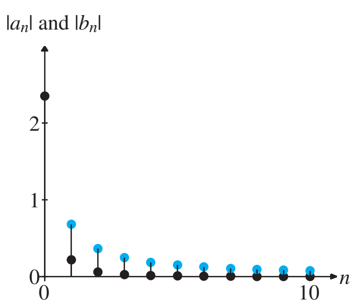


Figure 7 The coefficients of the complete Fourier series for $f(x) = e^x$.

The magnitude of the coefficients is plotted in Figure 7, with $|a_n|$ in black and $|b_n|$ in blue. The partial sum of order 3 is plotted in blue Figure 8.

The periodic extension of e^x satisfies the hypotheses of Corollary 2.4 and fails to be continuous only at the odd integers. Consequently, the Fourier series converges to the periodic extension everywhere except at the odd integers. ◆

Now we have three Fourier series that converge to $f(x) = e^x$ on the interval $(0, 1)$. The first in (3.5) contains only cosine terms. The second in (3.9) contains only sine terms. The third in (3.11) contains both sine and cosine terms. It is interesting to compare graphs of the partial sums in Figures 3, 6, and 8. The difference between the three is what happens outside of the interval $(0, 1)$. The cosine series converges to f_{ep}, the even periodic extension of f. The sine series converges to f_{op}, the odd periodic extension of f, except at the integers. And, finally, the full Fourier series converges to f_p, the periodic extension of f, except at the odd integers.

Of course, the same three series can be considered for any piecewise continuous function defined on an interval of the form $[-L, L]$.

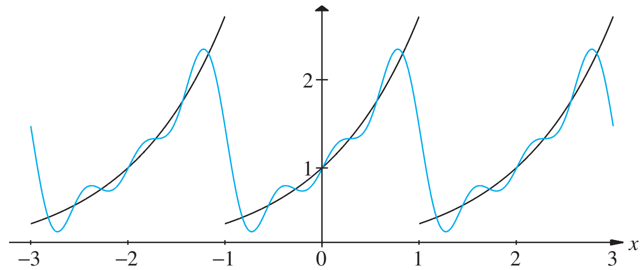


Figure 8 The partial sum S_3 of the complete Fourier series for $f(x) = e^x$ plotted over three periods.

EXERCISES

In exercises 1–4 sketch the graph of the odd extension of the function f. Also sketch the graph of the odd periodic extension with period 2 over three periods.

1. $f(x) = 1 - x$

2. $f(x) = 1 - 2x$

3. $f(x) = x^2 - 1$

4. $f(x) = x^2 - 2$

In exercises 5–8 sketch the graph of the even extension of the function f. Also sketch the graph of the even periodic extension with period 2 over three periods.

5. $f(x) = 1 - x$

6. $f(x) = 1 - 2x$

7. $f(x) = x^2 - 1$

8. $f(x) = x^2 - 2$

In Exercises 9 – 20 expand the given function in a Fourier cosine series valid on the interval $0 \le x \le \pi$. Plot the function and two partial sums of your choice over the interval $0 \le x \le \pi$. Plot the same partial sums and the function the series converges to over the interval $-3\pi \le x \le 3\pi$.

9. $f(x) = x$

10. $f(x) = \sin x$

11. $f(x) = \cos x$

12. $f(x) = 1$

13. $f(x) = \pi - x$

14. $f(x) = x^2$

15. $f(x) = x^3$

16. $f(x) = x^4$

17. $f(x) = \begin{cases} 1, & 0 \le x < \pi/2, \\ 0, & \pi/2 \le x \le \pi \end{cases}$

18. $f(x) = \begin{cases} x, & 0 \le x < \pi/2, \\ \pi/2, & \pi/2 \le x \le \pi \end{cases}$

19. $f(x) = x \cos x$

20. $f(x) = x \sin x$

In Exercises 21 – 32 expand the given function in a Fourier sine series valid on the interval $0 \le x \le \pi$. Plot the function and two partial sums of your choice over the interval $0 \le x \le \pi$. Plot the same partial sums and the function the series converges to over the interval $-3\pi \le x \le 3\pi$.

21. Same as Exercise 9

22. Same as Exercise 10

23. Same as Exercise 11

24. Same as Exercise 12

25. Same as Exercise 13

26. Same as Exercise 14

27. Same as Exercise 15

28. Same as Exercise 16

29. Same as Exercise 17

30. Same as Exercise 18

31. Same as Exercise 19

32. Same as Exercise 20

33. Show that the functions $\cos(n\pi x/L)$, $n = 0, 1, 2, \ldots$ are orthogonal on the interval $[0, L]$. This means that

$$\int_0^L \cos(n\pi x/L)\, \cos(p\pi x/L)\, dx = 0, \quad \text{if } p \ne n \quad .$$

Hint: Use Exercise 23.

