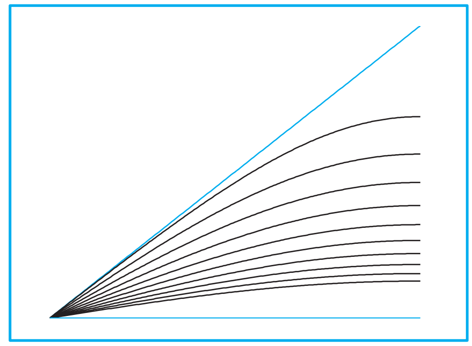
CHAPTER 13



Partial Differential Equations

We will now consider differential equations that model change where there is more than one independent variable. For example, the temperature in an object changes with time and with the position within the object. The rates of change lead to partial derivatives, and the equations relating them are called partial differential equations. The applications of the subject are many, and the types of equations that arise have a great deal of variety. We will limit our study to the equations that arise most frequently in applications. These model heat flow and simple waves. The differential equation models for heat flow and the vibrating string will be derived in Sections 1 and 3, where we will also describe some of their properties. We will then systematically study each of the equations, solving them in some cases using the method of separation of variables.

13.1 Derivation of the Heat Equation

Heat is a form of energy that exists in any material. Like any other form of energy, heat is measured in joules (1 J = 1 Nm). However, it is also measured in calories (1 cal = 4.184 J) or sometimes in British thermal units (1 BTU = 252 cal = 1.054 kJ).

The amount of heat within a given volume is defined only up to an additive constant. We will assume the convention of saying that the amount of heat is equal to 0 when the temperature is equal to 0. Suppose that ΔV is a small volume in which the temperature u is almost constant. It has been found experimentally that the amount of heat ΔQ in ΔV is proportional to the temperature u and to the mass $\Delta m = \rho \Delta V$, where ρ is the mass density of the material. Thus the amount of heat in ΔV is given by

$$\Delta Q = c\rho u \Delta V. \tag{1.1}$$

The new constant c is called the ***specific heat***. It measures the amount of heat required to raise 1 unit of mass of the material 1 degree of temperature. We will usually use the Celsius or Kelvin scales for temperature.

Let's consider a thin rod that is insulated along its length, as seen in Figure 1. If the length of the rod is L, the position along the rod is given by x, where $0 \le x \le L$. Since the rod is insulated, there is no transfer of heat from the rod except at its two ends. We may therefore assume that the temperature u depends only on x and on the time t.

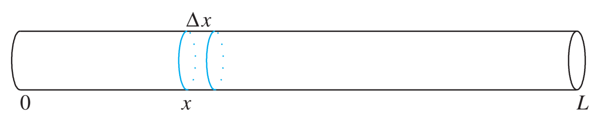


Figure 1 The variation of temperature in an insulated rod.

Consider a small section of the rod between x and $x + \Delta x$. Let S be the cross-sectional area of the rod. The volume of the section is $S\Delta x$, so (1.1) becomes

$$\Delta Q = c\rho u S \Delta x.$$

Therefore, the amount of heat at time t in the portion U of the rod defined by $a \le x \le b$ is given by the integral

$$Q(t) = S \int_a^b c\rho u(t, x)\, dx. \tag{1.2}$$

The specific heat and the density sometimes vary from point to point and more rarely with time as well. However, we will usually be dealing with homogeneous, time independent materials for which both the specific heat and the density are constants.

The heat equation models the flow of heat through the material. It is derived by computing the time rate of change of Q in two different ways. The first way is to differentiate (1.2). Differentiating under the integral sign, we get

$$\frac{dQ}{dt} = \frac{d}{dt} S \int_a^b c\rho u\, dx = S \int_a^b \frac{\partial}{\partial t}[c\rho u]\, dx.$$

Of course, if the specific heat and the density do not vary with time, this becomes

$$\frac{dQ}{dt} = S \int_a^b c\rho \frac{\partial u}{\partial t}\, dx. \tag{1.3}$$

The second way to compute the time rate of change of Q is to notice that, in the absence of heat sources within the rod, the quantity of heat in U can change only through the flow of heat across the boundaries of U at $x = a$ and $x = b$. The rate of heat flow through a section of the rod is called the ***heat flux*** through the section. Consider the section of the rod between $x = a$ and $a + \Delta x$. Experimental study of heat conduction reveals that the flow of heat across such a section has the following properties:

- Heat flows from hot positions to cold positions at a rate proportional to the difference in the temperatures on the two sides of the section. Thus the heat flux through the section is proportional to $u(a + \Delta x, t) - u(a, t)$.
- The heat flux through the section is inversely proportional to Δx, the width of the section.
- The heat flux through the section is proportional to the area of S of the boundary of the section.

Putting these three points together, we see that there is a coefficient C such that the heat flux *into* U at $x = a$ is given approximately by

$$-CS\frac{u(a + \Delta x, t) - u(a, t)}{\Delta x}. \tag{1.4}$$

The coefficient C is called the ***thermal conductivity***. It is positive since, if $u(a + \Delta x, t) > u(a, t)$, then the temperature is hotter inside U than it is outside, and the heat flows *out of* U at $x = a$. The thermal conductivity is usually constant, but it may depend on the temperature u and the position x.

If we let Δx go to 0 in (1.4), the difference quotient approaches $\partial u/\partial x$, and we see that the heat flux into U at $x = a$ is

$$-CS\frac{\partial u}{\partial x}(a, t). \tag{1.5}$$

The same argument at $x = b$ shows that the heat flux into U at $x = b$ is

$$CS\frac{\partial u}{\partial x}(b, t). \tag{1.6}$$

The total time rate of change of Q is the sum of the rates at the two ends. Using the fundamental theorem of calculus, this is

$$\frac{dQ}{dt} = S\left[C\frac{\partial u}{\partial x}(b, t) - C\frac{\partial u}{\partial x}(a, t)\right] = S\int_a^b \frac{\partial}{\partial x}\left(C\frac{\partial u}{\partial x}\right)dx. \tag{1.7}$$

If the thermal conductivity C is independent of x, this becomes

$$\frac{dQ}{dt} = CS\int_a^b \frac{\partial^2 u}{\partial x^2}\,dx. \tag{1.8}$$

In equations (1.3) and (1.8) we have two formulas for the rate of heat flow into U. Setting them equal, we see that

$$\int_a^b c\rho\frac{\partial u}{\partial t}\,dx = C\int_a^b \frac{\partial^2 u}{\partial x^2}\,dx \quad \text{or} \quad \int_a^b \left(c\rho\frac{\partial u}{\partial t} - C\frac{\partial^2 u}{\partial x^2}\right)dx = 0.$$

This is true for all $a < b$, which can be true only if the integrand is equal to 0. Hence,

$$c\rho\frac{\partial u}{\partial t} - C\frac{\partial^2 u}{\partial x^2} = 0$$

Table 1 Thermal diffusivities of common materials

Material	k (cm²/sec)	Material	k (cm²/sec)
Aluminum	0.84	Gold	1.18
Brick	0.0057	Granite	0.008–0.018
Cast iron	0.17	Ice	0.0104
Copper	1.12	PVC	0.0008
Concrete	0.004–0.008	Silver	1.70
Glass	0.0043	Water	0.0014

throughout the material. If we divide by $c\rho$, and set $k = C/c\rho$, the equation becomes

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0, \quad \text{or} \quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}. \tag{1.9}$$

The constant k is called the ***thermal diffusivity*** of the material. The units of k are $(\text{length})^2/\text{time}$. The values of k for some common materials are listed in Table 1.

Equation (1.9) is called the ***heat equation***. As we have shown, it models the flow of heat through a material and is satisfied by the temperature. It should be noticed that if we have a wall with height and width that are large in comparison to the thickness L, then the temperature in the wall away from its ends will depend only on the position within the wall. Consequently we have a one dimensional problem, and the variation of the temperature is modeled by the heat equation in (1.9)

A similar derivation shows that the diffusion of a substance through a liquid or a gas satisfies the same equation. In this case it is the concentration u that satisfies the equation. For this reason equation (1.9) is also referred to as the ***diffusion equation***.

Subscript notation for derivatives

We will find it useful to abbreviate partial derivatives by using subscripts to indicate the variable of differentiation. For example, we will write

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad u_{yx} = \frac{\partial^2 u}{\partial x \partial y}, \quad \text{and} \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}.$$

Using this notation, we can write the heat equation in (1.9) quite succinctly as

$$u_t = k u_{xx}.$$

The inhomogeneous heat equation

Equation (1.9) was derived under the assumption that there is no source of heat within the material. If there are heat sources, we can modify the model to accommodate them. If we look back to equation (1.7), which accounts for the rate of flow of heat into U, we see that we must modify the right-hand side to account for internal sources. We will assume that the heat source is spread throughout the material and that heat is being added at the rate of $p(u, x, t)$ thermal units per unit volume per second.

Notice that we allow the rate of heat inflow to depend on the temperature u, as well as on x and t. An example would be a rod that is not completely insulated along its length. Then heat would flow into or out of the rod along its length at a rate that is proportional to the difference between the temperature in U and the ambient temperature, so $p(u, x, t) = \alpha[u - T]$, where T is the ambient temperature.

Assuming there is a source of heat, equation (1.7) becomes

$$\frac{dQ}{dt} = CS\left[\frac{\partial u}{\partial x}(b, t) - \frac{\partial u}{\partial x}(a, t)\right] + S\int_a^b p(u, x, t)\,dx.$$

The rest of the derivation is unchanged, and in the end we get

$$c\rho\frac{\partial u}{\partial t} = C\frac{\partial^2 u}{\partial x^2} + p, \quad \text{or} \quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \frac{p}{c\rho}. \tag{1.10}$$

Because of the term involving p, equation (1.10) is called the ***inhomogeneous heat equation***, while equation (1.9) is called the ***homogeneous heat equation.***

Initial conditions

We have seen that ordinary differential equations have many solutions, and to determine a particular solution we specify initial conditions. The situation is more complicated for partial differential equations.

For example, specifying initial conditions for a temperature requires giving the temperature at each point in the material at the initial time. In the case of the rod this means that we give a function $f(x)$ defined for $0 \le x \le L$ and we look for a solution to the heat equation that also satisfies

$$u(x, 0) = f(x), \quad \text{for } 0 \le x \le L. \tag{1.11}$$

Types of boundary conditions

In addition to specifying the initial temperature, it will be necessary to specify conditions on the boundary of the material. For example, the temperature may be fixed at one endpoint of the rod as the result of the material being embedded in a source of heat kept at a constant temperature. The temperatures might well be different at the two ends of the rod. Thus if the temperature at $x = 0$ is T_0 and that at $x = L$ is T_L, then the temperature $u(x, t)$ satisfies

$$u(0, t) = T_0 \quad \text{and} \quad u(L, t) = T_L, \quad \text{for all } t. \tag{1.12}$$

Boundary conditions of the form in (1.12) specifying the value of the temperature at the boundary are called ***Dirichlet conditions***.

In other circumstances one or both ends of the rod might be insulated. This means that there is no flow of heat into or out of the rod at these points. According to the discussion leading to equation (1.5), this means that

$$\frac{\partial u}{\partial x} = 0 \tag{1.13}$$

at an insulated point. This type of boundary condition is called the ***Neumann condition***. A rod could satisfy a Dirichlet condition at one boundary point and a Neumann condition at the other.

There is a third condition that occurs, for example, when one end of the rod is poorly insulated from the exterior. According to Newton's law of cooling, the flow of heat across the insulation is proportional to the difference in the temperatures on the two sides of the insulation. If this is true at the endpoint $x = 0$, then arguing along the same lines as we did in the derivation of equation (1.5), we see that there is a positive number α such that

$$\frac{\partial u}{\partial x}(0, t) = \alpha(u(0, t) - T), \tag{1.14}$$

where T is the temperature outside the insulation and u is the temperature at the endpoint $x = 0$. Poor insulation at the endpoint $x = L$ leads in the same way to a boundary condition of the form

$$\frac{\partial u}{\partial x}(L, t) = -\beta(u(L, t) - T), \tag{1.15}$$

where $\beta > 0$. Boundary conditions of the type in (1.14) and (1.15) are called ***Robin conditions***.

Robin boundary conditions also arise when a solid wall meets a fluid or a gas. In such a case a thin boundary layer is formed, which shields the rest of the fluid or gas from the temperature in the wall. The constant β is sometimes called the ***heat transfer coefficient.***

Initial/boundary value problems

Putting everything together, we see that the temperature $u(x, t)$ in an insulated rod with Dirichlet boundary conditions must satisfy the heat equation together with initial and boundary conditions. The complete problem is to find a function $u(x, t)$ such that

$$\begin{aligned} u_t(x, t) &= ku_{xx}, \quad \text{for } 0 < x < L \text{ and } t > 0, \\ u(0, t) &= T_0, \quad \text{and} \quad u(L, t) = T_L, \quad \text{for } t > 0, \\ u(x, 0) &= f(x), \quad \text{for } 0 \le x \le L. \end{aligned} \tag{1.16}$$

The function $f(x)$ is the initial temperature distribution. The initial/boundary value problem is illustrated in Figure 2. As we have indicated, the Dirichlet boundary condition at each endpoint in (1.16) could be replaced with a Neumann or a Robin condition.

The maximum principle.

One of the major tenets of the theory of heat flow is that heat flows from hot areas to colder areas. From this starting point, physical reasoning allows us to conclude that the temperature $u(t, x)$ cannot get too hot or too cold in the region where it satisfies the heat equation. To be precise, let

$$m = \min_{0 \le x \le L} f(x) \quad \text{and} \quad M = \max_{0 \le x \le L} f(x).$$

Then, if $u(t, x)$ is a solution to the initial/boundary value problem in (1.16),

$$\min\{m, T_0, T_L\} \le u(t, x) \le \max\{M, T_0, T_L\} \quad \text{for } 0 \le t \text{ and } 0 \le x \le L.$$

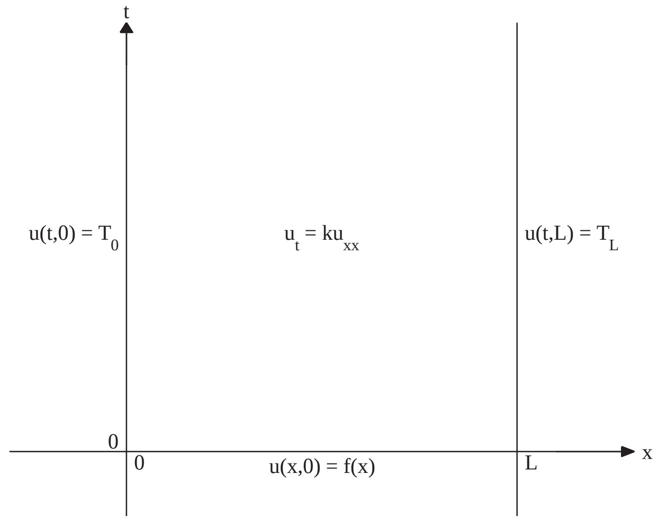


Figure 2 The initial/boundary value problem for the heat equation.

This result is called the ***maximum principle for the heat equation.*** In English it says that a temperature $u(t,x)$ defined for $0 \le t$ and $0 \le x \le L$ must achieve its maximum value (and its minimum value) on the boundary of the region where it is defined. Thus in Figure 2 The temperature $u(t,x)$ in the indicated half-strip must achieve its maximum and minimum values on the three lines which forms its boundary.

Linearity

If u and v are functions and α and β are constants, then

$$\frac{\partial}{\partial x}(\alpha u + \beta v) = \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial v}{\partial x}. \tag{1.17}$$

We will express this standard fact about $\partial/\partial x$ by saying that it is a ***linear operator***. It is an ***operator*** because it “operates” on a function u and yields another function $\partial u/\partial x$. That it is ***linear*** simply means that (1.17) is satisfied. It follows easily that more complicated differential operators, such as

$$\frac{\partial^2}{\partial x^2} \quad \text{and} \quad \frac{\partial^2}{\partial x \partial y}$$

are also linear. It then follows that the heat operator

$$\frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}$$

is linear. This implies the following theorem.

THEOREM 1.18 The homogeneous heat equation is a linear equation, meaning that if u and v satisfy

$$u_t = ku_{xx} \quad \text{and} \quad v_t = kv_{xx},$$

and α and β are constants, then the linear combination $w = \alpha u + \beta v$ satisfies $w_t = kw_{xx}$, so w is also a solution to the homogeneous heat equation.

We will make frequent use of Theorem 1.18. It will enable us to build up more complicated solutions as linear combinations of basic solutions.

EXERCISES

1. Suppose that the temperature at each point of a rod of length L is originally at $15°$. Suppose that starting at time $t = 0$ the left end is kept at $5°$ and the right end at $25°$. Write down the complete description of the initial/boundary value problem the temperature in the rod must obey.

2. Show that the temperature in the rod in Exercise 1 must satisfy $5 \le u(x, t) \le 25$ for $t \ge 0$ and $0 \le x \le L$.

3. Suppose the specific heat, density, and thermal conductivity depend on x, and are not constant. Show that the heat equation becomes

$$\frac{\partial}{\partial t}[c\rho u] = \frac{\partial}{\partial x}\left[C\frac{\partial u}{\partial x}\right].$$

4. If our rod is insulated at both ends, we would expect that the total amount of heat in the rod does not change with time. Show that this follows from equation (1.7).

5. Prove Theorem 1.18 by showing that $w_t = kw_{xx}$.

6. Solutions to the Dirichlet problem in (1.16) are unique. This means that if both u and v satisfy the conditions in (1.16), then $u(x, t) = v(x, t)$ for $t \ge 0$ and $0 \le x \le L$. Use the linearity of the heat equation and the maximum principle to prove this fact.

7. Suppose we have an insulated aluminum rod of length L. Suppose the rod is at a constant temperature of 15°K, and that starting at time $t = 0$, the left-hand end point is kept at 20°K and the right-hand endpoint is kept at 35°K. Provide the initial/boundary value problem that must be satisfied by the temperature $u(t, x)$.

8. Suppose we have an insulated gold rod of length L. Suppose the rod is at a constant temperature of 15°K, and that starting at time $t = 0$, the left-hand end point is kept at 20°K while the right-hand endpoint is kept insulated. Provide the initial/boundary value problem that must be satisfied by the temperature $u(t, x)$.

9. Suppose we have an insulated silver rod of length L. Suppose the rod is at a constant temperature of 15°K, and that starting at time $t = 0$, the right-hand end point is kept at 35°K while the left-hand endpoint is only partially insulated, so heat is lost there at a rate equal to 0.0013 times the difference between the temperature of the rod at this point and the ambient temperature $T = 15°$K. Provide the initial/boundary value problem that must be satisfied by the temperature $u(t, x)$.

13.2 Separation of Variables for the Heat Equation

We will start this section by solving the initial/boundary value problem

$$u_t(x,t) = k u_{xx}(x,t), \quad \text{for } t > 0 \text{ and } 0 < x < L, \tag{2.1}$$
$$u(0,t) = T_0 \quad \text{and} \quad u(L,t) = T_L, \quad \text{for } t > 0, \tag{2.2}$$
$$u(x,0) = f(x), \quad \text{for } 0 \le x \le L. \tag{2.3}$$

that we posed in (1.16).

Steady-state temperatures

It is useful for both mathematical and physical purposes to split the problem into two parts. We first find the steady-state temperature that satisfies the boundary conditions in (2.2). A ***steady-state*** temperature is one that does not depend on time. Then $u_t = 0$, so the heat equation (2.1) simplifies to $u_{xx} = 0$. Hence we are looking for a function $u_s(x)$ defined for $0 \le x \le L$ such that

$$\begin{aligned} &\frac{\partial^2 u_s}{\partial x^2}(x) = 0, \quad \text{for } 0 < x < L, \\ &u_s(0,t) = T_0 \quad \text{and} \quad u_s(L,t) = T_L, \quad \text{for } t > 0. \end{aligned} \tag{2.4}$$

The solution to this boundary value problem is easily found, since the general solution of the differential equation is $u_s(x) = Ax + B$, where A and B are arbitrary constants. Then the boundary conditions reduce to

$$u_s(0) = B = T_0 \quad \text{and} \quad u_s(L) = AL + B = T_L.$$

We conclude that $B = T_0$ and $A = (T_L - T_0)/L$, so the steady-state temperature is

$$u_s(x) = (T_L - T_0)\frac{x}{L} + T_0.$$

It remains to find $v = u - u_s$. It will be a solution to the heat equation, since both u and u_s are, and the heat equation is linear. The boundary and initial conditions that v satisfies can be calculated from those for u and u_s in (2.2), (2.3), and (2.4). Thus, $v = u - u_s$ must satisfy

$$\begin{aligned} &v_t(x,t) = k v_{xx}(x,t), \quad \text{for } 0 < x < L \text{ and } t > 0, \\ &v(0,t) = v(L,t) = 0, \quad \text{for } t > 0, \\ &v(x,0) = g(x) = f(x) - u_s(x), \quad \text{for } 0 \le x \le L. \end{aligned} \tag{2.5}$$

The most important fact is that the boundary conditions for v are $v(0,t) = v(L,t) = 0$. When the right-hand sides are equal to) we say that the boundary conditions are ***homogeneous***. This will make finding the solution a lot easier.

Having found the steady-state temperature u_s and the temperature v, the solution to the original problem is $u(x,t) = u_s(x) + v(x,t)$.

Solution with homogeneous boundary conditions

We will find the solution to the initial/boundary value problem with homogeneous boundary conditions in (2.5) using the technique of ***separation of variables***. It should be noted that separation of variables can only be used to solve an initial/boundary value problem when the boundary conditions are homogeneous. Since this is the first time we are using the technique, and since it is a technique we will use throughout this chapter, we will go through the process slowly. The basic idea of the method of separation of variables is to hunt for solutions in the product form

$$v(x,t) = X(x)T(t), \tag{2.6}$$

where $T(t)$ is a function of t and $X(x)$ is a function of x. We will insist that the product solution v satisfies the homogeneous boundary conditions. Since $0 = v(0,t) = X(0)T(t)$ for all $t > 0$, we conclude that $X(0) = 0$. A similar argument shows that $X(L) = 0$. This leads to a two-point boundary value problem for X that we will solve. In the end we will have found enough solutions of the factored form so that we will be able to solve the initial/boundary value problem in (2.5) using an infinite linear combination of them.

There are three steps to the method.

Step 1: Separate the PDE into two ODEs. When we insert $v = X(x)T(t)$ into the heat equation $v_t = kv_{xx}$, we get

$$X(x)T'(t) = kX''(x)T(t). \tag{2.7}$$

The key step is to separate the variables by bringing everything depending on t to the left, and everything depending on x to the right. Dividing (2.7) by $kX(x)T(t)$, we get

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}.$$

Since x and t are independent variables, the only way that the left-hand side, a function of t, can equal the right-hand side, a function of x, is if both functions are constant. Consequently, there is a constant that we will write as $-\lambda$, such that

$$\frac{T'(t)}{kT(t)} = -\lambda \quad \text{and} \quad \frac{X''(x)}{X(x)} = -\lambda,$$

or

$$T' + \lambda kT = 0 \quad \text{and} \quad X'' + \lambda X = 0. \tag{2.8}$$

The first equation has the general solution

$$T(t) = Ce^{-\lambda kt}. \tag{2.9}$$

We have to work a little harder on the second equation.

Step 2: Set up and solve the two-point boundary value problem. Since we insist that the solution X satisfies the homogeneous boundary conditions, the complete problem to be solved in finding X is

$$X'' + \lambda X = 0 \quad \text{with } X(0) = X(L) = 0. \tag{2.10}$$

Notice that the problem in (2.10) is not the standard initial value problem we have been solving up to now. There are two conditions imposed, but instead of both being imposed at the initial point $x = 0$, there is one condition imposed at each endpoint of the interval. Accordingly, this is called a ***two-point boundary value problem***. It is also called a ***Sturm Liouville problem***.[1]

Another point to be made is that the constant λ is still undetermined. Furthermore, as we will see, for most values of λ the only solution to (2.10) is the function that is identically 0. Solving a Sturm Liouville problem amounts to finding the numbers λ for which there are nonzero solutions to (2.10).

DEFINITION 2.11 A number λ is called an ***eigenvalue*** for the Sturm Liouville problem in (2.10) if there is a nonzero function X that solves (2.10). If λ is an eigenvalue, then any function that satisfies (2.10) is called an ***eigenfunction***.[2]

The solution to a Sturm Liouville problem like (2.10) is the list of its eigenvalues and eigenfunctions. Notice that because of the linearity of the differential equation in (2.10), any constant multiple of an eigenfunction is also an eigenfunction. We will usually choose the constant that leads to the least complicated form for the eigenfunction.

Let's return to the example in (2.10). We will first show that there are no negative eigenvalues. To see this, set $\lambda = -r^2$, where $r > 0$. The equation in (2.10) becomes $X'' - r^2X = 0$, which has general solution $X(x) = C_1e^{rx} + C_2e^{-rx}$. The boundary conditions are

$$\begin{aligned} 0 &= X(0) = C_1 + C_2 \\ 0 &= X(L) = C_1e^{rL} + C_2e^{-rL}. \end{aligned}$$

From the first equation, $C_2 = -C_1$. Inserting this into the second equation, we get

$$0 = C_1(e^{rL} - e^{-rL}).$$

Since $r \neq 0$, the factor in parenthesis on the right is nonzero. Hence $C_1 = 0$, which in turn implies that $C_2 = 0$, so the only solution is $X(x) = 0$. This means that λ is not an eigenvalue.[3]

This argument can be repeated if $\lambda = 0$. In this case the differential equation becomes $X'' = 0$, which has the general solution $X(x) = ax + b$, where a and b are constants. The boundary conditions become $0 = X(0) = b$ and $0 = X(L) = aL+b$, from which we easily conclude that $a = b = 0$.

[1] This is our first example of a Sturm Liouville problem. We will study them in some detail in Sections 6 and 7.

[2] You will observe that finding the eigenvalues and eigenfunctions of a Sturm Liouville problem is similar in many ways to finding the eigenvalues and eigenvectors of a matrix. It might be useful to compare the situation here with Section 9.1.

[3] This agrees with our physical intuition about heat flow. If there were a solution X with $\lambda < 0$, then according to (2.6) and (2.9), the product solution to the heat equation would be $v(x, t) = e^{-\lambda kt}X(x)$. If $\lambda < 0$, this solution would grow exponentially in magnitude as t increases. In fact, we notice experimentally that temperatures tend to remain stable over time in the absence of heat sources.

Next suppose that $\lambda > 0$ and set $\lambda = \omega^2$. Then the differential equation in (2.10) is $X'' + \omega^2 X = 0$, which has the general solution

$$X(x) = a\cos\omega x + b\sin\omega x.$$

For this solution the boundary condition $X(0) = 0$ becomes $a = 0$. Then the boundary condition $X(L) = 0$ becomes

$$b\sin\omega L = 0.$$

We are only interested in nonzero solutions, so we must have $\sin\omega L = 0$. This occurs if $\omega L = n\pi$ for some positive integer n. When this is true we have the eigenvalue $\lambda = \omega^2 = n^2\pi^2/L^2$. For any nonzero constant b, $X(x) = b\sin(n\pi x/L)$ is an eigenfunction. The simplest thing to do is to set $b = 1$.

In summary, the eigenvalues and eigenfunctions for the Sturm Liouville problem in (2.10) are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{and} \quad X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \text{for } n = 1, 2, 3, \ldots. \tag{2.12}$$

Finally, by incorporating (2.9) and (2.12), we get the product solutions,

$$v_n(x, t) = e^{-n^2\pi^2 kt/L^2}\sin\left(\frac{n\pi x}{L}\right), \quad \text{for } n = 1, 2, 3, \ldots, \tag{2.13}$$

to the heat equation, that also satisfy the boundary condition $v_n(0, t) = v_n(L, t) = 0$.

Step 3: Satisfying the initial condition. Having found infinitely many product solutions in (2.13), we can use the linearity of the heat equation (see Theorem 1.18) to conclude that any finite linear combination of them is also a solution. Hence, if b_n is a constant for each n, then for any N the function

$$v(x, t) = \sum_{n=1}^{N} b_n v_n(x, t) = \sum_{n=1}^{N} b_n e^{-n^2\pi^2 kt/L^2}\sin\left(\frac{n\pi x}{L}\right)$$

is a solution to the heat equation that satisfies the homogeneous boundary conditions. We are naturally led to consider the infinite series

$$v(x, t) = \sum_{n=1}^{\infty} b_n v_n(x, t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt/L^2}\sin\left(\frac{n\pi x}{L}\right). \tag{2.14}$$

We will assume that the coefficients b_n are such that this series converges, and that the resulting function v satisfies the heat equation and the homogeneous boundary conditions. These facts are true formally.[4] They are also true in the cases that we will consider, but we will not verify this. To do so requires some lengthy mathematical arguments that would not significantly add to our understanding of the issue.

[4] *Formally* means that we ignore the mathematical niceties of verifying that we can differentiate the function v by differentiating the terms in the infinite series.

Referring back to our original initial/boundary value problem in (2.5), we see that the function v defined in (2.14) satisfies everything except the initial condition $v(x, 0) = g(x) = f(x) - u_s(x)$. However, we have yet to determine the coefficients b_n. Using the series definition for v in (2.14), the initial condition becomes

$$g(x) = v(x, 0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{for } 0 \le x \le L. \tag{2.15}$$

Equation (2.15) will be recognized as the Fourier sine expansion for the initial temperature g. According to Section 12.3, and in particular equation (3.7), the values of b_n are given by

$$b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right) dx. \tag{2.16}$$

Substituting these values into (2.14) gives a complete solution to the homogeneous initial/boundary value problem in (2.5). As indicated previously, the function $u(x, t) = u_s(x) + v(x, t)$ satisfies the original initial/boundary value problem in equations (2.1), (2.2), and (2.3).

EXAMPLE 2.17 ◆ Suppose a rod of length 1 meter (100cm) is originally at 0°C. Starting at time $t = 0$, one end is kept at the constant temperature of 100°C, while the other is kept at 0°C. Find the temperature distribution in the rod as a function of time and position. Assume that the thermal diffusivity of the rod is $k = 1$ cm^2/sec.

If we use the meter as the unit of length, then $k = 0.0001$ m^2/sec. The temperature in the rod, $u(x, t)$, must solve the initial/boundary value problem

$$\begin{aligned} &u_t(x, t) = 0.0001\, u_{xx}(x, t), \quad \text{for } t > 0 \text{ and } 0 < x < 1,\\ &u(0, t) = 0 \quad \text{and} \quad u(1, t) = 100, \quad \text{for } t > 0,\\ &u(x, 0) = 0, \quad \text{for } 0 \le x \le 1. \end{aligned} \tag{2.18}$$

Following the discussion at the beginning of this section, we write the temperature distribution as $u = u_s + v$, where $u_s(x)$ is the steady-state temperature with the same boundary conditions as u, and v is a temperature with homogeneous boundary conditions, and the same initial condition as $u - u_s$. The steady-state temperature u_s must satisfy

$$u_s'' = 0 \quad \text{with} \quad u_s(0) = 0 \quad \text{and} \quad u_s(1) = 100.$$

We easily see that $u_s(x) = 100x$.

Then the temperature $v = u - u_s$ must satisfy

$$\begin{aligned} &v_t(x, t) = 0.0001 v_{xx}(x, t), \quad \text{for } t > 0 \text{ and } 0 < x < 1,\\ &v(0, t) = 0 \quad \text{and} \quad v(1, t) = 0, \quad \text{for } t > 0,\\ &v(x, 0) = -100x, \quad \text{for } 0 \le x \le 1. \end{aligned} \tag{2.19}$$

The boundary values are homogeneous, so we can use the formula for the solution in (2.14), with $k = 0.0001$ and $L = 1$, to get

$$v(x, t) = \sum_{n=1}^{\infty} b_n e^{-0.0001 n^2\pi^2 t} \sin n\pi x. \tag{2.20}$$

The coefficients are determined by the initial condition. Setting $t = 0$ in (2.20) and using $v(x, 0) = -100x$, we obtain

$$-100x = \sum_{n=1}^{\infty} b_n \sin n\pi x.$$

Therefore, the b_n are the Fourier sine coefficients of $-100x$ on the interval $(0, 1)$, which by (2.16) are

$$b_n = 2\int_0^1 (-100x)\sin n\pi x\,dx = -200\int_0^1 x\sin n\pi x\,dx = (-1)^n\frac{200}{n\pi}.$$

Thus,

$$v(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-0.0001n^2\pi^2 t}\sin n\pi x.$$

Finally, the temperature in the rod is

$$u(x,t) = u_s(x) + v(x,t) = 100x + \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-0.0001\,n^2\pi^2 t}\sin n\pi x. \quad (2.21)$$

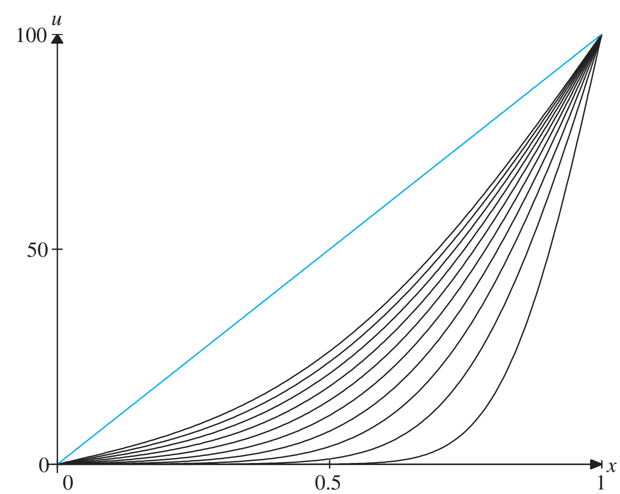


Figure 1 The temperature in the rod in Example 2.17.

The temperature is plotted in Figure 1. The initial temperature is $u(x, 0) = 0°\text{C}$. The steady-state temperature is plotted in blue. The black curves represent the temperature distribution after 200 second intervals. Notice how the temperature increases with time throughout the rod to the steady-state temperature. Heat flows from hot to cold, so to maintain the new temperature of 100°C at the right endpoint, heat must flow into the rod at this point. It then flows through the rod, raising the temperature in the process. Some heat has to flow out of the rod at the left endpoint to maintain the temperature there. Eventually the rod reaches steady state, at which point as much heat flows out of the rod at $x = 0$ as flows in at $x = 1$. ◆

The rate of convergence

The general term in the infinite series in equation (2.14) is

$$b_n e^{-n^2\pi^2 kt/L^2} \sin\left(\frac{n\pi x}{L}\right). \tag{2.22}$$

Since the sine function is bounded in absolute value by 1, this term is bounded by $|b_n|e^{-n^2\pi^2 kt/L^2}$. By the Riemann-Lebesgue lemma (see Theorem 2.10 in Section 12.2), the Fourier coefficient $b_n \to 0$ as $n \to \infty$. On the other hand, the exponential term $e^{-n^2\pi^2 kt/L^2} \to 0$ extremely rapidly as $n \to \infty$, at least if the product kt is relatively large. As a result the series in equation (2.14) converges rapidly for large values of the time t. The result is that the sum of the series in (2.14) can be accurately approximated by using relatively few of the terms of the infinite series. Sometimes one term is enough.

EXAMPLE 2.23 ◆ For the rod in Example 2.17 how many terms of series in (2.21) are needed to approximate the solution within one degree for $t = 10$, 100, and 1000. Estimate how long will it take before the heat in the rod is everywhere within 5° of the steady-state temperature?

The general term in the series in (2.21) is bounded by $200e^{-0.0001n^2\pi^2 t}/n\pi$. We will estimate the error by computing the first omitted term.[5] Thus we want to find the smallest integer n for which $200e^{-0.0001(n+1)^2\pi^2 t}/[(n+1)\pi] < 1$. Since we cannot solve this inequality for n, we compute the left-hand side for values of n and t until we get the correct values. For $t = 10$ we discover that we need 12 terms, while for $t = 100$ we need 5, and for $t = 1000$ one term will suffice.

For the temperature of the rod to be within 5° of the steady-state temperature, we will certainly need the first term in the infinite series in (2.21) to be less than 5. If we solve $200e^{-0.0001\pi^2 t}/\pi = 5$, we obtain $t = 2,578$ sec. We compute that for $t = 2,578$, the second term in the series is about 0.0012, so $t = 2,578$ sec is a good estimate. However, in view of the fact that we are ignoring terms, and an estimate is not expected to have four place accuracy, 2, 600 sec might be preferable, and since 2, 580 sec is 43 minutes, that might be even better. ◆

Insulated boundary points

As mentioned in Section 1, if the boundary points of the rod are insulated, there is no flow of heat through the endpoints of the rod, and the correct boundary conditions are the Neumann conditions $u_x(0,t) = 0 = u_x(L,t)$. The initial/boundary value problem to be solved is now

$$\begin{aligned} &u_t(x,t) = ku_{xx}(x,t), \quad \text{for } t > 0 \text{ and } 0 < x < L, \\ &u_x(0,t) = 0 \quad \text{and} \quad u_x(L,t) = 0, \quad \text{for } t > 0, \\ &u(x,0) = f(x), \quad \text{for } 0 \le x \le L. \end{aligned} \tag{2.24}$$

[5] This is a rough estimate and is not a usually a good idea. It is justified in this case because the terms are decreasing so rapidly.

We will use the method of separation of variables again, starting by looking for product solutions $u(x,t) = X(x)T(t)$. Notice that since the Neumann boundary conditions are homogeneous, it is not necessary to find the steady-state solution first.

Step 1: Separate the PDE into two ODEs. This first step is unchanged. The product $u(x,t) = X(x)T(t)$ is a solution only if the factors satisfy the differential equations

$$T' + \lambda k T = 0 \quad \text{and} \quad X'' + \lambda X = 0, \tag{2.25}$$

where λ is a constant. The first equation has the general solution

$$T(t) = Ce^{-\lambda k t}. \tag{2.26}$$

Step 2: Set up and solve the two-point boundary value problem. We will again insist that the product solution satisfy the boundary conditions. Since $0 = u_t(0,t) = X'(0)T(t)$ for all $t > 0$, we must have $X'(0) = 0$. A similar argument shows that $X'(L) = 0$, so we want to solve

$$X'' + \lambda X = 0 \quad \text{with } X'(0) = X'(L) = 0. \tag{2.27}$$

This is the two-point or Sturm Liouville boundary value problem for the Neumann problem. As before, we find that there are no negative eigenvalues. If $\lambda = 0$ the differential equation in (2.27) becomes $X'' = 0$, which has the general solution $X(x) = ax + b$. The first boundary condition is $0 = X'(0) = a$, leaving us with the constant function $X(x) = b$. This function also satisfies the second boundary $X'(L) = 0$, so $\lambda = 0$ is an eigenvalue. We will choose the simplest nonzero constant $b = 1$ and set $X_0(x) = 1$. The corresponding function in (2.26) is $T_0 = C$, which is also a constant. Once more we choose $C = 1$ so the resulting product solution to the heat equation is the constant function

$$u_0(x,t) = X_0(x)T_0(t) = 1.$$

For $\lambda > 0$, we set $\lambda = \omega^2$, where $\omega > 0$. Then the differential equation in (2.27) is $X'' + \omega^2 X = 0$, which has the general solution $X(x) = a\cos\omega x + b\sin\omega x$. The boundary condition $X'(0) = 0$ becomes $\omega b = 0$. Since $\omega > 0$, we have $b = 0$. Then the boundary condition $X'(L) = 0$ becomes

$$\omega a \sin\omega L = 0.$$

Since we are only interested in nonzero solutions, we must have $\sin\omega L = 0$. Therefore, $\omega L = n\pi$ for some positive integer n. When this is true we have $\lambda = \omega^2 = n^2\pi^2/L^2$, and $X(x) = a\cos(n\pi x/L)$. Again a can be any nonzero constant, and the simplest choice is $a = 1$.