34. Show that the functions $\sin(n\pi x/L)$, $n = 1, 2, 3, \ldots$ are orthogonal on the interval $[0, L]$. This means that

$$\int_0^L \sin(n\pi x/L)\, \sin(p\pi x/L)\, dx = 0, \quad \text{if } p \ne n \quad .$$

Hint: Use Exercise 23.

35. Show that

$$\int_0^1 \cos(2n\pi x) \sin(2k\pi x)\, dx = 0.$$

Hint: Use Exercise 23.

36. If $f(x)$ is continuous on the interval $0 \le x \le L$, show that its even periodic extension is continuous everywhere. Does this statement hold for the odd periodic extension? What additional condition(s) is (are) necessary to ensure that the odd periodic extension is everywhere continuous?

12.4 The Complex Form of a Fourier Series

If the piecewise continuous function f is periodic with period $2L$, then its Fourier series is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right], \tag{4.1}$$

where the coefficients are given by

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx, \quad \text{for } n \ge 0, \text{ and} \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx, \quad \text{for } n \ge 1. \end{aligned} \tag{4.2}$$

Sometimes it is useful to express the Fourier series in complex form using the complex exponentials, e^{inx} for $n = 0, \pm 1, \pm 2, \ldots$. This is possible because of the close connection between the complex exponentials and the trigonometric functions. We explored this connection in the appendix to this book and in Section 4.3. The most important facts to know about the complex exponential are Euler's formula

$$e^{iy} = \cos y + i \sin y, \tag{4.3}$$

which defines the exponential, and that all of the familiar properties of the real exponential remain true for the complex exponential.

If we write down Euler's formula with y replaced by $-y$, we get

$$e^{-iy} = \cos y - i \sin y. \tag{4.4}$$

Solving (4.3) and (4.4) for $\cos y$ and $\sin y$, we see that

$$\cos y = \frac{e^{iy} + e^{-iy}}{2} \quad \text{and} \quad \sin y = \frac{e^{iy} - e^{-iy}}{2i}. \tag{4.5}$$

Let's substitute these expressions into the Fourier series (4.1). The nth term in the sum is the component of f at the frequency $\omega_n = n\pi/L$, and it becomes

$$\begin{aligned} f_n(x) &= a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \\ &= \frac{a_n}{2}\left(e^{in\pi x/L} + e^{-in\pi x/L}\right) + \frac{b_n}{2i}\left(e^{in\pi x/L} - e^{-in\pi x/L}\right) \\ &= \frac{a_n - ib_n}{2} e^{in\pi x/L} + \frac{a_n + ib_n}{2} e^{-in\pi x/L} \\ &= \alpha_n e^{in\pi x/L} + \alpha_{-n} e^{-in\pi x/L}, \end{aligned} \tag{4.6}$$

where we have substituted

$$\alpha_n = \frac{a_n - ib_n}{2} \quad \text{and} \quad \alpha_{-n} = \frac{a_n + ib_n}{2}, \quad \text{for } n \geq 1. \tag{4.7}$$

We will also write the constant term as $f_0(x) = \alpha_0 = a_0/2$. Separating the positive and negative terms, the Fourier series can be written as

$$f(x) \sim \sum_{n=-\infty}^{\infty} \alpha_n e^{in\pi x/L}. \tag{4.8}$$

Notice that by (4.6), the component of f at frequency $\omega_n = n\pi/L$ is given by $f_n(x) = \alpha_n e^{i\omega_n x} + \alpha_{-n} e^{-i\omega_n x}$. As a result, when we talk in terms of low frequency components we have to consider the coefficients α_n and α_{-n} for small values of n.

We can use (4.2) to express the coefficients α_n in terms of the function f. For example, for $n \geq 1$ we have

$$\begin{aligned}\alpha_n &= \frac{a_n - ib_n}{2} \\ &= \frac{1}{2L}\int_{-L}^{L} f(x)\left[\cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right)\right] dx \\ &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\, dx.\end{aligned}$$

The corresponding formulas for $n = 0$ and for $n < 0$ can be computed in the same way, and we discover that

$$\alpha_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\, dx, \quad \text{for all } n. \tag{4.9}$$

It is important to notice that while α_n is the coefficient of $e^{in\pi x/L}$ in the Fourier series (4.8), it is $e^{-in\pi x/L}$ which appears in the integral in (4.9).

The series (4.8), with the coefficients computed using (4.9), is called the ***complex Fourier series*** for the function f. There are several differences between the Fourier series involving cosines and sines, given in Definition 1.20, and the Fourier series using complex exponentials presented here. First, the complex Fourier series involves a sum from $n = -\infty$ to $n = \infty$, rather than a sum from $n = 0$ to $n = \infty$. Next, for the complex Fourier series, there is one succinct formula (4.9) for the Fourier coefficients, rather than the two separate formulas for a_n and b_n in (1.18) and (1.19). For this reason, and also because computations using exponentials are easier than those using trigonometric functions, many scientists and engineers prefer to use the complex version of the Fourier series.