In summary, the eigenvalues and eigenfunctions for the Sturm Liouville problem in (2.27) are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{and} \quad X_n(x) = \cos\left(\frac{n\pi x}{L}\right), \quad \text{for } n = 0, 1, 2, 3, \ldots. \tag{2.28}$$

Notice that in the case $n = 0$, $X_0(x) = 1$, as we found earlier. For every nonnegative integer n we get the product solution

$$u_n(x,t) = e^{-n^2\pi^2 kt/L^2}\cos\left(\frac{n\pi x}{L}\right) \tag{2.29}$$

to the heat equation by using (2.26). Observe that this solution also satisfies the boundary conditions

$$\frac{\partial u_n}{\partial x}(0, t) = \frac{\partial u_n}{\partial x}(L, t) = 0.$$

Step 3: Satisfying the initial conditions. Having found infinitely many product solutions in (2.29), we can use the linearity of the heat equation (see Theorem 1.18) to conclude that any linear combination of the product solutions is also a solution. Hence if a_n is a constant for each n, the function

$$\begin{aligned} u(x, t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n u_n(x, t) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right) \end{aligned} \tag{2.30}$$

is formally a solution. Setting $u(x, 0) = f(x)$, we obtain the equation

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right). \tag{2.31}$$

This is the Fourier cosine expansion of f on the interval $0 \le x \le L$. From Section 3 of Chapter 12, and especially equation (3.2) in that section, we see that the coefficients a_n are given by

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx, \quad \text{for } n \ge 0. \tag{2.32}$$

Substituting these values into (2.30) gives a complete solution to the heat equation with Neumann boundary conditions.

Notice that each term in the infinite sum in (2.30) tends to 0 as $t \to \infty$. Using this and the definition of the coefficient a_0 we see that

$$\lim_{t\to\infty} u(x, t) = \frac{a_0}{2} = \frac{1}{L}\int_0^L f(x)\,dx.$$

Thus as t increases in an insulated rod, the temperature tends to a constant equal to the average of the initial temperature.

EXAMPLE 2.33 ◆ Suppose a rod of length 1 meter made from a material with thermal diffusivity $k = 1$ cm^2/sec is originally at steady state with its temperature maintained at 0°C at $x = 0$ and at 100°C at $x = 1$. (See Example 2.17.) Starting at time $t = 0$, both ends are insulated. Find the temperature distribution in the rod as a function of time and position. Find the constant temperature which is approached as $t \to \infty$. Estimate how long it will take for all portions of the rod to get to within 5°C of the final temperature?

According to our analysis in Example 2.17, the steady-state temperature at $t = 0$ is $f(x) = 100\,x$, with x measured in meters. This will be the initial temperature. With length measured in meters, $k = 0.0001\,\mathrm{m}^2/\mathrm{sec}$. Our new initial/boundary value problem is

$$\begin{aligned} u_t(x,t) &= 0.0001\,u_{xx}(x,t), \quad \text{for } t > 0 \text{ and } 0 < x < 1, \\ u_x(0,t) &= u_x(1,t) = 0, \quad \text{for } t > 0, \\ u(x,0) &= f(x) = 100\,x, \quad \text{for } 0 \le x \le 1. \end{aligned} \tag{2.34}$$

The solution as given in (2.30) with $k = 0.0001$ and $L = 1$ is

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-0.0001\,n^2\pi^2 t} \cos n\pi x. \tag{2.35}$$

The initial condition becomes

$$u(x,0) = 100\,x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x.$$

The a_n are the Fourier cosine coefficients of $100\,x$ on the interval $[0,1]$, so $a_0 = 100$, and

$$a_n = 2\int_0^1 100\,x \cos n\pi x\,dx = \begin{cases} 0, & \text{for } n > 0 \text{ even,} \\ -\dfrac{400}{n^2\pi^2}, & \text{for } n \text{ odd.} \end{cases}$$

Substituting into (2.35), using $n = 2p + 1$, we get the solution

$$u(x,t) = 50 - \frac{400}{\pi^2} \sum_{p=0}^{\infty} \frac{1}{(2p+1)^2} e^{-0.0001\,(2p+1)^2\pi^2 t} \cos(2p+1)\pi x. \tag{2.36}$$

Notice that each of the terms in the series, with the exception of the constant first term, include an exponential factor that approaches 0 as $t \to \infty$. Thus the temperature in the rod approaches the constant, steady-state temperature of 50°C as $t \to \infty$. Notice also that 50°C is the average of the initial temperature over the rod. This reflects the fact that the ends are insulated, and no heat flows into or out of the rod.

We suspect that one of the exponential terms (with $p = 0$) in equation (2.36) will suffice to find how long it takes for the temperature to be within 5° of the constant steady-state temperature. We solve $400\,e^{-0.0001\pi^2 t}/\pi^2 = 5$ to get $t = 2,120$ sec. We check that the contribution to the temperature of the $p = 1$ term is less than 3×10^{-8}, so $2,120$ sec is a good estimate.

The temperature is shown in Figure 2. The initial temperature $f(x) = 100\,x$ and the constant steady-state temperature of 50°C are shown plotted in blue. The black curves are the temperature profiles plotted at time intervals of 300s. ◆

EXERCISES

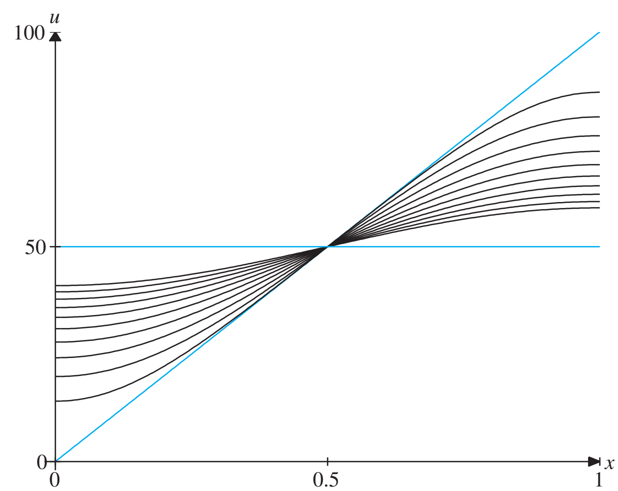


Figure 2 The temperature for the rod in Example 2.33.

1. Consider a rod 50 cm long with thermal diffusivity k cm^2/sec. Originally the rod is at a constant temperature of 100°C. Starting at time $t = 0$ the ends of the rod are immersed in an ice bath at temperature 0°C. Show that the temperature $u(x, t)$ in the rod for $t > 0$ is given by

$$u(x,t) = \sum_{p=0}^{\infty} \frac{400}{(2p+1)\pi} e^{-k(2p+1)^2\pi^2 t/2500} \sin\left(\frac{n\pi x}{50}\right). \tag{2.37}$$

If the rod is made of gold, find the thermal diffusivity in Table 1 on page 753, and estimate how long it takes the temperature in the rod to decrease everywhere to less than 10°C. How many terms in the series for u are needed to approximate the temperature within one degree at $t = 100$ sec. On one figure, plot the temperature versus x for $t = 0, 100, 200, 300, 400$.

2. Estimate how long it takes the temperature in the rod in Exercise 1 to decrease everywhere to less than 10°C if it is made of aluminum, silver, or PVC. For aluminum and silver, how many terms of the series in (2.37) are needed to approximate the temperature throughout the rod within 1° when $t = 100$ sec. For PVC, how many terms are needed to approximate the temperature throughout the rod within 1° when $t = 1$ day.

3. Consider a wall made of brick 10 cm thick, which separates a room in a house from the outside. The room is kept at 20°.

(a) Originally the outside temperature is 10°C and the temperature in the wall has reached steady-state. What is the temperature in the wall at this point?

(b) There is a sudden cold snap and the outside temperature drops to −10°C. Find the temperature in the wall as a function of position and time.

4. The wall of a furnace is 10 cm thick, and built from a refractory material with thermal diffusivity $k = 5 \times 10^{-5}$cm^2/sec. Originally there is no fire in the furnace and the temperature of the furnace and the outside are both 20°C. At

$t = 0$, a fire is lit and the inside of the furnace is quickly raised to 420°C. Find the temperature in the wall for $t > 0$.

In Exercises 5–8, find the temperature $u(t, x)$ in a rod modeled by the initial/boundary value problem

$$\begin{aligned} u_t(x,t) &= ku_{xx}(x,t), \quad \text{for } t > 0 \text{ and } 0 < x < L, \\ u(0,t) &= T_0 \quad \text{and} \quad u(L,t) = T_L, \quad \text{for } t > 0, \\ u(x,0) &= f(x), \quad \text{for } 0 \le x \le L. \end{aligned}$$

with the indicated values of the parameters.

5. $k = 4$, $L = 1$, $T_0 = 0$, $T_L = 0$, and $f(x) = x(1 - x)$

6. $k = 2$, $L = \pi$, $T_0 = 0$, $T_L = 0$, and $f(x) = \sin 2x - \sin 4x$

7. $k = 1$, $L = \pi$, $T_0 = 0$, $T_L = 0$, and $f(x) = \sin^2 x$

8. $k = 1$, $L = 1$, $T_0 = 0$, $T_L = 2$, and $f(x) = x$

In Exercises 9–12 use the temperature computed in the given exercise. Plot the initial temperature versus x and add the plots of the temperatue versus x for a number of time values like those in the text that show the significant portion of the change of the temperature. (Approximate the solution with an appropriate partial sum.) In addition, plot $y = u_x(0, t)$ and $y = u_x(L, t)$ as functions of t. Recall from (1.5) and (1.6) that these terms are proportional to the heat flux through the endpoints of the rod. Give a physical description of what is happening to the temperatue as time increases. Include the information from the graphs of the flux and the graphs of the solution.

9. Exercise 5

10. Exercise 6

11. Exercise 7

12. Exercise 8

In Exercises 13–18, find the temperature $u(t, x)$ in a rod modeled by the initial/boundary value problem

$$\begin{aligned} u_t(x,t) &= ku_{xx}(x,t), \quad \text{for } t > 0 \text{ and } 0 < x < L, \\ u_x(0,t) &= u_x(L,t) = 0, \quad \text{for } t > 0, \\ u(x,0) &= f(x), \quad \text{for } 0 \le x \le L. \end{aligned}$$

with the indicated values of the parameters. Plot the solution for a number of time values like those in the text that show the significant portion of the change of the temperature. Give a physical explanation of what is happening to the solution as time progresses.

13. $k = 1$, $L = 1$, and $f(x) = \begin{cases} x, & 0 \le x < 1/2, \\ (1 - x), & 1/2 \le x \le 1 \end{cases}$

14. $k = 1$, $L = 2$, and $f(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & 1 \le x \le 2 \end{cases}$

15. $k = 1$, $L = 1$, and $f(x) = \sin(\pi x)$

16. $k = 1$, $L = 1$, and $f(x) = \cos(\pi x)$

17. $k = 1$, $L = 2$, and $f(x) = \begin{cases} 0, & 0 \le x \le 1 \\ (x-1), & 1 < x \le 2 \end{cases}$

18. $k = 1/3$, $L = 2$, and $f(x) = x(2-x)$

In Exercises 19–21, we will consider heat flow in a rod of length L, where an internal heat source, given by $p(x)$, is present. As indicated in equation (1.10), this leads to the initial/boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} &= \frac{p(x)}{c\rho} \quad 0 < x < L,\ t > 0 \\ u(0,t) = A, &\quad u(L,t) = B, \\ u(x,0) = f(x) &\quad 0 \le x \le L, \end{aligned} \tag{2.38}$$

for the inhomogeneous heat equation, where f and p are given (known) functions of x and A and B are constants.

19. The corresponding steady-state solution is the function $v(x)$ that satisfies the partial differential equation and the boundary conditions. Show that $v(x)$ satisfies

$$v''(x) = -\frac{p(x)}{C}, \quad \text{with} \quad v(0) = A \quad \text{and} \quad v(L) = B.$$

(Remember that $k = C/c\rho$, where C is the thermal conductivity.) Suppose that $u_h(x,t)$ is the solution to the initial/boundary value problem

$$\begin{aligned} \frac{\partial u_h}{\partial t} - k\frac{\partial^2 u_h}{\partial x^2} &= 0 \quad 0 < x < L,\ t > 0, \\ u_h(0,t) = 0 &= u_h(L,t) \quad t > 0, \\ u_h(x,0) &= f(x) - v(x) \quad 0 \le x \le L. \end{aligned}$$

for the homogeneous heat equation. Show that the function $u(x,t) = u_h(x,t) + v(x)$ is a solution to the initial/boundary value problem in (2.38).

20. Use Exercise 19 to find the solution to the initial/boundary value problem in (2.38) with $k = 1$, $L = 1$, $p(x)/c\rho = 6x$, $A = 0$, $B = 1$, and $f(x) = \sin \pi x$.

21. Use Exercise 19 to find the solution to the initial/boundary value problem in (2.38) with $k = 1$, $L = 1$, $p(x)/c\rho = e^{-x}$, $A = 1$, $B = -1/e$, and $f(x) = \sin 2\pi x$.

13.3 The Wave Equation

We will start with the derivation of the wave equation in one space dimension. We will be modeling the vibrations of a wire or a string that is stretched between two points. A violin string is a very good example. We will also look at two techniques for solving the wave equation.

Derivation of the wave equation in one space variable

We assume the string is stretched from $x = 0$ to $x = L$. We are looking for the function $u(x, t)$ that describes the vertical displacement of the wire at position x and at time t. We assume the string is fixed at both endpoints, so $u(0, t) = u(L, t) = 0$ for all t. We will ignore the force of gravity, so at equilibrium we have $u(x, t) = 0$ for all x and t, which means that the string is in a straight line between the two fixed endpoints.

To derive the differential equation that models a vibrating string, we have to make some simplifying assumptions. In mathematical terms the assumptions amount to assuming that both $u(x, t)$, the displacement of the string, and $\partial u/\partial x$, the slope of the string, are small in comparison to L, the length of the string.

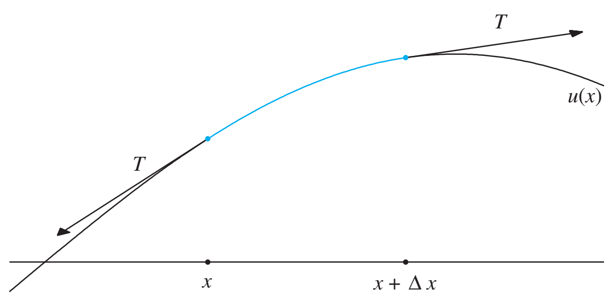


Figure 1 The forces acting on a portion of a vibrating string.

Consider the portion of the string above the small interval between x and $x + \Delta x$, as illustrated in blue in Figure 1. The forces acting on this portion come from the tension T in the string. The tension is a force that the rest of the string exerts on this particular part. For the portion in Figure 1, tension acts at the endpoints. We assume that the tension is so large that the string acts as if it were perfectly flexible and can bend without the requirement of a bending force. With that assumption, the tension acts tangentially to the string.

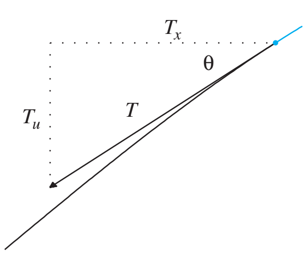


Figure 2 The resolution of the tension at the point *x*.

The tension at the point x is resolved into its horizontal and vertical components in Figure 2. We are assuming that the positive direction is upward. The vertical component is $T_u = -T \sin\theta$, and the horizontal component is $T_x = -T \cos\theta$. The slope of the graph of u at the point x is

$$\frac{\partial u}{\partial x} = \tan\theta.$$

We are assuming that the slope is very small, so θ is small. Therefore, $\cos\theta \approx 1$, and $\tan\theta \approx \sin\theta$. As a result, we have

$$T_u \approx -T\frac{\partial u}{\partial x}(x,t) \quad \text{and} \quad T_x \approx -T.$$

In a similar manner, we find that horizontal component of the force at $x + \Delta x$ is approximately T, which cancels the horizontal component at x. More interesting is the fact that the vertical component of the force at $x + \Delta x$ is approximately

$$T\frac{\partial u}{\partial x}(x+\Delta x,t),$$

so the total force acting in the vertical direction on the small portion of the string is

$$F \approx T\left(\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\right).$$

The length of the segment of string is close to Δx. If the string is uniform and has linear mass density ρ, then the mass of the segment is $m = \rho\Delta x$. The acceleration of the segment in the vertical direction is $\partial^2 u/\partial t^2$. By Newton's second law, we have $F = ma$, which translates into

$$\rho\Delta x\frac{\partial^2 u}{\partial t^2} \approx T\left(\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\right).$$

Dividing by Δx and taking the limit as Δx goes to 0, we have

$$\rho\frac{\partial^2 u}{\partial t^2} = T\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\right) = T\frac{\partial^2 u}{\partial x^2}.$$

If we set $c^2 = T/\rho$, the equation becomes

$$u_{tt} = c^2 u_{xx}. \tag{3.1}$$

This is the wave equation in one space variable. The constant c has dimensions length/time, so it is a velocity.

Notice that the homogeneous wave equation in (3.1) is linear. Once again we can build complicated solutions out of simpler ones.

Solution to the wave equation by separation of variables

Let's turn to the solution of the equation for the vibrating string. Since the wave equation is of order 2 in t, we are required to specify the initial velocity of the string as well as the initial displacement. Thus we are led to the initial/boundary value problem

$$\begin{aligned} u_{tt}(x,t) &= c^2 u_{xx}(x,t), \quad \text{for } 0 < x < L \text{ and } t > 0, \\ u(0,t) &= 0 \quad \text{and} \quad u(L,t) = 0, \quad \text{for } t > 0, \\ u(x,0) &= f(x) \quad \text{and} \quad u_t(x,0) = g(x), \quad \text{for } 0 \le x \le L. \end{aligned} \tag{3.2}$$

We will find the solution using separation of variables. Since the process is similar to that used in previous examples, we will omit some of the details. Notice that the boundary conditions in (3.2) are homogeneous, so we can proceed directly with the separation of variables. The starting point is to look for product solutions of the form $u(x,t) = X(x)T(t)$.

Step 1: Separate the PDE into two ODEs. Inserting $u(x, t) = X(x)T(t)$ into the wave equation and separating variables gives

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2T(t)}.$$

Since x and t are independent variables, each side of this equation must equal a constant, which we will denote by $-\lambda$. Thus the factors must satisfy the differential equations

$$X'' + \lambda X = 0 \quad \text{and} \quad T'' + \lambda c^2 T = 0. \tag{3.3}$$

Step 2: Set up and solve the two-point boundary value problem. The first equation in (3.3) together with the boundary condition $u(0, t) = 0 = u(L, t)$ implies that X must solve the two-point boundary value problem

$$X''(x) + \lambda X(x) = 0 \quad \text{with} \quad X(0) = 0 = X(L). \tag{3.4}$$

We have seen this Sturm Liouville problem before in (2.10). The solutions, given in (2.12), are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{and} \quad X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \qquad \text{for } n = 1, 2, 3, \ldots.$$

Step 3: Satisfying the initial conditions. With $\lambda_n = n^2\pi^2/L^2$, the second equation in (3.3) is

$$T'' + \left(\frac{cn\pi}{L}\right)^2 T = 0.$$

The functions $\cos(cn\pi t/L)$ and $\sin(cn\pi t/L)$ form a fundamental set of solutions. Consequently, we have found the product solutions

$$u_n(x, t) = \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{cn\pi t}{L}\right) \quad \text{and} \quad v_n(x, t) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{cn\pi t}{L}\right),$$

for $n = 1, 2, 3, \ldots$. Since the wave equation is linear, the function

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} [a_n u_n(x, t) + b_n v_n(x, t)] \\ &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[a_n \cos\left(\frac{cn\pi t}{L}\right) + b_n \sin\left(\frac{cn\pi t}{L}\right)\right] \end{aligned} \tag{3.5}$$

is a solution to the wave equation for any choice of the coefficients a_n and b_n that ensures that the series will converge. Further, $u(x, t)$ also satisfies the homogeneous boundary conditions.

The first initial condition is

$$f(x) = u(x, 0) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}.$$

To satisfy this condition, we choose the coefficients a_n to be

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \tag{3.6}$$

the Fourier sine coefficients for f. The second initial condition involves the derivative $u_t(x,t)$. Differentiating (3.5) term by term, we see that

$$u_t(x,t) = \sum_{n=1}^{\infty}\frac{cn\pi}{L}\sin\left(\frac{n\pi x}{L}\right)\left[-a_n\sin\left(\frac{cn\pi t}{L}\right) + b_n\cos\left(\frac{cn\pi t}{L}\right)\right].$$

The second initial condition now becomes

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} b_n\frac{cn\pi}{L}\sin\frac{n\pi x}{L}.$$

Therefore, $b_n cn\pi/L$ should be the Fourier sine coefficients for g, or

$$b_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx. \tag{3.7}$$

Inserting the values of a_n and b_n into (3.5) gives the complete solution to the wave equation.

Notice that every solution is an infinite linear combination of the product solutions

$$\sin\left(\frac{cn\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \quad \text{and} \quad \cos\left(\frac{cn\pi t}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

These solutions are periodic in time with frequency $\omega_n = nc\pi/L$. All of these frequencies are integer multiples of the *fundamental frequency* $\omega_1 = c\pi/L$. In music the contributions for $n > 1$ are referred to as *higher harmonics*. It is the fundamental frequency that our ears focus on, but the higher harmonics add body to the sound. This coupling of a fundamental frequency with the higher harmonics is thought to be accountable for the pleasing sound of a vibrating string. We will see later that the situation is different for the vibrations of a drum.

EXAMPLE 3.8 ◆ Suppose that a string is stretched and fixed at $x = 0$ and $x = \pi$. The string is plucked in the middle, which means that its shape is described by[6]

$$f(x) = \begin{cases} x, & \text{if } 0 \le x \le \pi/2, \\ \pi - x, & \text{if } \pi/2 \le x \le \pi. \end{cases}$$

At $t = 0$ the string is released with initial velocity $g(x) = 0$. Find the displacement of the string as a function of x and t. Assume that for this string we have $c = 0.002$.

[6] The wave equation was derived under the assumption that the displacement and the slopes were small. While this is not true of this and the other examples that we will examine, it is true for an initial displacement of, say, $0.001 \times f(x)$. Since the wave equation is linear, the solution with this initial condition is $0.001\times$ the solution we find in Example 3.8.

The solution is given by (3.5). We have only to find the coefficients a_n and b_n. Since $g(x) = 0$, we have $b_n = 0$. The coefficients a_n are the Fourier sine coefficients of f on the interval $(0, \pi)$, and they are given by

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

Inserting the definition of f, and evaluating the integral, we find that $a_n = 0$ if n is even, and if $n = 2k + 1$ is odd we have

$$a_{2k+1} = (-1)^k \frac{4}{\pi(2k+1)^2}.$$

Substituting into (3.5), we see that

$$u(x,t) = \sum_{k=0}^{\infty} (-1)^k \frac{4}{\pi(2k+1)^2}\sin(2k+1)x \cdot \cos 0.002(2k+1)t \tag{3.9}$$

is the solution. ◆

The rate of convergence

The general term in the series in equation (3.5) is

$$\sin\left(\frac{n\pi x}{L}\right)\left[a_n \cos\left(\frac{cn\pi t}{L}\right) + b_n \sin\left(\frac{cn\pi t}{L}\right)\right]. \tag{3.10}$$

The first factor, $\sin(n\pi x/L)$ is bounded in absolute value by 1. We can express the second factor in terms of its amplitude and phase,

$$a_n \cos\left(\frac{cn\pi t}{L}\right) + b_n \sin\left(\frac{cn\pi t}{L}\right) = A_n \cos\left(\frac{cn\pi t}{L} + \phi_n\right), \tag{3.11}$$

where the amplitude $A_n = \sqrt{a_n^2 + b_n^2}$. Thus, the general term in (3.10) is bounded by A_n for all $t > 0$. We can judge the convergence of the solution in equation (3.5) by the rate of convergence of $\sum_{n=1}^{\infty} A_n$. Notice that the rate of convergence of $\sum_{n=1}^{\infty} A_n$ does not change as t increases.

EXAMPLE 3.12 ◆ The displacement of the string in Example 3.8 is given by the series in (3.9). How many terms must be included if we approximate the solution by the sum including all terms satisfying $A_{2k+1} > 0.01$? How many if we include all terms satisfying $A_{2k+1} > 0.001$?

We see that $A_{2k+1} = 4/[\pi(2k+1)^2]$. For any acceptable error e, we have $A_{2k+1} < e$ if $k > 1/\sqrt{\pi e} - 1/2$. Thus for an acceptable error of $e = 0.01$ we must keep all terms with $k \le 5$, and for $e = 0.001$ terms with $k \le 17$ are needed. ◆

Comparing Examples 2.23 and 3.12, we see that many more terms are needed to get the required accuracy for solutions to the wave equation than are needed for solutions to the heat equation. The exponential decay of the terms in the solution to the heat equation makes the series converge much faster.

D'Alembert's solution

Let's examine another approach to solving the wave equation in one space variable. We start by finding all solutions to the wave equation

$$u_{tt}(x,t) = c^2 u_{xx}(x,t) \tag{3.13}$$

without worrying about initial or boundary conditions. We do this by introducing new variables $\xi = x + ct$ and $\eta = x - ct$. By the chain rule,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}.$$

Similarly, we have $u_t = c\left[u_\xi - u_\eta\right]$. Differentiating once more using the chain rule, we see that

$$u_{xx} = [u_\xi + u_\eta]_x = [u_\xi + u_\eta]_\xi + [u_\xi + u_\eta]_\eta = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}.$$

Similarly, $u_{tt} = c^2[u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}]$. Therefore, $u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta}$. Consequently, in the new variables the wave equation has the form $u_{\xi\eta} = 0$.

If we read this equation as

$$\frac{\partial}{\partial \eta} u_\xi = 0,$$

we can integrate to find that

$$u_\xi(\xi,\eta) = H(\xi),$$

where $H(\xi)$ is an arbitrary[7] function of ξ. We can now integrate once more to find that

$$u(\xi,\eta) = \int H(\xi)\, d\xi + G(\eta),$$

where $G(\eta)$ is an arbitrary function of η. If we set $F(\xi) = \int H(\xi)\, d\xi$, we find that

$$u(\xi,\eta) = F(\xi) + G(\eta),$$

where F and G are arbitrary functions.

In terms of the original variables, we see that every solution to the wave equation (3.13) has the form

$$u(x,t) = F(x + ct) + G(x - ct), \tag{3.14}$$

where F and G are arbitrary functions. It is easily verified that any function of the form in (3.14) is a solution to the wave equation. The general solution to the wave equation in (3.14) is called the ***d'Alembert solution***.

[7] Although the argument used requires that H is a differentiable function, it is really true that H can be an arbitrary function. The same is true for the functions F and G that follow.

Traveling waves

If we choose $F = 0$ in (3.14), we see that $u(x, t) = G(x - ct)$ is a solution to the wave equation. Let's get an idea of what this solution looks like. Figure 3 shows the graph of a function $G(x)$ that is a nonzero bump centered at $x = 0$. Figure 4 shows the graph of $G(x - ct)$, where $t > 0$ is fixed. Notice that the graph of $G(x - ct)$ is now centered at $x = ct$. From this we see that as t increases, the solution $u(x, t) = G(x - ct)$ to the wave equation has a graph versus x that is a bump moving to the right as t increases. Furthermore, since the wave has moved a distance ct in time t, it is moving to the right with speed c.

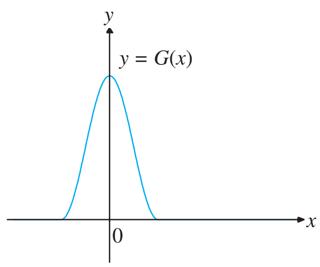


Figure 3 The graph of $G(x)$.

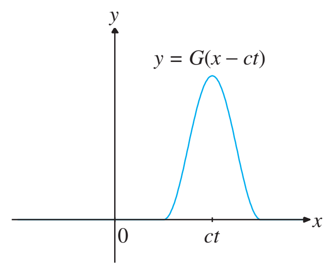


Figure 4 The graph of $G(x - ct)$ for $t > 0$.

Similarly, the solution $F(x + ct)$ represents a wave moving to the left with speed c as t increases. We will call solutions of the form $G(x - ct)$ and $F(x + ct)$ ***traveling waves***.

As a result, we see that the d'Alembert solution in (3.14) represents the general solution to the wave equation (3.13) as the sum of two traveling waves, one moving to the right with speed c and the other moving to the left with speed c.

Solving the initial/boundary value problem

The d'Alembert solution in (3.14) can be used to find the solution to the initial/boundary value problem that we encountered in (3.2). To make the argument somewhat easier to follow, we will make the assumption that the initial velocity is 0, so the initial/boundary value problem we will solve is

$$\begin{aligned} u_{tt}(x, t) &= c^2 u_{xx}(x, t), \quad \text{for } 0 < x < L \text{ and } t > 0, \\ u(0, t) = 0 \quad &\text{and} \quad u(L, t) = 0, \quad \text{for } t > 0, \\ u(x, 0) = f(x) \quad &\text{and} \quad u_t(x, 0) = 0, \quad \text{for } 0 \le x \le L. \end{aligned} \tag{3.15}$$

In the process we will gain additional information about the solution.

We start with a d'Alembert solution $u(x, t) = F(x+ct)+G(x-ct)$ from (3.14). We will use the initial and boundary conditions in (3.15) to find out what F and G have to be. We will assume that F and G are defined for all values of x. Observe that $u_t(x, t) = c[F'(x + ct) - G'(x - ct)]$. Therefore, the initial conditions imply that

$$\begin{aligned} f(x) = u(x, 0) &= F(x) + G(x), \quad \text{and} \\ 0 = u_t(x, 0) &= c[F'(x) - G'(x)], \end{aligned}$$

for $0 \le x \le L$. The second equation can be integrated to yield $F(x) - G(x) = C$, where C is a constant. Solving these two linear equations, we get $F(x) = [f(x) + C]/2$ and $G(x) = [f(x) - C]/2$ for $0 \le x \le L$. When we substitute into (3.14), we see that the constant cancels, so we may as well take $C = 0$. Thus we have

$$F(x) = G(x) = \frac{1}{2} f(x), \quad \text{for } 0 \le x \le L. \tag{3.16}$$

Next we use the boundary conditions. Setting $x = 0$ in (3.14), and using (3.16), we obtain $0 = u(0, t) = F(ct) + F(-ct)$, or $F(-ct) = -F(ct)$ for $t > 0$. Consequently, F must be an odd function. From (3.16) we get

$$F(x) = \frac{1}{2} f_o(x), \quad \text{for } -L \le x \le L, \tag{3.17}$$

where f_o is the odd extension of f.[8]

The second boundary condition is

$$0 = u(L, t) = F(L + ct) + F(L - ct).$$

If we set $ct = y + L$ in this formula, we get $F(y + 2L) + F(-y) = 0$. Using the fact that F is odd, this becomes

$$F(y + 2L) = -F(-y) = F(y).$$

This means that F must be periodic with period $2L$. Building on (3.17), we conclude that

$$F(x) = \frac{1}{2} f_{op}(x), \quad \text{for all } x \in \mathbf{R}.$$

where f_{op} is the odd periodic extension of f to the whole real line. Thus the solution to the initial/boundary value problem in (3.15) is

$$u(x, t) = \frac{1}{2}\left[f_{op}(x + ct) + f_{op}(x - ct)\right]. \tag{3.18}$$

EXAMPLE 3.19 ◆ Suppose that a string of length 1 m originally has the shape of the graph on the left in Figure 5, and has initial velocity 0. Assuming that $c = 1\text{m/sec}$, find the displacement of the string as a function of x and t.

The mathematical formula for the function f is given by

$$f(x) = \begin{cases} x - 3/8, & \text{for } 3/8 \le x \le 1/2 \\ 5/8 - x, & \text{for } 1/2 \le x \le 5/8 \\ 0, & \text{otherwise.} \end{cases}$$

According to the previous discussion, the solution is given by (3.18). The graph of the odd extension of f is given on the right in Figure 5.

Figure 6 shows the displacement of the string at several times. Notice how the initial wave splits into a forward wave and a backward wave, which then reflect when they hit the boundary points at $x = 0$ and $x = 1$. ◆

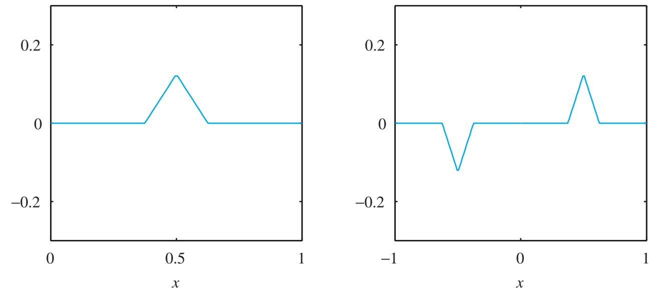


Figure 5 The initial displacement $f(x)$ for the string in Example 3.19, and its odd extension.

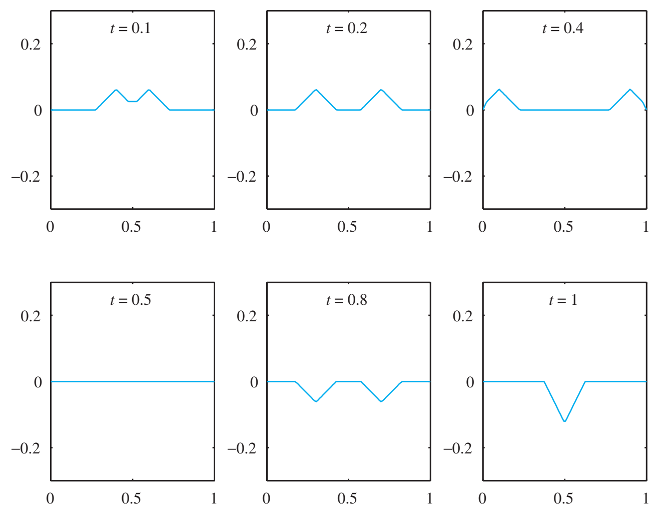


Figure 6 The displacement of the string in Example 3.19 at several times.

EXERCISES

In Exercises 1–6, use Fourier series to find the displacement $u(x,t)$ of the string of length L with fixed endpoints, initial displacement $u(x,0) = f(x)$, and initial velocity $u_t(x,0) = g(x)$. Assume that $c = 1$.

1. $f(x) = x(1-x)/4$, $g(x) = 0$, and $L = 1$

2. $f(x) = \begin{cases} x/10, & \text{for } 0 \le x \le 5 \\ 1 - x/10, & \text{for } 5 \le x \le 10, \end{cases}$ $g(x) = 0$, and $L = 10$

3. $f(x) = 0$, $g(x) = 1$, and $L = 1$

4. $f(x) = 0$, $g(x) = \begin{cases} -1/2, & \text{for } 0 \le x \le 1/2 \\ 1/2, & \text{for } 1/2 \le x \le 1, \end{cases}$ and $L = 1$

[8] We studied periodic extensions in Section 12.2 and odd and even extensions in Section 12.3.

5. $f(x) = 0$, $g(x) = \begin{cases} 1, & \text{for } 1 \le x \le 2 \\ 0, & \text{otherwise,} \end{cases}$ and $L = 3$

6. $f(x) = x(1 - x)/4$, $g(x) = -1$ and $L = 1$.

In Exercises 7–8, use the d'Alembert solution (3.14) to find the displacement $u(x, t)$ of the string of length L with fixed endpoints, initial displacement $u(x, 0) = f(x)$, and initial velocity $u_t(x, 0) = 0$. Sketch the solution as a function of x, $0 \le x \le L$, for the specific values of t that are given.

7. $c = 2$, $f(x) = \sin \pi x$, and $L = 1$. Plot $u(x, t)$ as a function of x for $t = 0,\ 1/8,\ 1/4,\ 1/2,\ 3/4,\ 1$.

8. $c = 1$, $L = 10$, and $f(x) = \begin{cases} 0, & \text{for } 0 \le x \le 5, \\ x - 5, & \text{for } 5 < x \le 6, \\ 7 - x, & \text{for } 6 < x \le 7, \\ 0, & \text{for } 7 < x \le 10. \end{cases}$

Plot $u(x, t)$ as a function of x for $t = 2,\ 4,\ 6,\ 8,\ 10,\ 12$.

9. Suppose that we have a string of length $L = 1$ with fixed endpoints, and $c = 1$. In this section we discussed two methods of finding the displacement $u(x, t)$ of the string with initial displacement $u(x, 0) = f(x)$, and initial velocity $u_t(x, 0) = g(x) = 0$. The first solution is

$$u_1(x, t) = \sum_{n=1}^{\infty} a_n \sin n\pi x \cos n\pi t,$$

where a_n are the Fourier sine coefficients for f, i.e. $f(x) = \sum_n a_n \sin n\pi x$. The second solution is d'Alembert's solution,

$$u_2(x, t) = \frac{1}{2}\left(f_{op}(x + t) + f_{op}(x - t) \right).$$

Show that these two solutions are the same. (*Hint:* Use the trigonometric identity $\sin A \cos B = [\sin(A + B) + \sin(A - B)]/2$ to transform u_1 into u_2.)

10. Use the method of separation of variables to find the general solution for the initial/boundary value problem

$$\begin{aligned} u_{tt}(x, t) + u_t(x, t) + u(x, t) &= u_{xx}(x, t), \quad \text{for } 0 < x < 1 \text{ and } t > 0, \\ u(0, t) = 0 &= u(1, t) \quad \text{for } t > 0, \\ u(x, 0) &= f(x) \quad \text{for } 0 \le x \le 1, \\ u_t(x, 0) &= 0 \quad \text{for } 0 \le x \le 1. \end{aligned}$$

Express the solution in terms of the Fourier sine coefficients of the function f on the interval $0 \le x \le 1$. The differential equation in this problem is called the ***telegraph equation.***

11. D'Alembert's solution can also be used to find the displacement $u(x,t)$ of a string with fixed endpoints having initial displacement $u(x,0) = 0$ and initial velocity $u_t(x,0) = g(x)$. Follow the derivation of the solution in (3.18) to show that the solution is given by

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g_{op}(s)\,ds,$$

where g_{op} is the odd periodic extension of g. (*Hint:* At some point it will be necessary to know that the derivative of an even function is odd, and vice versa.)

12. Use Exercise 11 to find the displacement $u(x,t)$ of a string of length L with fixed endpoints, where $c = 1$, $u(x,0) = 0$ and ,

$$u_t(x,0) = g(x) = \begin{cases} 0 & \text{for } 0 \le x < 1, \\ 2 & \text{for } 1 \le x \le 2, \\ 0 & \text{for } 2 < x \le 3. \end{cases}$$

Plot $u(x,t)$ as a function of x for $t = 0,\ 0.25,\ 0.5,\ 1.5,\ 4.5,\ 6$.

13. Use Exercise 11 and the solution in (3.18) to show that the displacement $u(x,t)$ of a string of length L with fixed endpoints having initial displacement $u(x,0) = f(x)$ and initial velocity $u_t(x,0) = g(x)$ is

$$u(x,t) = \frac{1}{2}\left(f_{op}(x+ct) + f_{op}(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g_{op}(s)\,ds.$$

14. The displacement of a wire or string that is stretched horizontally between two fixed endpoints actually satisfies the equation

$$u_{tt} = c^2 u_{xx} - g, \tag{3.20}$$

where g is the aceleration due to gravity. Usually the force of gravity is ignored because it is so much smaller than the tension in the string. In this exercise we will consider a string of length L, and include gravity.

(a) Find a steady-state solution v. This means that v is independent of t and satisfies equation (3.20) and the boundary conditions. If $u(x,t)$ is a solution to (3.20), what equation does $w(x,t) = u(x,t) - v(x)$ satisfy?

(b) Use separation of variables to find the solution $u(x,t)$ to (3.20) which satisfies the boundary conditions $u(0,t) = u(L,t) = 0$ and the initial conditions $u(x,0) = 0 = u_t(x,0)$ for $0 \le x \le L$.

15. The total energy in a vibrating string is

$$E(t) = \frac{1}{2}\int_0^L \left[\rho u_t^2 + T u_x^2\right] dx. \tag{3.21}$$

Show that if $u(0,t) = 0 = u(L,t)$ for all $t > 0$, then $E(t)$ is constant. Thus, the energy in the string is conserved. (*Hint:* Differentiate (3.21) under the integral. Then use the wave equation and prove that $u_t(0,t) = 0 = u_t(L,t)$ for all $t > 0$.)

16. If you pluck a violin string, and then finger the string, fixing it precisely in the middle, the tone increases by one octave. In mathematical terms this means that the frequency is doubled. Explain why this happens.

17. Our derivation of the wave equation ignored any damping effects of the medium in which the string is vibrating. If damping is taken into account, the equation becomes

$$u_{tt} = c^2 u_{xx} - 2ku_t, \tag{3.22}$$

where k is a damping constant which we will assume satisfies $0 < k < \pi c/L$, where L is the length of the string.