EXAMPLE 4.10 ◆ Find the complex Fourier series for the function $f(x) = e^x$ on the interval $[-1, 1]$.

This is the function we examined in Examples 3.4, 3.8, and 3.10. For this function it is much easier to compute the complex Fourier coefficients than the real ones. The nth coefficient is

$$\begin{aligned}\alpha_n &= \frac{1}{2}\int_{-1}^{1} e^x e^{-in\pi x}\,dx \\ &= \frac{1}{2}\int_{-1}^{1} e^{(1-in\pi)x}\,dx \\ &= \frac{1}{2(1-in\pi)}\left[e^{1-in\pi} - e^{-1+in\pi}\right] \\ &= \frac{(-1)^n}{2(1-in\pi)}(e-1/e).\end{aligned}$$

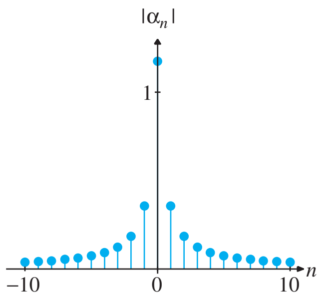


Figure 1 The coefficients of the complex Fourier series for $f(x) = e^x$.

The last identity follows since $e^{in\pi} = e^{-in\pi} = (-1)^n$.

The magnitude of the coefficients is plotted in Figure 1. Notice that we included negative indices. The complex Fourier series is

$$e^x \sim \frac{e-1/e}{2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{1-in\pi}e^{in\pi x} \quad \text{for } -1 \le x \le 1. \qquad ◆$$

Relation between the real and complex Fourier series

We derived the complex Fourier series from the real series. In doing so we found that the complex coefficients can be computed from the real coefficients using (4.7). In turn, we can solve these relationships for the real coefficients in terms of the complex coefficients, getting

$$a_0 = 2\alpha_0, \quad a_n = \alpha_n + \alpha_{-n}, \quad \text{and} \quad b_n = i(\alpha_n - \alpha_{-n}), \quad \text{for } n \ge 1. \tag{4.11}$$

These equations simplify somewhat if the function f is real valued. In that case $\overline{f(x)} = f(x)$, so

$$\begin{aligned}\overline{\alpha_n} &= \overline{\frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx} = \frac{1}{2L}\int_{-L}^{L}\overline{f(x)}\cdot\overline{e^{-in\pi x/L}}\,dx \\ &= \frac{1}{2L}\int_{-L}^{L} f(x)e^{in\pi x/L}\,dx = \alpha_{-n}.\end{aligned}$$

Consequently, if f is real valued,

$$a_n = \alpha_n + \overline{\alpha_n} = 2\,\mathrm{Re}\,\alpha_n \quad \text{and} \quad b_n = i(\alpha_n - \overline{\alpha_n}) = -2\,\mathrm{Im}\,\alpha_n.$$

EXAMPLE 4.12 ◆ Compute the coefficients of the real Fourier series for the function $f(x) = e^x$ on the interval $[-1, 1]$.

We computed the complex coefficients in Example 4.10 and found that

$$\alpha_n = \frac{(-1)^n}{2(1-in\pi)}(e-1/e).$$

Since the function is real valued, we can use (4.11) to find that

$$a_n = 2\,\mathrm{Re}\,\alpha_n = \frac{(-1)^n(e-1/e)}{1+n^2\pi^2} \quad \text{and}$$
$$b_n = -2\,\mathrm{Im}\,\alpha_n = \frac{(-1)^{n+1}n\pi(e-1/e)}{1+n^2\pi^2}.$$

◆

EXERCISES

1. Show that the complex Fourier coefficients for an even, real-valued function are real. Show that the complex Fourier coefficients for an odd, real-valued function are purely imaginary (i.e., their real parts are zero).

In Exercises 2–11 find the complex Fourier series for the given function on the interval $[-\pi, \pi]$.

2. $f(x) = x$

3. $f(x) = |x|$

4. $f(x) = \begin{cases} -1, & -\pi \le x < 0, \\ 1, & 0 \le x \le \pi \end{cases}$

5. $f(x) = \begin{cases} 0, & -\pi \le x < 0, \\ 1, & 0 \le x \le \pi \end{cases}$

6. $f(x) = x^2$

7. $f(x) = e^{bx}$

8. $f(x) = x^3$

9. $f(x) = \pi - x$

10. $f(x) = |\cos x|$

11. $f(x) = |\sin x|$

12. Two complex valued function f and g are said to be orthogonal on the interval $[a, b]$ if $\int_a^b f(x)\overline{g(x)}\,dx = 0$. Show that The functions e^{ipx} and e^{iqx} are orthogonal on $[-\pi, \pi]$ if p and q are different integers.

13. Use the method of proof of Theorem 1.17, and Exercise 12 to show that if $f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{inx}$ for $-\pi \le x \le \pi$, then

$$\alpha_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx.$$

12.5 The Discrete Fourier Transform and the FFT

Suppose that $f(t)$ is piecewise continuous for $0 \le t \le 2\pi$.[9] Then

$$f(t) \sim \sum_{k=-\infty}^{\infty} \alpha_k e^{ikt}, \quad \text{where} \quad \alpha_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}\,dt. \tag{5.1}$$

The Fourier coefficients α_k are often too difficult to compute exactly. In such a case it is useful to approximate the coefficients using a numerical integration technique such as the trapezoid rule.

We remind you that for a function F defined on the interval $[0, 2\pi]$, the trapezoid rule for approximating the integral $\int_0^{2\pi} F(t)\,dt$ with step size $h = 2\pi/N$ is

$$\int_0^{2\pi} F(t)\,dt \approx h\left[\frac{1}{2}F(0) + F(h) + F(2h) + \cdots + F((N-1)h) + \frac{1}{2}F(Nh)\right].$$

If $F(t)$ is 2π-periodic, then $F(0) = F(2\pi) = F(Nh)$, and the preceding formula becomes

$$\int_0^{2\pi} F(t)\,dt \approx h\sum_{j=0}^{N-1} F(jh) = \frac{2\pi}{N}\sum_{j=0}^{N-1} F(2\pi j/N).$$

Applying this formula to the integral for the Fourier coefficient in (5.1), we get

$$\alpha_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}\,dt \approx \frac{1}{N}\sum_{j=0}^{N-1} f(2\pi j/N)e^{-2\pi ijk/N}. \tag{5.2}$$

Let's set

$$y_j = f(2\pi j/N) \quad \text{and} \quad w = e^{2\pi i/N}.$$

Then $\overline{w} = e^{-2\pi i/N}$, and the approximation becomes

$$\alpha_k \approx \frac{1}{N}\sum_{j=0}^{N-1} y_j \overline{w}^{jk}. \tag{5.3}$$

The sum on the right side of equation (5.3) involves the discrete values $y_j = f(2\pi j/N)$. The values of $f(t)$ for $t \ne 2\pi j/N$ are ignored. It is a common occurrence in the digital age to replace a time dependent function with such a discrete sample of that function. For example, $f(t)$ may represent a music signal that we want to transmit over the internet. The internet, or any other computer network, allows only discrete signals, so to transmit the music we replace the continuous signal $f(t)$ with the discrete sample $y_j = f(2\pi j/N)$ for $j = 0, 1, 2, \dots, N-1$.

In many digital applications signals arise that are not represented by a continuous function at all. Instead, they arise as discrete values y_j at a discrete set of times t_j. Such a signal is illustrated in Figure 1. Here, the horizontal axis represents time, which has been divided into many small time intervals.

[9] In most applications of the material in this section, the function f represents a time dependent signal. Consequently, we will use t instead of x as the independent variable.

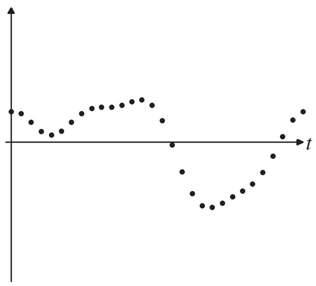


Figure 1 A discrete signal.

The discrete Fourier transform

Even for discrete signals such as that illustrated in Figure 1, it is often useful to consider the transform we found in equation (5.3). We will assume that the signal is an infinite sequence $y = \{y_j | -\infty < j < \infty\}$ that is ***periodic with period** N*, meaning that $y_{N+j} = y_j$ for all j. Wherever the sequence y_k comes from, the transform on the right-hand side of (5.3) is important.

DEFINITION 5.4 Let $y = \{y_j\}$ be a sequence of complex numbers that is periodic with period N. The ***discrete Fourier transform*** of y is the sequence $\hat{y} = \{\hat{y}_k\}$, where

$$\hat{y}_k = \sum_{j=0}^{N-1} y_j e^{-2\pi ikj/N} = \sum_{j=0}^{N-1} y_j \overline{w}^{jk}, \quad \text{for } -\infty < k < \infty. \tag{5.5}$$

For the last expression in (5.5) we use the notation $w = e^{2\pi i/N}$, so that $\overline{w} = e^{-2\pi i/N}$.