(a) Find all product solutions $u(x, t) = X(x)T(t)$ to (3.22) which satisfy the boundary conditions $u(0, t) = u(L, t) = 0$ for $t > 0$.

(b) Find a series representation for the solution $u(x, t)$ which satisfies the boundary conditions and the initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$.

13.4 Laplace's Equation

So far we have considered partial differential equations where there was only one spatial dimension. Now we want to begin to study situations where there is more than one. Our discussion will be limited to the three most important examples, Laplace's equation, the heat equation, and the wave equation.

The Laplacian operator and Laplace's equation

The Laplacian operator is a part of all of the partial differential equations we will discuss. The discussion naturally begins with the gradient. In two spatial dimensions the gradient of a function $u(x, y)$ is

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)^T.$$

For a function $u(x, y, z)$ of three variables we have

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)^T.$$

In greater generality, for a function $u(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbf{R}^n$, the gradient is the vector

$$\nabla u(\mathbf{x}) = \left(\frac{\partial u}{\partial x_1}(\mathbf{x}), \frac{\partial u}{\partial x_2}(\mathbf{x}), \ldots, \frac{\partial u}{\partial x_n}(\mathbf{x})\right)^T.$$

This equation also defines the gradient as a vector valued differential operator, which we write as

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)^T.$$

Notice that in dimension $n = 1$, the gradient is just the ordinary derivative,

$$\nabla u = \frac{du}{dx}.$$

The ***Laplacian operator*** or, more simply, the ***Laplacian***, is roughly the "square" of the gradient operator. It is denoted by ∇^2, and it is defined by[9]

$$\nabla^2 u = \nabla \cdot \nabla u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}.$$

Observe that the dot in $\nabla \cdot \nabla$ is the vector dot product. Thus the Laplacian operator is the square of the gradient operator if we are using the dot product. Using the subscript notation for derivatives, the notation specializes in low dimensions to

$$\begin{aligned} \nabla^2 u(x) &= u_{xx}(x), \quad \text{for } n = 1, \\ \nabla^2 u(x, y) &= u_{xx}(x, y) + u_{yy}(x, y), \quad \text{for } n = 2, \\ \nabla^2 u(x, y, z) &= u_{xx}(x, y, z) + u_{yy}(x, y, z) + u_{zz}(x, y, z), \quad \text{for } n = 3. \end{aligned}$$

The equation

$$\nabla^2 u(\mathbf{x}) = 0 \tag{4.1}$$

is called ***Laplace's*** equation. We have seen that in one space dimension, steady-state temperatures satisfy Laplace's equation. This is true in two or three dimensions as well. There are many other applications. For example, a conservative force $\mathbf{F}$ has a potential u, which is a function for which $\mathbf{F} = -\nabla u$. If in addition the force is divergence free, then the potential u satisfies Laplace's equation. In particular, this applies to an electric force in regions of space where there are no charges present, or to a gravitational force in regions where there is no mass.

A solution to Laplace's equation is called a ***harmonic function***. Laplace's equation and harmonic functions are widely studied by mathematicians, both for their important applications and because of their intrinsic interest.

The heat equation

In one space dimension temperatures satisfy the heat equation $u_t = ku_{xx}$. If we replace u_{xx} by the Laplacian of u, the same is true in higher dimensions. Thus, if $u(\mathbf{x}, t)$ represents the temperature at a point $\mathbf{x}$ in space and at time t, then u satisfies the ***heat equation***

$$\frac{\partial u}{\partial t}(\mathbf{x}, t) = k\nabla^2 u(\mathbf{x}, t), \tag{4.2}$$

where k is a constant called the ***thermal diffusivity***. In low dimensions we can write the heat equation as

$$u_t = k(u_{xx} + u_{yy}), \text{ for } n = 2, \quad \text{and} \quad u_t = k(u_{xx} + u_{yy} + u_{zz}), \text{ for } n = 3.$$

A steady-state temperature is a temperature which does not depend on t. Notice that for a steady-state temperature u, the heat equation in (4.2) reduces to Laplace's equation (4.1).

[9] Many mathematicians and scientists use the notation $\Delta u = \nabla^2 u$. However, we will follow the usage that we think is most common.

The wave equation

In one space dimension the displacement of a vibrating string satisfies the wave equation $u_{tt} = c^2 u_{xx}$. Once again, if we replace u_{xx} by the Laplacian of u, then wave phenomena in higher dimensions satisfy the same equation. The wave equation is the equation

$$u_{tt} = c^2 \nabla^2 u,$$

where c is a constant that has the dimensions of velocity. The wave equation describes a variety of oscillatory behavior. For example, in two dimensions it describes the motion of a drum head. In three dimensions it describes electromagnetic waves.

Linearity

Laplace's equation, the heat equation, and the wave equation are all linear equations. We will use this to build up more and more complicated solutions as linear combinations of more basic solutions.

Boundary conditions for Laplace's equation

In this section and the next we will find solutions to Laplace's equation in a rectangle and in a disk in the plane $\mathbf{R}^2$. We could do this with any of the boundary conditions we discussed for the heat equation in Section 1. For example, the ***Dirichlet problem*** is to solve the boundary value problem

$$\begin{aligned} \nabla^2 u(x, y) = u_{xx} + u_{yy} = 0, &\quad \text{for } (x, y) \in D, \\ u(x, y) = f(x, y), &\quad \text{for } (x, y) \in \partial D, \end{aligned} \tag{4.3}$$

where D is a region in $\mathbf{R}^2$ and ∂D is its boundary. The boundary condition $u(x, y) = f(x, y)$ is called a ***Dirichlet condition***. The problem of finding a function u satisfying (4.3) for a given f defined on the boundary ∂D is called the ***Dirichlet problem***. Notice that being able to solve the Dirichlet problem means that the steady-state temperature in a region D is completely determined by the temperature on the boundary ∂D.

If the boundary of the region ∂D is insulated, there is no flow of heat across the boundary. This means that the temperature is not varying in the direction normal to the boundary. Let $\mathbf{n}(x, y)$ denote the vector of length 1 at the point $(x, y) \in \partial D$, which is orthogonal to the boundary at (x, y) and points out of D. The vector $\mathbf{n}$ is called the ***unit exterior normal*** to the boundary of D, and $\partial u/\partial \mathbf{n} = \nabla u \cdot \mathbf{n}$ is called the ***normal derivative*** of u. Since the temperature is not varying in the direction $\mathbf{n}$, $\partial u/\partial \mathbf{n} = 0$. More generally, we can specify the normal derivative at each point of the boundary. Then we would have

$$\frac{\partial u}{\partial \mathbf{n}}(x, y) = g(x, y), \quad \text{for } (x, y) \in \partial D, \tag{4.4}$$

where g is a function defined on the boundary of D. This is called a ***Neumann condition***. If we replace the Dirichlet condition in (4.3) with the Neumann condition, the problem is called the ***Neumann problem***. We could also impose a ***Robin condition***

$$\frac{\partial u}{\partial \mathbf{n}}(x, y) - \alpha(x, y)u(x, y) = h(x, y), \quad \text{for } (x, y) \in \partial D, \tag{4.5}$$

where α and h are functions defined on the boundary of D, to get the ***Robin problem***.

The maximum principle for harmonic functions

Harmonic functions are solutions to Laplace's equation and therefore represent steady-state temperatures. Since heat flows from hot areas to colder areas, a steady-state temperature cannot be higher at one point than it is everywhere around it. Therefore, a solution to Laplace's equation cannot have a local maximum (or a local minimum). This fact is referred to as the ***maximum principle for harmonic functions.***

If $u(x, y)$ is a solution to the Dirichlet problem (4.3) in a region D, then it follows from the maximum principle that u achieves its maximum and minimum values on the boundary ∂D. This is also sometimes called the maximum principle.

The mean value property of harmonic functions

Suppose that u is a harmonic function in a region D. Suppose also that $\mathbf{p} = (x_0, y_0)^T \in D$, and that $r > 0$ is so small that the disk U of radius r and center $\mathbf{p}$ is completely contained in D. Then ***the mean value property of harmonic functions*** states that the value $u(\mathbf{p})$ is the average of u over U. In other words,

$$u(\mathbf{p}) = \frac{1}{\pi r^2} \int_U u(x, y)\, dx\, dy.$$

If you think of u as a steady-state temperature, you find that the temperature at any point must be the average of the temperatures in any disk centered at that point. Clearly, this fact reflects the fact that heat flows from hot to cold.

Solution on a rectangle with Dirichlet boundary conditions

We shall consider the Dirichlet problem for the rectangle

$$D = \{(x, y) \mid 0 < x < a \text{ and } 0 < y < b\}$$

illustrated in Figure 1.

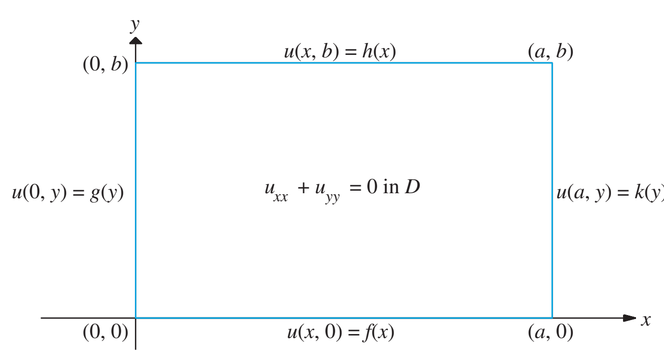


Figure 1 The Dirichlet problem for the rectangle D.

The boundary conditions specify the temperature u on each of the four sides as indicated in Figure 1. The full statement of the Dirichlet problem is

$$\begin{aligned} u_{xx}(x, y) + u_{yy}(x, y) &= 0, \quad \text{for } (x, y) \in D, \\ u(x, 0) = f(x) \quad \text{and} \quad u(x, b) &= h(x), \quad \text{for } 0 \le x \le a, \\ u(0, y) = g(y) \quad \text{and} \quad u(a, y) &= k(y), \quad \text{for } 0 \le y \le b, \end{aligned} \tag{4.6}$$

where f, g, h, and k are given functions.

We will reduce the problem to one that can be solved using separation of variables by imposing homogeneous boundary conditions on two opposite sides of the rectangle and the correct boundary condition from (4.6) on the remaining sides. The problem is to find u such that

$$\begin{aligned} u_{xx}(x, y) + u_{yy}(x, y) &= 0, \quad \text{for } (x, y) \in D, \\ u(x, 0) = f(x) \quad \text{and} \quad u(x, b) &= h(x), \quad \text{for } 0 \le x \le a, \\ u(0, y) = 0 \quad \text{and} \quad u(a, y) &= 0, \quad \text{for } 0 \le y \le b. \end{aligned} \tag{4.7}$$

There is the similar problem where homogeneous boundary conditions are imposed on the top and bottom of the rectangle, which can be solved using the same technique. Using the linearity of Laplace's equation, the sum of the two is the solution to (4.6).

We start the separation of variables by looking for product solutions of the form $u(x, y) = X(x)Y(y)$. We want u to satisfy the homogeneous boundary conditions, which means that $X(0) = X(a) = 0$. Substituting $u(x, y) = X(x)Y(y)$ into Laplace's equation, we obtain

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

Upon separating variables in the usual way, we obtain the differential equations

$$X'' + \lambda X = 0 \quad \text{and} \quad Y'' - \lambda Y = 0. \tag{4.8}$$

The function X must satisfy the homogeneous boundary conditions, so we want to solve the Sturm Liouville problem

$$X'' + \lambda X = 0 \quad \text{with } X(0) = X(a) = 0. \tag{4.9}$$

This is the same problem that arose in our study of the heat equation with Dirichlet conditions (see (2.10)). The eigenvalues and eigenfunctions are

$$\lambda_n = \frac{n^2\pi^2}{a^2} \quad \text{and} \quad X_n(x) = \sin\left(\frac{n\pi x}{a}\right), \quad \text{for } n = 1, 2, 3, \ldots. \tag{4.10}$$

The factor Y satisfies the differential equation $Y'' - \lambda Y = 0$. We now know that λ is one of the eigenvalues, so let's write $\lambda = \omega^2$, where $\omega = n\pi/a$. Then the equation $Y'' - \omega^2 Y = 0$ has the fundamental set of solutions $e^{\omega y}$ and $e^{-\omega y}$. While these are the standard solutions, it will be convenient to use

$$\sinh \omega y = \frac{e^{\omega y} - e^{-\omega y}}{2} \quad \text{and} \quad \sinh \omega(y - b) = \frac{e^{\omega(y-b)} - e^{-\omega(y-b)}}{2}.$$

These functions are linear combinations of $e^{\omega y}$ and $e^{-\omega y}$, so they are solutions to the equation $Y'' - \omega^2 Y = 0$. They are not multiples of each other, so they are linearly independent, and therefore form a fundamental set of solutions. The advantage for us is that $\sinh \omega y$ vanishes at $y = 0$, while $\sinh \omega(y - b)$ vanishes at $y = b$. This fact will facilitate finding the solution that satisfies the inhomogeneous boundary conditions.

Thus, for each positive integer n, we set $\omega = n\pi/a$ and we get two product solutions to Laplace's equation

$$\begin{aligned} u_n(x, y) &= \sinh\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \quad \text{and} \\ v_n(x, y) &= \sinh\left(\frac{n\pi(y - b)}{a}\right) \sin\left(\frac{n\pi x}{a}\right), \end{aligned} \tag{4.11}$$

that satisfy the homogeneous part of the boundary conditions.

Using the linearity of Laplace's equation, the function

$$\begin{aligned} u(x, y) &= \sum_{n=1}^{\infty} a_n u_n(x, y) + \sum_{n=1}^{\infty} b_n v_n(x, y) \\ &= \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \\ &\quad + \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi(y - b)}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \end{aligned} \tag{4.12}$$

is a solution to Laplace's equation for any constants a_n and b_n for which the series converges. In addition u satisfies the homogeneous part of the boundary conditions.

The coefficients a_n and b_n are chosen to satisfy the inhomogeneous boundary conditions in (4.7). For $y = 0$ the first sum in (4.12) vanishes, so

$$f(x) = u(x, 0) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi(-b)}{a}\right) \sin\left(\frac{n\pi x}{a}\right).$$

This will be recognized as the Fourier sine expansion of $f(x)$, so using equation (3.7) of Section 3 in Chapter 12, we have

$$b_n \sinh\left(\frac{n\pi(-b)}{a}\right) = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx. \tag{4.13}$$

Using the fact that sinh is an odd function, we solve for

$$b_n = \frac{-2}{a \sinh(n\pi b/a)} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx.$$

Similarly, the boundary condition at $y = b$ requires

$$a_n = \frac{2}{a \sinh(n\pi b/a)} \int_0^a h(x) \sin\left(\frac{n\pi x}{a}\right) dx. \tag{4.14}$$

EXAMPLE 4.15 ◆ Find the steady-state temperature $u(x, y)$ in a square plate 1 m on a side where $u(x, 1) = x - x^2$ for $0 \le x \le 1$, and $u(x, y) = 0$ on the other three sides.

The square is the rectangle in Figure 1 with $a = b = 1$. The boundary temperatures are given by $f = g = k = 0$, and $h(x) = x - x^2$. The solution is given by (4.12). Since $f(x) = 0$, (4.13) implies that $b_n = 0$. To compute a_n we use (4.14). We compute that

$$\int_0^1 (x - x^2) \sin n\pi x \, dx = \frac{2}{n^3\pi^3}(1 - \cos n\pi) = \begin{cases} \dfrac{4}{n^3\pi^3}, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

This can be accomplished by integrating by parts twice. Then, from (4.12) and (4.14) we conclude that

$$u(x, y) = \sum_{k=0}^{\infty} \frac{8}{(2k+1)^3\pi^3 \sinh(2k+1)\pi} \sin(2k+1)\pi x \cdot \sinh(2k+1)\pi y.$$

Truncating the above sum at $k = 10$, an approximate solution is graphed in Figure 2. Note that the solution agrees with the graph of $h(x) = x - x^2$ on the part of the boundary where $y = 1$. The boundary values of the other three sides are all zero, as specified in (4.7). ◆

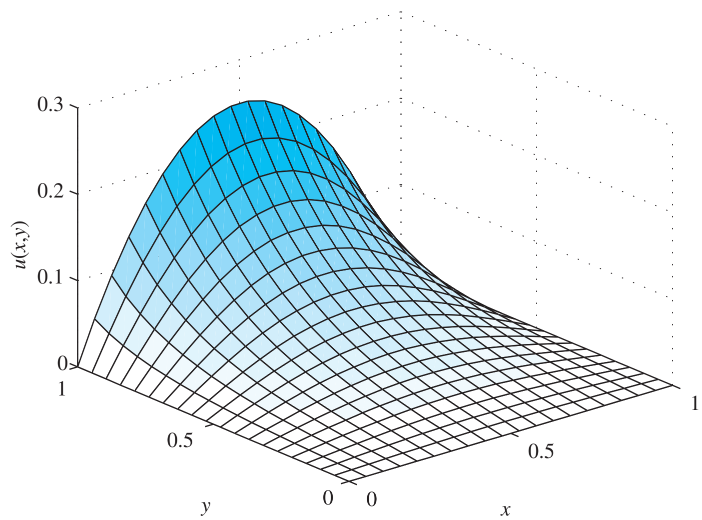


Figure 2 An approximate solution to the Dirichlet problem in Example 4.15.

EXERCISES

1. Consider a rectangular metal plate $a = 1$ m wide and $b = 2$ m long, as shown in Figure 1. Suppose that the temperature is 10°C on the bottom edge, and 0°C on the others. Find the steady-state temperature $u(x, y)$ throughout the plate.

2. Consider a rectangular metal plate $a = 10$ cm wide and $b = 25$ cm long, as shown in Figure 1. Suppose that the temperature is $u(x, 2) = 20$°C on the top edge, $u(x, 0) = (100°\text{C})x$ on the bottom, and 0°C on the others. Find the steady-state temperature $u(x, y)$ throughout the plate.

3. Show that when $f = h = 0$, the general solution of the boundary value problem in (4.6) is

$$u(x, y) = \sum_{n=1}^{\infty} \left[a_n \sinh \frac{n\pi x}{b} + b_n \sinh \frac{n\pi(x-a)}{b} \right] \sin \frac{n\pi y}{b},$$

where

$$a_n = \frac{2}{b \sinh(n\pi a/b)} \int_0^b k(y) \sin(n\pi y/b)\, dy$$

and

$$b_n = \frac{-2}{b \sinh(n\pi a/b)} \int_0^b g(y) \sin(n\pi y/b)\, dy.$$

4. Suppose that you have a square plate for which the temperature on one side is kept at a uniform temperature of 100°, and the other three sides are kept at 0°. What is the temperature in the middle of the square? (*Hint:* Don't do any compuation of series. Use physical intuition, the symmetry of the square, and the linearity of the Laplacian.)

Exercises 5–10 are concerned with the boundary value problem in (4.6) in the rectangle D of width a and height b shown in Figure 1. Compute the solution for the given boundary functions f, g, h, and k. Draw a hand sketch of what you think the graph of u over D should be and then compare with a computer drawn graph of the exact solution or of the first 10 terms or so of the solution.

5. $a = b = 1$, $f = g = k = 0$, and

$$h(x) = \begin{cases} x, & \text{for } 0 \le x \le 1/2, \\ 1 - x, & \text{for } 1/2 < x \le 1 \end{cases}$$

6. $a = 2$, $b = 1$, $f = g = k = 0$, and

$$h(x) = \begin{cases} 1, & \text{for } 0 \le x \le 1, \\ -1, & \text{for } 1 < x \le 2 \end{cases}$$

7. $a = 1$, $b = 1$, $f(x) = \sin(2\pi x)$, and $g = h = k = 0$

8. $a = 1$, $b = 2$, $f(x) = \sin^2(\pi x)$, and $g = h = k = 0$

9. $a = 1$, $b = 2$, $f(x) = -1$, $h(x) = 1$, and $g = k = 0$

10. $a = b = 1$, $g = k = 0$, $f(x) = \sin(2\pi x)$, and

$$h(x) = \begin{cases} x, & \text{for } 0 \le x \le 1/2, \\ x - 1, & \text{for } 1/2 < x \le 1 \end{cases}$$

11. (a) Consider the rectangle $D_L = \{(x, y) \mid 0 < x < 1 \quad \text{and} \quad 0 < y < L\}$. Compute the solution of the boundary value problem

$$\begin{aligned} u_{xx}(x, y) + u_{yy}(x, y) &= 0 \quad \text{for } (x, y) \in D \\ u(x, 0) = f(x) \quad \text{and} \quad u(x, L) &= 0 \quad \text{for } 0 \le x \le 1, \\ u(0, y) = 0 \quad \text{and} \quad u(a, y) &= 0 \quad \text{for } 0 \le y \le L, \end{aligned}$$

where $f(x)$ is a piecewise differentiable function on the interval $0 \le x \le 1$, with the Fourier sine series $f(x) \sim \sum_{n=0}^{\infty} B_n \sin(n\pi x)$.