An important property of $w = e^{2\pi i/N}$ is that $w^N = e^{2\pi i} = 1$. Of course, it follows that $\overline{w}^N = 1$. From this we see that

$$\hat{y}_{k+N} = \sum_{j=0}^{N-1} y_j \overline{w}^{j(k+N)} = \sum_{j=0}^{N-1} y_j \overline{w}^{jk} \overline{w}^{jN} = \sum_{j=0}^{N-1} y_j \overline{w}^{jk} = \hat{y}_k.$$

Thus the discrete Fourier transform is also periodic of period N.

Let's look back at equation (5.3), where we used the trapezoid rule to approximate α_k, the kth Fourier coefficient of the function f. Using (5.5), we can now write (5.3) as

$$\alpha_k \approx \frac{\hat{y}_k}{N}. \tag{5.6}$$

It follows from the Riemann–Lebesgue lemma that $\alpha_k \to 0$ as $k \to \pm\infty$. On the other hand, the sequence $\hat{y}_k$ is periodic. This implies that (5.3) is not a good approximation for large k. In fact, The trapezoid rule algorithm used to approximate the integral in (5.2) loses accuracy as the integrand $f(t)e^{ikt}$ becomes more oscillatory as the frequency (and index) k increases. Therefore we would expect equation (5.3) to provide a good approximation only for k that are relatively small compared to N.

There is another, related factor to consider. We have previously talked of the importance of the low frequency components of a function. When we use the complex Fourier series, this means that we include both α_k and α_{-k} for small values of k. By (5.6) and the periodicity of the sequence $\hat{y}$,

$$\alpha_{-k} \approx \frac{\hat{y}_{-k}}{N} = \frac{\hat{y}_{N-k}}{N}.$$

Therefore, when considering small frequency components while using the discrete Fourier transform, we must include both $\hat{y}_k$ and $\hat{y}_{N-k}$ for small nonnegative values of the index k.

EXAMPLE 5.7 ◆ Use the discrete Fourier transform to compute approximately the first 64 Fourier coefficients of the function

$$f(t) = e^{-t^2/10}[\sin 2t + 2\cos 4t + 0.4 \sin t \sin 10t]$$

on the interval $[0, 2\pi]$.

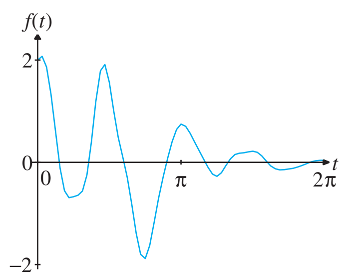


Figure 2 The function in Example 5.7.

The function f is plotted in Figure 2. Because of the terms involving $\sin 2t$ and $\cos 4t$, we would expect that the Fourier coefficients of order 2 and 4 would be large. With $N = 64$, we set $y_j = f(2\pi j/N)$ for $0 \le j \le N-1$. Then we use the fast Fourier transform function in Matlab to compute the discrete Fourier transform $\hat{y}$. The magnitude of the $\hat{y}_k$ is plotted in Figure 3. Indeed, the coefficients corresponding to $k = 4$ and $k = 60 = N-4$ are the largest. Notice how the coefficients with index k and $N-k$ are largest for small k. Thus the coefficients corresponding to small frequency components dominate. ◆

Now let's look at (5.5) and restrict ourselves to $k = 0, 1, \ldots, N-1$. The kth equation expresses $\hat{y}_k$ as a linear combination of $\{y_j \mid 0 \le j \le N-1\}$. These N equations can be expressed as the single matrix equation

$$\begin{pmatrix} \hat{y}_0 \\ \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_{N-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \overline{w} & \overline{w}^2 & \cdots & \overline{w}^{N-1} \\ 1 & \overline{w}^2 & \overline{w}^4 & \cdots & \overline{w}^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \overline{w}^{N-1} & \overline{w}^{2(N-1)} & \cdots & \overline{w}^{(N-1)^2} \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{N-1} \end{pmatrix}. \tag{5.8}$$

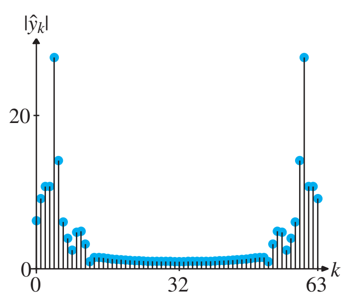


Figure 3 The discrete Fourier transform of the discretization of the function in Example 5.7.