(b) Consider the infinite strip $D = \{(x, y) \mid 0 < x < 1 \quad \text{and} \quad 0 < y < \infty\}$. Find the temperature $u(x, y)$ on D which is equal to 0 on the infinite sides and satiasfies $u(x, 0) = f(x)$ for $0 \le x \le 1$. (*Hint:* Use part (a) to solve on the rectangle with bounds $0 \le x \le L$, and $u(L, y) = 0$. Then let L increase to ∞ and find the limiting temperature.)

12. Show that the steady-state temperature in a region D is completely determined by the temperature on the boundary ∂D. In other words, if $u(x, y)$ and $v(x, y)$ are two possible steady-state temperatues which satisfy $u(x, y) = v(x, y)$ for every point $(x, y) \in \partial D$, then $u(x, y) = v(x, y)$ at every point $(x, y) \in D$. (*Hint:* Consider $w = u - v$ and apply the maximum principle.)

13.5 Laplace's Equation on a Disk

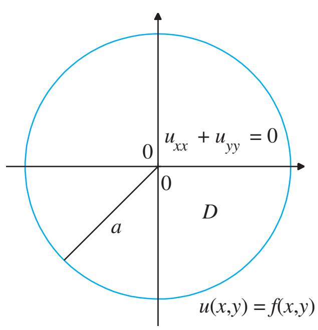


Figure 1 The Dirichlet problem on the disk.

Now we turn our attention to finding steady-state temperatures in regions with circular symmetry. One example is a metal disk. Another is a pipe, which has a cross-section which is a ring or annulus, described mathematically as the region between two concentric circles.

Finding a steady-state temperature involves solving the Dirichlet problem (4.3). Therefore, for a metal disk D of radius a centered at the origin, we want to find u such that

$$\begin{aligned} u_{xx} + u_{yy} &= 0, \quad \text{if } x^2 + y^2 < a^2, \\ u(x, y) &= f(x, y), \quad \text{if } x^2 + y^2 = a^2, \end{aligned} \tag{5.1}$$

where f is a function defined on the boundary of the disk. The geometry is illustrated in Figure 1.

The Laplacian in other coordinate systems

Since D is a circular domain, our problem will be more easily solved if we use polar coordinates r and θ. We will derive the form of the Laplacian in polar coordinates in some detail so the derivation can be a model to be used with other coordinate systems.

The original Cartesian coordinates x and y and the polar coordinates r and θ are related by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned} \quad \text{and} \quad \begin{aligned} r^2 &= x^2 + y^2 \\ \tan\theta &= y/x. \end{aligned}$$

Differentiating $r^2 = x^2 + y^2$, we get $2r\,\partial r/\partial x = 2x$. Solving this equation for the partial derivative and then doing the same calculation for the y-derivative, we get

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \quad \text{and} \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta.$$

In the same way, by differentiating $\tan\theta = y/x$ we find that

$$\frac{\partial\theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r} \quad \text{and} \quad \frac{\partial\theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}.$$

If u is a function, then the chain rule implies that

$$u_x = u_r \cdot \frac{\partial r}{\partial x} + u_\theta \cdot \frac{\partial\theta}{\partial x} = u_r \cdot \cos\theta - u_\theta \cdot \frac{\sin\theta}{r}.$$

Differentiating once more using the chain rule, we see that

$$\begin{aligned} u_{xx} &= \frac{\partial}{\partial r}\left[u_r \cdot \cos\theta - u_\theta \cdot \frac{\sin\theta}{r}\right]\cos\theta \\ &\quad - \frac{\partial}{\partial\theta}\left[u_r \cdot \cos\theta - u_\theta \cdot \frac{\sin\theta}{r}\right]\frac{\sin\theta}{r} \\ &= u_{rr} \cdot \cos^2\theta - 2u_{r\theta} \cdot \frac{\sin\theta\cos\theta}{r} + u_{\theta\theta} \cdot \frac{\sin^2\theta}{r^2} \\ &\quad + u_r \cdot \frac{\sin^2\theta}{r} + 2u_\theta \cdot \frac{\sin\theta\cos\theta}{r^2}. \end{aligned}$$

In exactly the same way we compute that

$$u_{yy} = u_{rr} \cdot \sin^2\theta + 2u_{r\theta} \cdot \frac{\sin\theta\cos\theta}{r} + u_{\theta\theta} \cdot \frac{\cos^2\theta}{r^2} + u_r \cdot \frac{\cos^2\theta}{r} - 2u_\theta \cdot \frac{\sin\theta\cos\theta}{r^2}.$$

Thus

$$\nabla^2 u = u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}. \tag{5.2}$$

Using this technique, we can find the form of the Laplacian in any coordinate system, although the details can be tedious. We will state two more results. Cylindrical coordinates are defined by the relations

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned} \qquad \text{and} \qquad \begin{aligned} r^2 &= x^2 + y^2 \\ \tan\theta &= y/x \\ z &= z. \end{aligned}$$

In cylindrical coordinates the Laplacian operator has the form

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}. \tag{5.3}$$

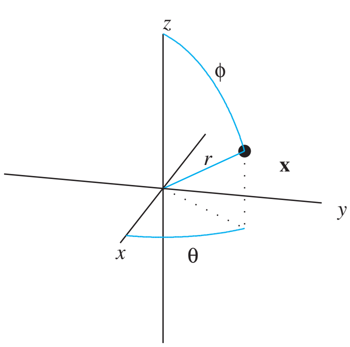


Figure 2 Spherical coordinates of a point in $\mathbf{R}^3$.

Spherical coordinates r, θ, and ϕ are illustrated in Figure 2. They are related to the Cartesian coordinates by

$$x = r\cos\theta\sin\phi, \qquad r^2 = x^2 + y^2 + z^2,$$
$$y = r\sin\theta\sin\phi, \quad \text{and} \quad \tan\theta = y/x,$$
$$z = r\cos\phi \qquad \tan\phi = \sqrt{x^2+y^2}\big/z$$

The expression for the Laplacian in spherical coordinates is

$$\nabla^2 u = \frac{1}{r^2}(r^2 u_r)_r + \frac{1}{r^2\sin\phi}(\sin\phi\cdot u_\phi)_\phi + \frac{1}{r^2\sin^2\phi}u_{\theta\theta}. \tag{5.4}$$

The Dirichlet problem on the disk

Using polar coordinates, the Dirichlet problem in (5.1) becomes

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \quad \text{for } 0 \le r < a, \\ u(a,\theta) &= f(\theta), \quad \text{for } 0 \le \theta \le 2\pi. \end{aligned} \tag{5.5}$$

The function f is supposed to be defined on the circle of radius $r = a$. Since this circle is parameterized by $\theta \to (a\cos\theta, a\sin\theta)$, we can consider f to be a function of θ. Since it is really a function of $\sin\theta$ and $\cos\theta$, which are periodic with period 2π, the function f must be 2π-periodic.

We solve the problem using separation of variables in polar coordinates by looking for product functions of the form $u(r,\theta) = R(r)T(\theta)$, which are solutions to Laplace's equation. Just like f, T must be 2π-periodic.

When we insert the function $u(r,\theta) = R(r)T(\theta)$ into Laplace's equation, we obtain

$$\left[R''(r) + \frac{1}{r}R'(r)\right]T(\theta) + \frac{1}{r^2}R(r)T''(\theta) = 0.$$

We multiply by $r^2/[R(r)T(\theta)]$ to separate the r variable from the θ variable, obtaining

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{T''(\theta)}{T(\theta)}.$$

Both sides must be equal to a constant, λ, so we obtain the following two equations:

$$r^2R''(r) + rR'(r) - \lambda R(r) = 0 \quad \text{and} \quad T''(\theta) + \lambda T(\theta) = 0. \tag{5.6}$$

The function T must be 2π-periodic. Therefore, it must solve the Sturm Liouville problem

$$T''(\theta) + \lambda T(\theta) = 0 \quad \text{with } T \ 2\pi\text{-periodic}. \tag{5.7}$$

Notice that the boundary condition is different than in previous Sturm Liouville problems. These conditions are called ***periodic boundary conditions***.

Let's first look for nonzero solutions to (5.7) with $\lambda < 0$. We write $\lambda = -s^2$, with $s > 0$. The differential equation becomes $T'' - s^2T = 0$, which has the general solution $T(\theta) = Ae^{s\theta} + Be^{-s\theta}$. However, no function of this type is periodic, so there are no nonzero solutions for $\lambda < 0$.

If $\lambda = 0$, the differential equation in (5.7) is $T'' = 0$, which has the general solution $T(\theta) = A + B\theta$. Since T must be 2π-periodic, we conclude that $B = 0$.

Thus, for $\lambda = 0$ the nonzero solutions are any constant function, that is a multiple of the function

$$c_0(\theta) = 1.$$

For $\lambda > 0$ we set $\lambda = \omega^2$, where $\omega > 0$. Then the differential equation has the form $T'' + \omega^2 T = 0$, which has the general solution

$$T(\theta) = A\cos\omega\theta + B\sin\omega\theta.$$

Since this function must be 2π-periodic, we conclude that ω must be a positive integer. Thus, any linear combination of the functions

$$c_n(\theta) = \cos n\theta \quad \text{and} \quad s_n(\theta) = \sin n\theta$$

will be a solution to the Sturm Liouville problem in (5.7).

To sum up, the eigenvalues for the Sturm Liouville problem in (5.7) are $\lambda_n = n^2$, for n any nonnegative integer. The corresponding eigenfunctions are the single function $c_0(\theta) = 1$ for $n = 0$, and the pair of functions $c_n(\theta) = \cos n\theta$ and $s_n(\theta) = \sin n\theta$ for $n \geq 1$.

In view of the fact that $\lambda = n^2$, where n is a nonnegative integer, the differential equation for R in (5.6) becomes

$$r^2 R'' + rR' - n^2 R = 0. \tag{5.8}$$

This is a special case of Euler's equation, which we studied in Section 11.3. A fundamental set of solutions is

$$\begin{aligned} r^0 = 1 \quad &\text{and} \quad \ln r, \quad \text{for } n = 0,\\ r^n \quad &\text{and} \quad r^{-n}, \quad \text{for } n \geq 1. \end{aligned} \tag{5.9}$$

However, there is a hidden boundary condition. We are really looking at functions defined on the disk, and the point $r = 0$ corresponds to the center of the disk. We want our solutions[10] to be bounded there, so the solutions $\ln r$ for $n = 0$ and r^{-n} for $n > 0$ are not viable. Consequently, we are led to the solution

$$R_n(r) = r^n, \quad \text{for } n \geq 0.$$

The corresponding product solutions to Laplace's equation are $u_0(r, \theta) = 1$ and

$$\begin{aligned} u_n(r,\theta) &= R_n(r)c_n(\theta) = r^n \cos n\theta \quad \text{and}\\ v_n(r,\theta) &= R_n(r)s_n(\theta) = r^n \sin n\theta \end{aligned}$$

for $n \geq 1$. Since the Laplacian is linear, the function

$$\begin{aligned} u(r,\theta) &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n u_n(r,\theta) + B_n v_n(r,\theta)\\ &= \frac{A_0}{2} + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \end{aligned} \tag{5.10}$$

[10] It is good to remember that our solutions represent temperatures.

is a solution to Laplace's equation on the disk for any constants A_n and B_n for which the series converges.

The boundary condition $u(a, \theta) = f(\theta)$ now becomes

$$f(\theta) = u(a, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} a^n \left(A_n \cos n\theta + B_n \sin \theta\right).$$

This is the complete Fourier series for the boundary function f. According to Theorem 1.11 in Chapter 12, the coefficients must be

$$A_n = \frac{1}{a^n \pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta, \quad \text{for } n \geq 0,$$

$$B_n = \frac{1}{a^n \pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta, \quad \text{for } n \geq 1.$$

Inserting these values into (5.10) yields the solution to the Laplace equation (5.2).

EXAMPLE 5.11 ◆ A beer can of radius 1 inch is full of beer and lies on its side, halfway submerged in the snow (see Figure 3). The snow keeps the bottom half of the beer can at 0°C while the sun warms the top half of the can to 1°C. Find the steady-state temperature inside the can.

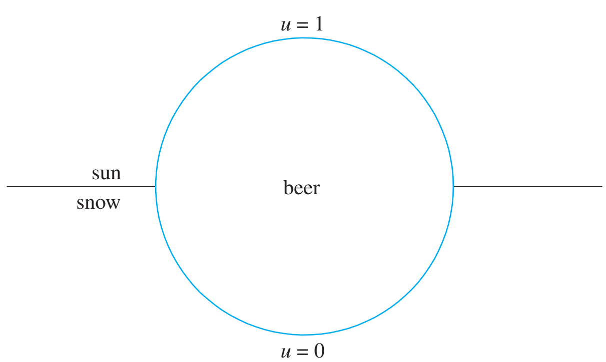


Figure 3 The can of beer in Example 5.11.

The boundary of the beer can is a circle of radius 1 inch, which in polar coordinates can be described by the equations $r = 1$ and $0 \leq \theta \leq 2\pi$. The temperature function on the boundary is given by

$$f(\theta) = \begin{cases} 1, & \text{for } 0 \leq \theta \leq \pi, \\ 0, & \text{for } \pi < \theta < 2\pi. \end{cases}$$

Thus, we wish to solve (5.1) with $a = 1$ and f as given.

We need to compute the Fourier coefficients of f. First, we have

$$A_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi} 1\,d\theta = 1.$$

Next, for $n \geq 1$,

$$A_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta = \frac{1}{\pi}\int_0^{\pi} \cos n\theta\,d\theta = 0.$$

Finally,

$$B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta = \frac{1}{\pi}\int_0^{\pi} \sin n\theta\,d\theta = \frac{1-(-1)^n}{n\pi}.$$

If n is even, $B_n = 0$, and if $n = 2k+1$ is odd, $B_{2k+1} = 2/[(2k+1)\pi]$. Substituting the coefficients into (5.10), we see that the solution to (5.2) is

$$u(r,\theta) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2r^{2k+1}}{\pi(2k+1)}\sin(2k+1)\theta.$$

Figure 4 shows the graph of the partial sum of the solution up to $k = 20$ over the unit disk.

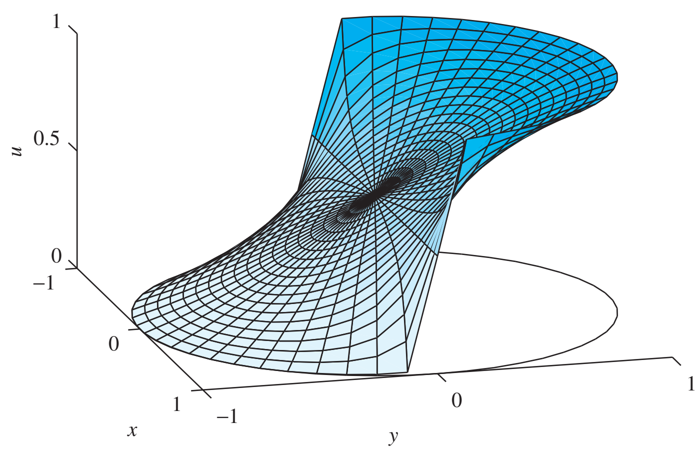


Figure 4 The temperature of the beer in Example 5.11.

The vertical axis is the temperature of the beer in the can. Notice that the graph of the temperature on the boundary of the disk jumps from 0 to 1 halfway around the disk as is consistent with the temperature of the surface of the beer can. ◆

EXERCISES

1. Verify that the Laplacian has the form in (5.3) in cylindrical coordinates.
2. Verify that the Laplacian has the form in (5.4) in spherical coordinates.

3. Suppose that the temperature on the surface of the beer can in Example 5.11 is $f(x, y) = 1 + y$, where y is the distance above the snow. Find the temperature throughout the can. (*Hint:* This Exercise is easier than it might look. Keep in mind that you want to solve the Dirichlet problem in (5.1))

4. Suppose that the snow keeps the temperature on the bottom half of the surface of the beer can in Example 5.11 at 0°C, but this time suppose the vertical sun's rays keep the very top of the can at 1°C and that the temperature at any other point on the top half of the can is proportional to the sine of the angle between the sun's rays and the tangent line to the can. Give an intuitive argument why this is a reasonable model for the temperature on the boundary (*Hint:* look at how the intensity of the sunlight on the can's surface depends on the angle between the sun's rays and the tangent to the can). Compute the temperature at any point on the inside of the beer can. Draw a hand sketch of what you think is the graph of the solution (over the region $0 \le r \le 1$ and $0 \le \theta \le 2\pi$) and then compare with the graph of the first 10 terms or so of the computed solution.

In Exercises 5–8, find the temperature in a disk of radius 1, with the given temperature $f(\theta)$, for $0 \le \theta \le 2\pi$.

5. $f(\theta) = \sin^2 \theta$

6. $f(\theta) = \cos^2 \theta$

7. $f(\theta) = \theta(2\pi - \theta)$

8. $f(\theta) = \sin\theta \cos\theta$

If we are looking for the steady-state temperature in a ring shaped plate, then the hidden boundary condition used to eliminate half of the solutions from (5.9) does not come into play. Exercises 9–11 deal with this situation.

9. Consider a plate that is ring shaped. Its boundary consists of two concentric circles with radii $a < b$. Suppose that the inner circle is kept at a uniform temperature T_1 and the outer circle at a uniform temperature of T_2. Find the temperature throughout the plate. (*Hint:* Since in polar coordinates the temperature on the boundary does not depend on θ, you can conclude that the steady-state temperature doesn't either.)

10. Suppose that the outer boundary, where $r = b$, of the ring shaped plate in Exercise 9 is insulated, and the inner boundary, where $r = a$, is kept at the uniform temperature T. Find the steady-state temperature in the plate. (*Hint:* According to (4.4), since the plate is insulated at $r = b$, the normal derivative of the temperature u is equal to 0 there. With the circular symmetry we have, this means that $u_r = 0$.)

11. Consider once more the plate in Exercise 9. Now suppose that the temperature on the inner boundary is uniformly equal to 0°, and on the outer boundary is given by $f(x, y)$. Find the steady-state temperature throughout the plate.

12. Suppose we have a semicircular plate of radius a. Suppose that the temperature on the flat base is kept at 0°, while on the curved portion the temperature is described by $f(x, y)$. Find the steady-state temperature through out the plate.

13. Suppose that we have a plate that is shaped like a piece of pie. It is a segment of a circle of radius a, with angle θ_0. Suppose the temperature is fixed at $0°$ along the flat portions of the boundary, and is given by $f(\theta)$ for $0 \le \theta \le \theta_0$ along the curved portion. Find the steady-state temperature through out the plate.

13.6 Sturm Liouville Problems

One of the steps in the method of separation of variables is the solution of a Sturm Liouville problem. The prototypical example appeared in (2.10). It was to find numbers λ and nonzero functions X defined on the interval $[0, L]$ for which

$$-X'' = \lambda X \quad \text{with} \quad X(0) = X(L) = 0. \tag{6.1}$$

We have rewritten the differential equation in the form we will adopt in this section. The solutions to the Sturm-Liouville problem in (6.1) were found in (2.12). They are

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{and} \quad X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \text{for } n = 1, 2, 3, \dots .$$

We saw another example using the same differential equation, but with different boundary conditions in (2.27). Let's look at some more examples.