It will be useful to use vector notation. We will set

$$\begin{aligned} \mathbf{y} &= (y_0, y_1, \ldots, y_{N-1})^T, \quad \text{and} \\ \hat{\mathbf{y}} &= (\hat{y}_0, \hat{y}_1, \ldots, \hat{y}_{N-1})^T. \end{aligned}$$

With this notation, equation (5.8) becomes

$$\hat{\mathbf{y}} = F\mathbf{y}, \tag{5.9}$$

where

$$F = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \overline{w} & \overline{w}^2 & \cdots & \overline{w}^{N-1} \\ 1 & \overline{w}^2 & \overline{w}^4 & \cdots & \overline{w}^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \overline{w}^{N-1} & \overline{w}^{2(N-1)} & \cdots & \overline{w}^{(N-1)^2} \end{pmatrix}. \tag{5.10}$$

The inverse discrete Fourier transform

Equation (5.8) gives the formula for computing the discrete Fourier coefficients in terms of the original discrete signal. Many applications require the reverse operation, the computation of the original discrete signal, y_k, from its discrete Fourier coefficients, $\widehat{y}_k$. Therefore, we would like to solve for the y_k in equation (5.8) or, equivalently, we need to find the inverse of the matrix F in (5.10).

Computing the inverse of F is somewhat difficult, so we will simply give the result. Consider the complex conjugate of F

$$\overline{F} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{N-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{N-1} & w^{2(N-1)} & \cdots & w^{(N-1)^2} \end{pmatrix}.$$

Direct computation shows that

$$\overline{F} \cdot F = NI \quad \text{or} \quad F^{-1} = \frac{1}{N}\overline{F}.$$

The computation of $\overline{F} \cdot F$ is not too difficult. For example, when $N = 3$,

$$\overline{F} \cdot F = \begin{pmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w^2 & w^4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \overline{w} & \overline{w}^2 \\ 1 & \overline{w}^2 & \overline{w}^4 \end{pmatrix}.$$

An explicit computation shows that this matrix product is $3I$. For example, the $(2, 1)$-entry of this matrix product is

$$1 + w + w^2 = \frac{1 - w^3}{1 - w} = 0,$$

since $w^3 = 1$. On the other hand, the $(2, 2)$-entry is

$$1 + |w|^2 + |w|^4 = 3.$$

We summarize this discussion in the next theorem.

THEOREM 5.11 The original signal y_j, $j = 0, \ldots, N-1$, can be computed from its discrete Fourier transform, $\widehat{y}_k$, $k = 0, \ldots, N-1$, using

$$y_j = \frac{1}{N}\sum_{k=0}^{N-1} \widehat{y}_k w^{jk}, \quad \text{for } -\infty < j < \infty.$$

We can write this in matrix form as

$$\mathbf{y} = \frac{1}{N}\overline{F}\,\widehat{\mathbf{y}}. \tag{5.12}$$

Noise filtering

Practical applications of this theorem are numerous. We will mention two. The first involves filtering noise from a signal. When a signal is transmitted, it is often corrupted by interference from background radiation or other sources. The corrupted part of the signal is called noise. In many applications, the noise appears with a certain frequency range that is different from the dominant frequencies of the original signal. To filter out noise, a discrete Fourier transform of the signal is computed using (5.8). Then, the Fourier coefficients, $\widehat{y}_k$, corresponding to the noisy, undesirable frequencies are set equal to zero. The signal is then recomputed from the new Fourier coefficients using equation (5.12). Since the frequency components corresponding to the noise have been removed, the resulting signal should contain much less noise than the original.

Frequently, noise occurs at relatively high frequencies. Suppose that we add the noise term $N(t) = 2\sin(50t)$ to the function in Example 5.7. The resulting signal is $g(t) = f(t) + N(t)$, and it is plotted in Figure 4. It is difficult to see that the signal of interest is the function $f(t)$ plotted in Figure 2. We set $y_j = g(2\pi j/N)$ for $0 \le j \le N-1$, with $N = 256$, and take the discrete Fourier transform. The result is plotted in Figure 5. Notice the large terms at $k = 50$ and $k = 206 = N - 50$. To eliminate the high frequency noise, we "zero out" the high frequency coefficients by setting $\widehat{y}_k = 0$ for $13 \le k \le N - 13 = 243$, and compute the inverse transform. The resulting function is plotted in blue in Figure 6, while the original function f is plotted in black. The graphical comparison shows that we have effectively recovered the wanted signal from the noisy one. The most significant difference occurs at the two endpoints. This is a result of Gibb's phenomenon. Since f_p, the periodic extension of f, is not continuous, we have to expect this.

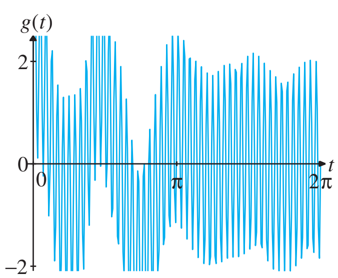


Figure 4 A signal in the presence of high frequency noise.