EXAMPLE 6.2 ◆ Consider a rod of length L that is kept at a constant temperature of $0°$ at $x = 0$ and is insulated at $x = L$. The temperature $u(x, t)$ in the rod satisfies the initial/boundary value problem

$$\begin{aligned} u_t(x,t) &= ku_{xx}(x,t), \quad \text{for } t > 0 \text{ and } 0 < x < L,\\ u(0,t) &= 0 \quad \text{and} \quad u_x(L,t) = 0, \quad \text{for } t > 0,\\ u(x,0) &= f(x), \quad \text{for } 0 \le x \le L, \end{aligned}$$

where $f(x)$ is the temperature in the rod at time $t = 0$. If we look for a product solution $u(x, t) = X(x)T(t)$, the function X must satisfy

$$-X'' = \lambda X \quad \text{with } X(0) = X'(L) = 0. \tag{6.3}$$

This is a Sturm Liouville problem of a type we have not seen before. ◆

EXAMPLE 6.4 ◆ Suppose for the same rod that the insulation at $x = L$ is poor and slowly leaks heat (see (1.14) and (1.15)). Then the boundary condition at that point will be a Robin condition. Assuming for simplicity that the ambient temperature is $T = 0$, the temperature satisfies the initial/boundary value problem

$$\begin{aligned} u_t(x,t) &= ku_{xx}(x,t), \quad \text{for } t > 0 \text{ and } 0 < x < L,\\ u(0,t) &= 0 \quad \text{and} \quad u_x(L,t) + \gamma u(L,t) = 0, \quad \text{for } t > 0,\\ u(x,0) &= f(x), \quad \text{for } 0 \le x \le L, \end{aligned}$$

where γ is a positive constant. This time, if we look for a product solution $u(x, t) = X(x)T(t)$, the function X must satisfy

$$-X'' = \lambda X \quad \text{with } X(0) = 0 \text{ and } X'(L) + \gamma X(L) = 0. \tag{6.5}$$

Again this is a Sturm Liouville problem of a type we have not seen before. ◆

EXAMPLE 6.6 ◆ Suppose that the insulation along the length of the rod in Example 6.2 is not perfect, and slowly leaks heat. In this case, by an argument similar to that in Section 1, we are led to the initial/boundary value problem

$$u_t(x,t) = k\left[u_{xx}(x,t) - q(x)u(x,t)\right], \quad \text{for } t > 0 \text{ and } 0 < x < L,$$
$$u(0,t) = 0 \quad \text{and} \quad u_x(L,t) = 0, \quad \text{for } t > 0,$$
$$u(x,0) = f(x), \quad \text{for } 0 \le x \le L,$$

where $q(x) > 0$ is a measure of the leakiness of the insulation. Again let's look for a product solution $u(x,t) = X(x)T(t)$. We find that X must satisfy

$$-X'' + qX = \lambda X \quad \text{with } X(0) = X'(L) = 0. \tag{6.7}$$

This Sturm Liouville problem involves a different differential operator than the previous two. ◆

As these examples indicate, there is a large variety of initial/boundary value problems that might be solved using the method of separation of variables. Each of these problems leads to a Sturm Liouville problem. In this section and the next we will study these problems in general. In addition to providing us with additional techniques to solve initial/boundary value problems, the study of Sturm Liouville problems in general will provide us with important insights into much of what we have studied both in this chapter and in Chapter 12.

The differential operator

We will assume that the differential equation in the Sturm Liouville problem has the form

$$-(p\phi')' + q\phi = \lambda w\phi, \tag{6.8}$$

where p, q, and w are functions of x for $a \le x \le b$. It will be convenient to use operator notation in this section. The differential operators we will deal with have the form

$$L\phi = -(p\phi')' + q\phi. \tag{6.9}$$

Using this notation, the differential equation in (6.8) can be written as

$$L\phi = \lambda w\phi. \tag{6.10}$$

The function $w(x)$ is called the ***weight function***.

In equations (6.1), (6.3), and (6.5), the differential operator is $L\phi = -\phi''$. Hence $p(x) = 1$ and $q(x) = 0$. In equation (6.7) the differential operator is $L\phi = -\phi'' + q\phi$, so again $p(x) = 1$ but now $q(x) > 0$. In all four cases the differential equation is written as $L\phi = \lambda\phi$, so the weight function is $w(x) = 1$.

In this section we will only consider ***nonsingular*** Sturm Liouville problems.

DEFINITION 6.11 A Sturm-Liouville problem involving the equation

$$L\phi = -(p\phi')' + q\phi = \lambda w\phi,$$

is nonsingular if

- the coefficient $p(x)$ and its derivative $p'(x)$ are both continuous on $[a, b]$, and $p(x) > 0$ for $a \le x \le b$,
- the coefficient $q(x)$ is piecewise continuous on $[a, b]$, and
- the weight function w is continuous and positive on $[a, b]$.

These conditions can be relaxed, but only at the endpoints a and b. If so, that endpoint is said to be ***singular***. For example, if $p(a) = 0$, the endpoint a is singular. We will discuss some singular Sturm Liouville problems later in this chapter.

Operators of the form (6.9) are said to be ***formally self-adjoint***. The most important property of formally self-adjoint operators is in the following proposition.

PROPOSITION 6.12 Let L be a differential operator of the type in (6.9). If f and g are two functions defined on (a, b) that have continuous second derivatives, then

$$\int_a^b Lf \cdot g\,dx = \int_a^b f \cdot Lg\,dx + p(fg' - f'g)\Big|_a^b. \tag{6.13}$$

We will leave the proof to Exercise 12.

The property of formally self-adjoint operators displayed in Proposition 6.12 is not true for most differential operators. This property is the main reason for limiting our consideration to formally self-adjoint operators. In Exercise 13 we will exhibit a differential operator that does not have this property.

The assumption that our operator is formally self-adjoint might seem too restrictive. We could consider the more general operator

$$M\phi = -P\phi'' - Q\phi' + R\phi.$$

However, assuming that $P(x) > 0$ for all $x \in (a, b)$, it is always possible to find a function μ so that the operator $L = \mu M$ is formally self-adjoint. To see this, notice that

$$\begin{aligned} \mu M\phi &= -\mu P\phi'' - \mu Q\phi' + \mu R\phi, \quad \text{while} \\ L\phi &= -p\phi'' - p'\phi' + q\phi. \end{aligned}$$

For these to be equal, it is necessary to find functions $\mu(x)$, $p(x)$, and $q(x)$ so that

$$p = \mu P, \quad p' = \mu Q, \quad \text{and} \quad q = \mu R.$$

The quotient of the first two equations gives us the linear differential equation $p' = (Q/P)p$, which can be solved to find p. Then we take $\mu = p/P$ and $q = \mu R$. This process is exemplified in Exercises 6–9.

The boundary conditions

In each of the examples at the beginning of this section, we imposed two boundary conditions. We will do so in general. The most general boundary condition has the form

$$B\phi = \alpha\phi(a) + \beta\phi'(a) + \gamma\phi(b) + \delta\phi'(b) = 0.$$

Notice that the condition mixes the values of ϕ and ϕ' at the two endpoints. We will consider only one pair of boundary conditions that mix the endpoints in this way. These are

$$B_1\phi = \phi(a) - \phi(b) = 0 \quad \text{and} \quad B_2\phi = \phi'(a) - \phi'(b) = 0. \tag{6.14}$$

We will call these the ***periodic boundary conditions***, because they are satisfied by a function ϕ that is periodic with period $b - a$.

Boundary conditions that involve only one endpoint are called ***unmixed***. We will consider pairs of unmixed boundary conditions where one of the conditions applies to each endpoint. The most general unmixed boundary conditions have the form

$$B_1\phi = \alpha_1\phi'(a) + \beta_1\phi(a) = 0 \quad \text{and} \quad B_2\phi = \alpha_2\phi'(b) + \beta_2\phi(b) = 0, \tag{6.15}$$

where α_1, α_2, β_1, and β_2 are constants. In order that the boundary conditions be meaningful, we will insist that the vectors (α_1, β_1), and (α_2, β_2) are nonzero.

The general unmixed boundary condition in (6.15) splits into three cases, depending on the coefficients. For the endpoint a they are

1. $\phi(a) = 0$. The Dirichlet condition (if $\alpha_1 = 0$).
2. $\phi'(a) = 0$. The Neumann condition (if $\beta_1 = 0$).
3. $\phi'(a) + \gamma_1\phi(a) = 0$. The Robin condition (if neither α_1 nor β_1 is equal to 0).

You should compare these conditions with our discussion of boundary conditions in Sections 1 and 4.

The eigenvalues and eigenfunctions

The Sturm Liouville problem for a given operator L, weight function w, and boundary conditions B_1 and B_2 on an interval (a, b) is to find all numbers λ and nonzero functions ϕ such that

$$\begin{aligned} L\phi &= \lambda w\phi \quad \text{on the interval } (a, b), \text{ and} \\ B_1\phi &= B_2\phi = 0. \end{aligned} \tag{6.16}$$

Any number λ for which there is a nonzero function ϕ satisfying (6.16) is called an ***eigenvalue*** of the Sturm Liouville problem. If λ is an eigenvalue, then any function ϕ that satisfies (6.16) is called an associated ***eigenfunction***.[11] Thus, our problem is to find all of the eigenvalues and eigenfunctions for a Sturm Liouville boundary value problem. Notice that if c_1 and c_2 are constants and ϕ_1 and ϕ_2 are eigenfunctions,

[11] You are encouraged to compare the discussion here of eigenvalues and eigenfunctions with the discussion in Section 9.1 of eigenvalues and eigenvectors of a matrix.

then, because L is linear, $L(c_1\phi_1 + c_2\phi_2) = c_1 L\phi_1 + c_2 L\phi_2 = \lambda w(c_1\phi_1 + c_2\phi_2)$, so any linear combination of eigenfunctions is also an eigenfunction. In particular, any constant multiple of an eigenfunction is also an eigenfunction. We will usually choose the constant for which the eigenfunction has the simplest algebraic form.

We have seen two examples of the solution to a Sturm Liouville problem in (2.9) and (2.27). Let's look at another.

EXAMPLE 6.17 ◆ Find the eigenvalues and eigenfunctions for the operator $L\phi = -\phi''$ on the interval $[-\pi, \pi]$ with periodic boundary conditions, and with weight $w = 1$.

We analyzed what is essentially the same problem in Section 5. See equation (5.7) and the following text. The eigenvalues and eigenfunctions are

$$\lambda_0 = 0 \quad \text{with} \quad c_0(x) = 1, \quad \text{and}$$
$$\lambda_n = n^2 \quad \text{with} \quad c_n(x) = \cos nx \quad \text{and} \quad s_n(x) = \sin nx, \quad \text{for } n \geq 1.$$

We will leave the details to Exercise 14. ◆

Properties of eigenvalues and eigenfunctions

The ***multiplicity*** of an eigenvalue is the number of linearly independent eigenfunctions associated to it. Notice that the positive eigenvalues in Example 6.17 have multiplicity 2, while all of the eigenvalues in (2.12) and (2.27) have multiplicity 1.

Suppose that λ is any number, and let's look at the differential equation $L\phi = -(p\phi')' + q\phi = \lambda w\phi$. If we write this out and rearrange it, we get

$$p\phi'' + p'\phi' + (\lambda w - q)\phi = 0. \tag{6.18}$$

This is a second order, linear differential equation. From Section 4.1 we know that it has a fundamental set of solutions consisting of two linearly independent functions $\phi_1(x)$ and $\phi_2(x)$. The general solution is the linear combination

$$\phi(x) = A\phi_1(x) + B\phi_2(x). \tag{6.19}$$

For any λ, the boundary conditions put constraints on the coefficients A and B in (6.19). As we have seen in our examples, for most values of λ the two boundary conditions will imply that both A and B are equal to zero, and the only solution is the zero function. Thus, most numbers λ are not eigenvalues.

In Example 6.17, the boundary conditions were satisfied by all solutions to the differential equation, so they did not constrain the coefficients A and B at all. The multiplicity is 2. In general, if an eigenvalue has multiplicity 2, then there are two linearly independent solutions to (6.16). These two solutions form a fundamental set of solutions to the differential equation, so every solution to the differential equation also solves the boundary conditions. The only other possibility is that the multiplicity is 1. In this case, the two boundary conditions put the same nontrivial constraint on A and B.

Our examples show that eigenvalues are rare, and the next theorem shows that this is true in general.

THEOREM 6.20 The eigenvalues for a nonsingular Sturm Liouville problem with either unmixed or periodic boundary conditions, repeated according to their multiplicity, form a sequence of real numbers

$$\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \quad \text{where} \quad \lambda_n \to \infty. \tag{6.21}$$

For each eigenvalue λ_n there is an associated eigenfunction which we will denote by $\phi_n(x)$. If we have a repeated eigenvalue, $\lambda_n = \lambda_{n+1}$, the eigenfunctions ϕ_n and ϕ_{n+1} can be chosen to be linearly independent. The eigenfunctions are all real valued.

The proof of Theorem 6.20 requires techniques that are beyond the scope of this book, so we will not present it.[12]

Example 6.17 shows that eigenvalues with multiplicity 2 do occur. However, it is rare, as is shown by the following result.

PROPOSITION 6.22 Suppose that one of the boundary conditions for a nonsingular Sturm Liouville problem is unmixed. Then every eigenvalue has multiplicity 1.

The proof of Proposition 6.22 is left to Exercise16. Under the hypotheses of Proposition 6.22, the eigenvalues satisfy

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots \quad \text{and} \quad \lambda_n \to \infty, \tag{6.23}$$

instead of the less restrictive inequalities in (6.21).

There is one additional fact that will frequently speed our search for eigenvalues and eigenfunctions.

PROPOSITION 6.24 Suppose that we have a nonsingular Sturm Liouville problem

$$\begin{aligned} L\phi &= -(p\phi')' + q\phi = \lambda w\phi, \\ B_1\phi &= \alpha_1\phi'(a) + \beta_1\phi(a) = 0, \\ B_2\phi &= \alpha_2\phi'(b) + \beta_2\phi(b) = 0, \end{aligned}$$

where

(a) $q(x) \ge 0$ for $a \le x \le b$, and
(b) the boundary conditions on a function ϕ imply that $p\phi\phi'|_a^b \le 0$.

Then all of the eigenvalues are nonnegative. If $\lambda = 0$ is an eigenvalue, the corresponding eigenfunctions are the constant functions.

Proof Suppose that λ is an eigenvalue and ϕ is an associated eigenfunction. Consider the following computation, where the last step involves an integration by parts.

$$\begin{aligned} \lambda \int_a^b \phi^2(x)\, w(x)\, dx &= \int_a^b \phi(x) \cdot [\lambda w(x)\phi(x)]\, dx = \int_a^b \phi(x) \cdot L\phi(x)\, dx \\ &= -\int_a^b \phi(p\phi')'\, dx + \int_a^b q\phi^2\, dx \\ &= \int_a^b p(x)\phi'(x)^2\, dx - p\phi\phi'\big|_a^b + \int_a^b q(x)\phi(x)^2\, dx \end{aligned} \tag{6.25}$$

[12] See *Linear Differential Operators* by M. A. Naimark.

Since the Sturm-Liouville problem is nonsingular, the coefficient $p(x)$ is always positive. By our hypotheses, the coefficient $q(x)$ is nonnegative, and the term $-p\phi\phi'|_a^b \geq 0$. Hence all of the terms on the right-hand side of the equation are nonnegative. Consequently,

$$\lambda \int_a^b \phi^2(x)\, w(x)\, dx \geq 0,$$

and since the weight function $w(x)$ is positive, we conclude that $\lambda \geq 0$.

If $\lambda = 0$ is an eigenvalue, then all three terms on the right-hand side of (6.25) must be equal to 0. In particular, $\int_a^b p(x)\phi'(x)^2\, dx = 0$. Since $p(x) > 0$, this means that $\phi'(x) = 0$, so ϕ is a constant function.

Let's end the section by finding the eigenvalues and eigenfunctions for the Sturm Liouville problems in Examples 6.2, 6.4, and 6.6.

EXAMPLE 6.26 ◆ Find the eigenvalues and eigenfunctions for the Sturm Liouville problem in Example 6.2.

Let's rewrite equation (6.3) with $X(x)$ replaced by $\phi(x)$ to get

$$-\phi'' = \lambda\phi \quad \text{for } x \in (0, L), \text{ with} \quad \phi(0) = 0 = \phi'(L).$$

The coefficients are $p = 1$ and $q = 0$. Thus q is nonnegative, and the boundary condition implies that $p\phi\phi' = \phi\phi'$ vanishes at each endpoint. Therefore, by Proposition 6.24, all of the eigenvalues are nonnegative. If $\lambda = 0$ is an eigenvalue, then the eigenfunction ϕ is a constant. However, $\phi(0) = 0$, so $\phi(x) = 0$. Thus $\lambda = 0$ is not an eigenvalue.

Thus all eigenvalues are positive. If we set $\lambda = \omega^2$, where $\omega > 0$, the differential equation becomes $\phi'' + \omega^2\phi = 0$. The general solution is $\phi(x) = A\cos\omega x + B\sin\omega x$. The first boundary condition says that $0 = \phi(0) = A$. Then the second boundary condition says that $\phi'(L) = \omega B\cos\omega L = 0$. Since we are looking for nonzero solutions, $B \neq 0$. Hence we must have $\cos\omega L = 0$. This is true only if $\omega L = \pi/2 + n\pi = (2n+1)\pi/2$, where n is a nonnegative integer. Thus our eigenvalues and eigenfunctions are

$$\lambda_n = \frac{(2n+1)^2\pi^2}{4L^2} \quad \text{and} \quad \phi_n(x) = \sin\frac{(2n+1)\pi x}{2L}, \quad \text{for } n = 0, 1, 2, \ldots.$$

The first five eigenfunctions are shown in Figure 1. ◆

EXAMPLE 6.27 ◆ Find the eigenvalues and eigenfunctions for the Sturm Liouville problem in Example 6.4.

Let's rewrite (6.5) as

$$-\phi'' = \lambda\phi \quad \text{for } x \in (0, L), \text{ with} \quad \phi(0) = 0 = \phi'(L) + \gamma\phi(L).$$

Again $p = 1$ and $q = 0$. We have $p(0)\phi(0)\phi'(0) = 0$, while $p(L)\phi(L)\phi'(L) = -\gamma\phi(L)^2 \leq 0$, since $\gamma > 0$. Therefore Proposition 6.24 shows that all of the

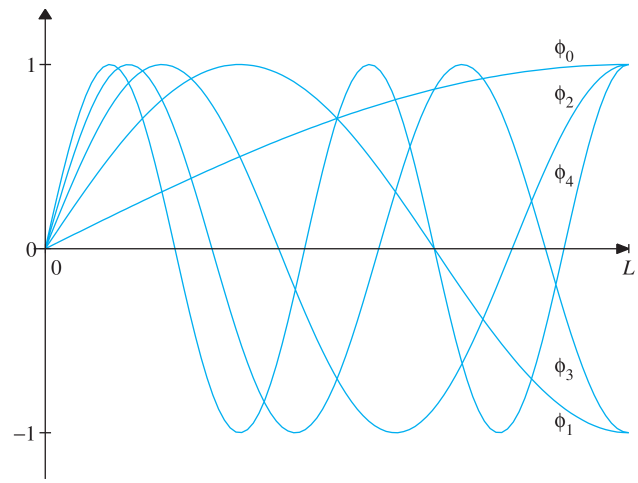


Figure 1 The first five eigenfunctions for the Sturm Liouville problem in Example 6.26.

eigenvalues are nonnegative. If $\lambda = 0$ is an eigenvalue then the eigenfunction ϕ is a constant. However, since $\phi(0) = 0$, $\phi(x) = 0$, so $\lambda = 0$ is not an eigenvalue. For $\lambda > 0$, we write $\lambda = \omega^2$, where $\omega > 0$. The differential equation has the general solution $\phi(x) = A\cos\omega x + B\sin\omega x$, where A and B are arbitrary constants. The first boundary condition implies that $0 = \phi(0) = A$. Hence $\phi(x) = B\sin\omega x$. The second boundary condition implies that

$$0 = \phi'(L) + \gamma\phi(L) = B\left[\omega\cos\omega L + \gamma\sin\omega L\right].$$

Since $\phi(x) \neq 0$, $B \neq 0$. Hence the second factor must vanish. Dividing by $\cos\omega L$ and rearranging, we get

$$\tan\omega L = -\frac{\omega}{\gamma}. \tag{6.28}$$

For those values of ω that solve equation (6.28), $\lambda = \omega^2$ is an eigenvalue and $\phi(x) = \sin\omega x$ is an associated eigenfunction.