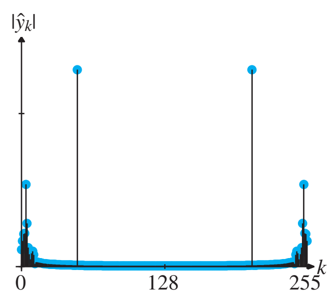


Figure 5 The discrete Fourier transform of the noisy signal.

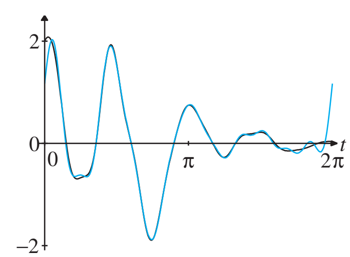


Figure 6 The result of filtering out the high frequencies.

Data compression

A second application involves data compression. The goal is to store or transmit a signal using the fewest possible bits of data. One way to accomplish this is to store or transmit only the dominant Fourier coefficients of a given signal. In view of the Riemann–Lebesgue Lemma, Theorem 2.10, only a finite number of Fourier coefficients are dominant, since these coefficients get very small as the frequency

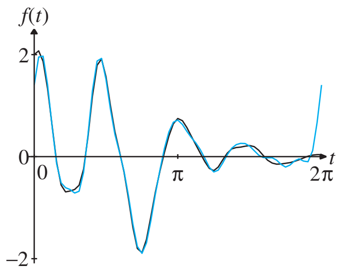


Figure 7 The result of removing small coefficients.

gets large. Thus, a compression routine can be implemented in a three-step process. First, we compute the discrete Fourier coefficients using equation (5.8). Then we set all of the small Fourier coefficients equal to zero, storing only the dominant Fourier coefficients. Finally, to recover the compressed signal, use equation (5.12). What constitutes "small" depends on the application and the tolerance for error. There is a trade-off between the number of Fourier coefficients that are set equal to zero and the accuracy of the compressed signal. The larger the amount of compression, the more coefficients that are set equal to zero, and the greater the difference between the compressed signal and the original signal.

As an example, we set all coefficients for the function in Example 5.7 equal to 0 that were smaller than $1/10$ of the largest coefficient. This resulted in 21 nonzero coefficients. Again we computed the inverse transform, and plotted the result in blue in Figure 7. The function $f(t)$ is plotted in black. The comparison shows that there is loss, but not a great deal.

The fast Fourier transform

Calculating the discrete Fourier transform using equation (5.5) or (5.8) involves lots of computations. Computing each $\widehat{y}_k$ using (5.5) requires the sum of N products of two numbers. We will call the combination of a multiplication and an addition a ***multiply-add***, and we will refer to it as an MA. Thus computing each $\widehat{y}_k$ requires N MAs. Computing the complete discrete Fourier transform means computing $\widehat{y}_k$ for $0 \le k \le N-1$. This requires N^2 MAs.

The computation can be speeded up using the multiplicative nature of N. Suppose that $N = pq$, where the factors p and q are both bigger than 1. The index in the sum in (5.5) can be written as $j = \alpha p + \beta$, where $0 \le \alpha \le q-1$ and $0 \le \beta \le p-1$. In terms of α and β, the sum in (5.5) becomes the double sum

$$\widehat{y}_k = \sum_{\beta=0}^{p-1}\sum_{\alpha=0}^{q-1} y_{\alpha p+\beta}\overline{w}^{(\alpha p+\beta)k} = \sum_{\beta=0}^{p-1}\left(\sum_{\alpha=0}^{q-1} y_{\alpha p+\beta}\overline{w}^{\alpha pk}\right)\overline{w}^{\beta k}. \tag{5.13}$$

We will isolate the inner sum by setting

$$\widehat{y}_{\beta,k} = \sum_{\alpha=0}^{q-1} y_{\alpha p+\beta}\overline{w}^{\alpha pk}. \tag{5.14}$$

Then (5.13) becomes

$$\widehat{y}_k = \sum_{\beta=0}^{p-1} \widehat{y}_{\beta,k}\overline{w}^{\beta k}, \quad \text{for } 0 \le k \le N-1. \tag{5.15}$$

The idea is to compute $\widehat{y}_{\beta,k}$ first using (5.14), and then compute the Fourier transform using (5.15). The savings in the computation comes from realizing that $\widehat{y}_{\beta,k}$ is periodic in k with period q. To see this, we first remember that $\overline{w}^N = 1$ and $N = pq$. Then we have

$$\widehat{y}_{\beta,k+q} = \sum_{\alpha=0}^{q-1} y_{\alpha p+\beta}\overline{w}^{\alpha p(k+q)} = \sum_{\alpha=0}^{q-1} y_{\alpha p+\beta}\overline{w}^{\alpha pk} = \widehat{y}_{\beta,k}.$$

Thus we only need to compute $\widehat{y}_{\beta,k}$ for $0 \le \beta \le p-1$ and $0 \le k \le q-1$. Since computing each $\widehat{y}_{\beta,k}$ requires q MAs, computing all of them requires $pq \cdot q = pq^2 = Nq$ MAs. Now computing the N components of the Fourier transform using (5.15) requires Np additional MAs, for a total of $N(p+q)$. If N, p, and q are all large numbers, the sum $p+q$ is much smaller than the product $N = pq$.

The process outlined in the previous paragraph can be iterated if N has more factors. If $N = p_1 \cdot p_2 \cdot \ldots \cdot p_n$, the number of MAs required is reduced to $N(p_1 + p_2 + \cdots + p_n)$. This algorithm for computing the discrete Fourier transform is called the ***fast Fourier transform (FFT)***. Clearly the FFT works best if N has a large number of very small factors, the best being when N is a power of 2. This is the most commonly used case. When $N = 2^L$ the FFT can compute the Fourier coefficients with only about $N \cdot 2L = 2N \log_2 N$ MAs. For example, if $N = 2^{10} = 1024$, the FFT requires only about 20,000 MAs versus the one million or so that are required using (5.8). The savings get more impressive as N gets larger. A similar FFT routine exists for computing the inverse discrete Fourier transform. The mathematical computer programs Matlab, Mathematica, and Maple all have built-in commands for the FFT and inverse FFT.

EXERCISES

All of these exercises are designed to be done with a mathematical computer program such as Matlab, Maple, or Mathematica.

1. Consider the function

$$f(t) = e^{-t^2/10}\left(\cos 2t + 2\sin 4t + 0.4\cos 2t\cos 40t\right).$$

For what values of n would you expect the Fourier coefficients to be largest? Why? Compute the coefficients numerically through $n = 50$ and see if you are right. (You can use a fast Fourier transform algorithm with $N = 256$ to do this if you wish.) Plot the partial sum of the Fourier series of order $n = 6$ and compare with the plot of the original $f(x)$.

2. Consider the function

$$g(t) = e^{-t^2/8}\left[\cos 2t + 2\sin 4t + 0.4\cos 2t\cos 10t\right],$$

for $0 \le t \le 2\pi$. Compute numerically the partial sum of the Fourier series of order $N = 25$. Zero out any coefficients that have absolute value smaller than 10% of the maximum. Plot the resulting series and compare with the original function $g(t)$. Try experimenting with different tolerances (other than 10%).

3. Show that if $y = \{y_m\}$ is a sequence of real numbers that is periodic with period N and $\widehat{y}$ is the discrete Fourier transform of y, then the complex conjugate of $\widehat{y}_m$ is $\widehat{y}_{N-m}$. (As a result, when m is small relative to N, $\widehat{y}_{N-m}$ has to be considered a low frequency coefficient, since it is equal to the conjugate of $\widehat{y}_m$, which is approximately equal to the conjugate of the mth Fourier coefficient.)

The next three problems require the use of the fast Fourier transform on a computer (e.g., Maple or Matlab's FFT routine).

4. ***Filtering*** Let

$$f(t) = e^{-t^2/10}\left(\sin(2t) + 2\cos(4t) + 0.4\sin(t)\sin(50t)\right).$$

Discretize f by setting $y_k = f(2k\pi/256)$, for $k = 0 \ldots 255$. Use the fast Fourier transform to compute $\widehat{y}_k$ for $0 \le k \le 255$. According to Exercise 3, the low-frequency coefficients are $\widehat{y}_0 \ldots \widehat{y}_m$ and $\widehat{y}_{256-m} \ldots \widehat{y}_{255}$ for some low value of m. Filter out the high-frequency terms by setting $\widehat{y}_k = 0$ for $m \le k \le 255-m$ with $m = 6$. Apply the inverse fast Fourier transform to this new set of $\widehat{y}_k$ to compute the y_k (now filtered); plot the new values of y_k and compare with the original function. Experiment with other values of m.

5. ***Compression*** Let tol $= 0.01$. In Exercise 4, if $|\widehat{y}_k| <$ tol $\times$ M, where $M = \max_{0\le k\le 255} |\widehat{y}_k|$, set $\widehat{y}_k$ equal to zero. Apply the inverse fast Fourier transform to this new set of $\widehat{y}_k$ to compute the y_k. Plot the new values of y_k and compare with the original function. Experiment with other values of tol. Keep track of the percentage of Fourier coefficients that have been filtered out. Matlab's sort command is useful for finding a value for tol in order to filter out a specified percentage of coefficients.

6. Repeat the previous two exercises over the interval $0 \le t \le 1$ with the function

$$\begin{aligned} f(t) = {} & -52t^4 + 100t^3 - 49t^2 + 2 + N(100(t - 1/3)) \\ & + N(200(t - 2/3)) \end{aligned}$$

where $N(t) = te^{-t^2}$.