Equation (6.28) cannot be solved exactly, but it can be solved to any desired degree of accuracy using numerical methods. To see what the solutions look like, let's first simplify the equation somewhat by setting $\theta = \omega L$, and $\alpha = 1/(\gamma L)$. Then (6.28) becomes

$$\tan\theta = -\alpha\theta. \tag{6.29}$$

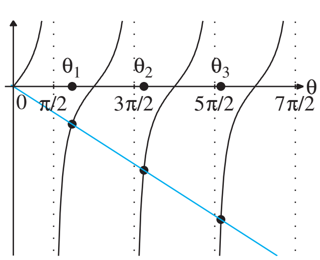


Figure 2 The solutions to $\tan\theta = -\alpha\theta$.

In Figure 2, we plot $f(\theta) = \tan\theta$ in black and $g(\theta) = -\alpha\theta$ in blue. The points where the two graphs intersect correspond to the values of θ that solve (6.29). From Figure 2 we see that there are infinitely many solutions to (6.29), which we will write as the increasing sequence θ_j for $j = 1, 2, 3, \ldots$. Again from Figure 2, we see that

$$(j - 1/2)\pi < \theta_j < j\pi,$$

and that θ_j gets closer to $(j - 1/2)\pi$ as j increases. For each j, $\omega_j = \theta_j/L$. This leads to the eigenvalues and eigenfunctions

$$\lambda_j = \omega_j^2 = \frac{\theta_j^2}{L^2} \quad \text{and} \quad \phi_j(x) = \sin \omega_j x = \sin \frac{\theta_j x}{L}, \quad \text{for } j = 1, 2, 3, \ldots. \tag{6.30}$$

For the case when $L = \gamma = 1$, the first five eigenfunctions are plotted in Figure 3.

◆

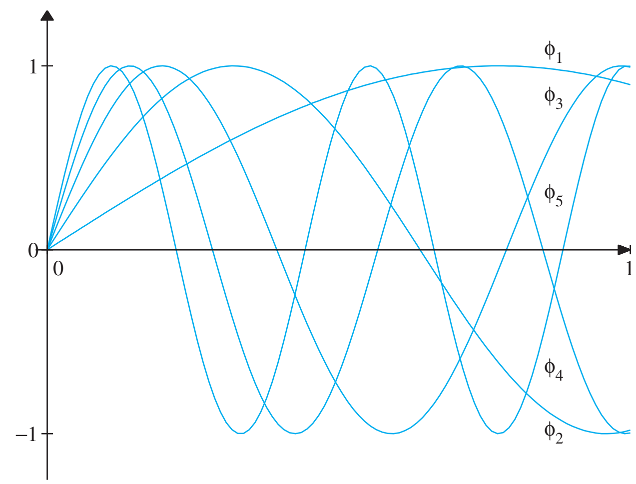


Figure 3 The first five eigenfunctions for the Sturm Liouville problem in Example 6.27.

EXAMPLE 6.31 ◆ Find the eigenvalues and eigenfunctions for the Sturm Liouville problem in Example 6.6.

Let's rewrite (6.7) as

$$-\phi'' + q\phi = \lambda\phi, \quad \text{for } x \in (0, L), \text{ with } \phi(0) = \phi'(L) = 0. \tag{6.32}$$

If the coefficient q is not constant, we will not usually be able to find solutions explicitly, but Theorem 6.20 guarantees that they exist. However, if q is a positive constant, we can rewrite the differential equation in (6.32) as

$$-\phi'' = (\lambda - q)\phi.$$

Since $\lambda - q$ is a constant, we see that the problem is almost the same as that in Example 6.26. To be precise, $\lambda - q$ must be an eigenvalue for the Sturm Liouville problem in Example 6.26, with the corresponding eigenfunction. Thus the eigenvalues and eigenfunctions are

$$\lambda_n = q + \frac{(2n+1)^2\pi^2}{4L^2} \quad \text{and} \quad \phi_n(x) = \sin \frac{(2n+1)\pi x}{2L}, \quad \text{for } n = 0, 1, 2, \ldots.$$

◆

EXERCISES

1. Which of the following operators are formally self-adjoint?

(a) $L\phi = \phi'' + \phi'$
(b) $L\phi = x\phi'' + \phi'$
(c) $L\phi = x\phi'' + 2\phi'$
(d) $L\phi = \cos x\, \phi'' + \sin x\, \phi'$
(e) $L\phi = \sin x\, \phi'' + \cos x\, \phi'$
(f) $L\phi = (1 - x^2)\phi'' - 2x\phi'$

In Exercises 2–5 find the eigenvalues and eigenfunctions for the given Sturm Liouville problem.

2. $-\phi'' = \lambda\phi$ with $\phi'(0) = \phi'(1) = 0$

3. $-\phi'' = \lambda\phi$ with $\phi'(0) = \phi(1) = 0$

4. $-\phi'' = \lambda\phi$ with $\phi'(0) = \phi'(1) + \phi(1) = 0$

5. $-\phi'' = \lambda\phi$ with $\phi'(0) - \phi(0) = \phi(1) = 0$

In Exercises 6–9, use the procedure given after the statement of Proposition 6.12 to transform the given differential equation into a formally self-adjoint equation.

6. $\phi'' + 4\phi' + \lambda\phi = 0$

7. $2x\phi'' + \lambda\phi = 0$

8. $x(x-1)\phi'' + 2x\phi' + \lambda\phi = 0$

9. $x^2\phi'' - 2x\phi' + \lambda\phi = 0$

10. Following the lead of Example 6.27, show how to graphically find the eigenvalues for the Sturm Liouville problem

$$-\phi'' = \lambda\phi \quad \text{with} \quad \phi'(0) - \phi(0) = \phi'(1) + \phi(1) = 0.$$

Find the eigenfunctions as well.

11. Consider the Sturm Liouville problem

$$-\phi'' = \lambda\phi \quad \text{with} \quad \phi(0) = \phi'(1) - a\phi(1) = 0,$$

where $a > 0$.

(a) Show that this problem does not satisfy the hypotheses of Proposition 6.24.

(b) Show that all eigenvalues are positive if $0 < a < 1$, that 0 is the smallest eigenvalue if $a = 1$, and that there is one negative eigenvalue if $a > 1$.

12. Prove Proposition 6.12. (*Hint:* Start with $\int_a^b Lf \cdot g\,dx$, insert the definition of L in (6.9), and then integrate by parts twice.)

13. Not all differential operators have the property in Proposition 6.12.

(a) Use Proposition 6.12 to show that if L is a formally self-adjoint operator, then

$$\int_a^b Lf \cdot g\,dx = \int_a^b f \cdot Lg\,dx$$

for any two functions f and g that vanish at both endpoints.

(b) Consider the operator $L\phi = \phi'' + \phi'$ on the interval $[0, 1]$. Show that the integral identity in part (a) is not true for L with $f(x) = x(1 - x)$ and $g(x) = x^2(1 - x)$, and therefore L does not have the property in Proposition 6.12.

14. Verify that the eigenvalues and eigenfunctions for the Sturm Liouville problem in Example 6.17 are those listed there.

15. Show that if $u(x, t) = X(x)T(t)$ is a product solution of the differential equation in Example 6.6, together with the boundary conditions, then X must be a solution to the Sturm Liouville problem in (6.7).

16. Prove Proposition 6.22. (*Hint:* Suppose that the boundary condition at $x = a$ is unmixed. Let λ be an eigenvalue and suppose that ϕ_1 and ϕ_2 are eigenfunctions. Let W be the Wronskian of ϕ_1 and ϕ_2. Use the boundary condition to show that $W(a) = 0$. Then use Proposition 1.26 of Section 4.1.)

13.7 Orthogonality and Generalized Fourier Series

You may have noticed that the eigenfunctions in the examples of Sturm Liouville problems in Sections 2, 3, and 4 were the bases of Fourier sine and cosine series. In addition, in Example 6.17 we have a Sturm Liouville problem for which the eigenfunctions are the basis of complete Fourier series. You may have asked yourself if the eigenfunctions of other Sturm Liouville problems lead to similar expansions. In this section we will carry out the derivation of such series.

Inner products, and orthogonality

The key idea in the derivation of Fourier series in Chapter 12 was the notion of orthogonality. It will also be important here, and it is time to put the idea into its proper framework. This involves the use of an inner product.

DEFINITION 7.1 Suppose that f and g are piecewise continuous functions on the interval $[a, b]$. The ***inner product*** of f and g with weight function $w(x) > 0$ is defined to be

$$(f, g)_w = \int_a^b f(x)g(x)\,w(x)\,dx. \tag{7.2}$$

If $w(x) = 1$, we will denote $(f, g)_w$ by (f, g). Thus

$$(f, g) = \int_a^b f(x)g(x)\,dx. \tag{7.3}$$

Notice that

$$(f, g)_w = (f, wg) = (wf, g). \tag{7.4}$$

Some elementary properties of the inner product are easily discovered. First, the inner product is ***symmetric***, which means that

$$(f, g)_w = (g, f)_w.$$

Second, the inner product is ***linear in each component***. For example, if a and b are constants, then

$$(af + bg, h)_w = a(f, h)_w + b(g, h)_w.$$

Finally, the inner product is ***positive***, meaning that

$$(f, f)_w > 0, \quad \text{unless} \quad f(x) = 0.$$

Our first result using this new definition will throw some light on why we write our differential operator in the formally self-adjoint form of (6.9).

PROPOSITION 7.5 Suppose that we have a nonsingular Sturm Liouville equation

$$L\phi = -(p\phi')' + q\phi = \lambda w\phi, \tag{7.6}$$

together with the unmixed boundary conditions

$$\begin{aligned} B_1\phi &= \alpha_1\phi'(a) + \beta_1\phi(a) = 0, \\ B_2\phi &= \alpha_2\phi'(b) + \beta_2\phi(b) = 0. \end{aligned} \tag{7.7}$$

If f and g are two functions defined on $[a, b]$ that have continuous second derivatives and satisfy the boundary conditions, then

$$(Lf, g) = (f, Lg). \tag{7.8}$$

Proof According to Proposition 6.12 we have

$$\int_a^b Lf \cdot g\, dx = \int_a^b f \cdot Lg\, dx + p(fg' - f'g)\big|_a^b.$$

Hence, to prove (7.8) we need to show that $p(fg' - f'g)\big|_a^b = 0$ if both f and g satisfy the boundary conditions. In fact, $fg' - f'g$ is equal to 0 at each of the endpoints. We will show this for the endpoint $x = a$. Since both f and g satisfy the boundary condition at $x = a$, we get the system of equations $\alpha_1 f'(a) + \beta_1 f(a) = 0$ and $\alpha_1 g'(a) + \beta_1 g(a) = 0$. In matrix form this can be written

$$\begin{pmatrix} f'(a) & f(a) \\ g'(a) & g(a) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the vector $(\alpha_1, \beta_1)^T$ is nonzero, the determinant of the matrix, $f'(a)g(a) - f(a)g'(a)$, must be equal to 0, as we wanted to show.

The property of the boundary value problem with unmixed boundary conditions expressed in (7.8) is critical to the theory that we are presenting. It is important to the proof of Theorem 6.20, as well as to the results that follow. We will say that a boundary value problem is ***self-adjoint*** if $(Lf, g) = (f, Lg)$ for any two functions that satisfy the boundary conditions.

DEFINITION 7.9 Two real valued functions f and g defined on the interval $[a, b]$ are said to be ***orthogonal with respect to the weight*** w if

$$(f, g)_w = \int_a^b f(x)g(x)\,w(x)\,dx = 0.$$

You will notice that this is the sense in which we used the term orthogonal for Fourier series in Chapter 12. The eigenfunctions of a Sturm Liouville problem have orthogonality properties similar to those we discovered for the sines and cosines in Chapter 12.

PROPOSITION 7.10 Suppose that ϕ_j and ϕ_k are eigenfunctions of the Sturm Liouville problem defined by (7.7) associated to different eigenvalues $\lambda_j \neq \lambda_k$. Then ϕ_j and ϕ_k are orthogonal with respect to the weight w.

Proof Since ϕ_j and ϕ_k are eigenfunctions associated to λ_j and λ_k, we have

$$L\phi_j = \lambda_j w\phi_j \quad \text{and} \quad L\phi_k = \lambda_k w\phi_k.$$

Hence

$$\begin{aligned}(L\phi_j, \phi_k) &= \lambda_j(w\phi_j, \phi_k) = \lambda_j(\phi_j, \phi_k)_w \quad \text{and}\\ (L\phi_k, \phi_j) &= \lambda_k(w\phi_k, \phi_j) = \lambda_k(\phi_k, \phi_j)_w.\end{aligned}$$

Using these equations, the properties of the inner product, and Proposition 7.5, we have

$$\lambda_j(\phi_j, \phi_k)_w = (L\phi_j, \phi_k) = (\phi_j, L\phi_k) = \lambda_k(\phi_j, \phi_k)_w.$$

Thus

$$(\lambda_j - \lambda_k)(\phi_j, \phi_k)_w = 0.$$

Since $\lambda_j - \lambda_k \neq 0$, we must have $(\phi_j, \phi_k)_w = 0$, so ϕ_j and ϕ_k are orthogonal with respect to the weight w.

Generalized Fourier series

In Chapter 12 we saw how the orthogonality properties of the sines and cosines led to Fourier series expansions for functions. The orthogonality result in Proposition 7.10 will allow us to find an analog to Fourier series based on the eigenfunctions of a Sturm Liouville problem. You are encouraged to observe the similarity of this development with that for Fourier series in Chapter 12.

First we assume that a function can be expressed as an infinite linear combination of eigenfunctions, and derive a formula for the coefficients.

PROPOSITION 7.11 Suppose that $\{\phi_n \mid n = 1, 2, \ldots\}$ is the sequence of orthogonal eigenfunctions for a nonsingular Sturm Liouville problem on the interval $[a, b]$. Suppose that

$$f(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots = \sum_{n=1}^{\infty} c_n\phi_n(x) \tag{7.12}$$

for $a < x < b$. Then

$$c_n = \frac{(f, \phi_n)_w}{(\phi_n, \phi_n)_w} = \frac{\int_a^b f(x)\phi_n(x)\, w(x)\, dx}{\int_a^b \phi_n^2(x)\, w(x)\, dx}. \tag{7.13}$$

Proof If we compute the inner product of f and ϕ_k using (7.12) and Proposition 7.10, we get[13]

$$(f, \phi_k)_w = \left(\sum_{n=1}^{\infty} c_n \phi_n, \phi_k \right)_w = \sum_{n=1}^{\infty} c_n (\phi_n, \phi_k)_w = c_k (\phi_k, \phi_k)_w,$$

from which the result follows.

Given a piecewise continuous function f on $[a, b]$, we can evaluate the inner products $(f, \phi_n)_w$ and $(\phi_n, \phi_n)_w$, and therefore the coefficients c_n in (7.13). Then we can write down the infinite series

$$f(x) \sim c_1\phi_1(x) + c_2\phi_2(x) + \cdots = \sum_{n=1}^{\infty} c_n \phi_n(x). \tag{7.14}$$

DEFINITION 7.15 The series in (7.14) with coefficients given by (7.13) is called the ***generalized Fourier series*** for the function f. The coefficients c_n are called the ***generalized Fourier coefficients*** of f.

Two questions immediately come to mind. Does the series converge? If the series converges, does it converge to the function f? The answers are almost the same as for Fourier series.

THEOREM 7.16 Suppose that $\{\phi_n \mid n = 1, 2, \ldots\}$ is the sequence of orthogonal eigenfunctions for a nonsingular Sturm Liouville problem on the interval $[a, b]$. Suppose also that f is a piecewise continuous function on the interval $[a, b]$.

1. If the left- and right-hand derivatives of f exist at a point $x_0 \in (a, b)$, then the generalized Fourier series in (7.14) converges at x_0 to
$$\frac{f(x_0^+) + f(x_0^-)}{2}.$$
2. If the right-hand derivative of f exists at a and f satisfies the boundary condition at a, then the series converges at a to $f(a)$.
3. If the left-hand derivative of f exists at b and f satisfies the boundary condition at b, then the series converges at b to $f(b)$.

[13] we are quietly assuming that the series for f converges fast enough that we can distribute the sum out of the inner product.

Notice that if f is continuous at a point $x_0 \in (a, b)$ and is differentiable there, then the generalized Fourier series converges to $f(x_0)$ at x_0 by part 1 of Theorem 7.16.

EXAMPLE 7.17 ◆ Find the generalized Fourier series for the function $f(x) = 100x/L$ on the interval $[0, L]$ using the eigenfunctions of the Sturm Liouville problem in Example 6.26. Discuss the convergence properties of the series.

The eigenfunctions are $\phi_n(x) = \sin((2n+1)\pi x/(2L))$ and the weight function is $w(x) = 1$. Hence,

$$(\phi_n, \phi_n) = \int_0^L \sin^2 \frac{(2n+1)\pi x}{2L}\, dx = \frac{L}{2}.$$

Next, using integration by parts, we get

$$(f, \phi_n) = \frac{100}{L} \int_0^L x \cdot \sin \frac{(2n+1)\pi x}{2L}\, dx = (-1)^n \frac{400L}{(2n+1)^2\pi^2}.$$

Consequently, the coefficients are

$$c_n = \frac{(f, \phi_n)}{(\phi_n, \phi_n)} = (-1)^n \frac{800}{(2n+1)^2\pi^2}, \tag{7.18}$$

and the generalized Fourier series is

$$f(x) = \frac{100x}{L} = \sum_{n=0}^{\infty} (-1)^n \frac{800}{(2n+1)^2\pi^2} \sin \frac{(2n+1)\pi x}{2L}.$$

Theorem 7.16 guarantees that the series will converge to $f(x)$ for x in the open interval $(0, L)$. Since f satisfies the boundary condition at $x = 0$, convergence there is also guaranteed. In fact, at $x = 0$ we have $f(0) = 0$, and each term of the series is also equal to 0, so the series converges there. The sum of the first two terms and the sum of the first eight terms of the generalized Fourier series are shown in blue in Figure 1, while the function f is shown in black. The series seems to converge very rapidly for all values of x. It appears that the series converges to $f(L) = 100$ at $x = L$, although this is not guaranteed by Theorem 7.16. Notice, however, that the error seems to be greatest at this endpoint. ◆

The results of Example 7.17 will enable us to solve the initial/boundary value problem for the heat equation in the next example.

EXAMPLE 7.19 ◆ Suppose a rod of length L is at steady state, with the temperature maintained at $0°$ at the left-hand endpoint, and at $100°$ at the right-hand endpoint. At time $t = 0$, the heat source at the right-hand endpoint is removed and that point is insulated. Find the temperature in the rod as a function of t and x. Suppose that the thermal diffusivity is $k = 1$.

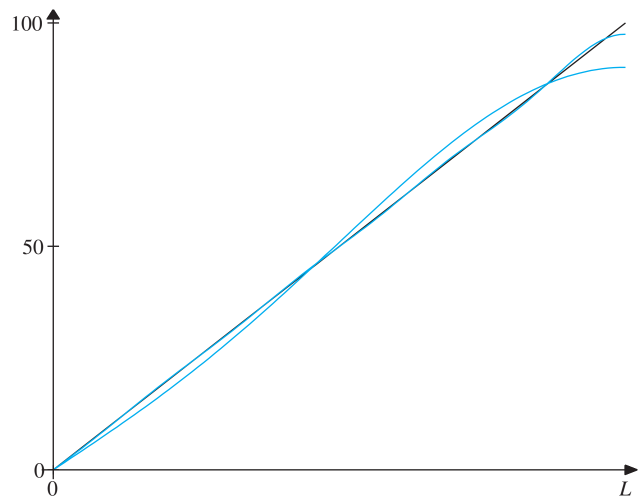


Figure 1 The partial sums of orders 2 and 8 for the generalized Fourier series in Example 7.17.

The initial temperature distribution in the rod is $100x/L$. The temperature at $x = 0$ is maintained at 0°. The end at $x = L$ is insulated, so we have a Neumann boundary condition there. Thus we need to solve the initial/boundary value problem

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \quad \text{for } t > 0 \text{ and } 0 < x < L, \\ &u(0,t) = 0 \quad \text{and} \quad u_x(L,t) = 0, \quad \text{for } t > 0, \\ &u(x,0) = 100x/L, \quad \text{for } 0 \le x \le L. \end{aligned} \tag{7.20}$$

Notice that the boundary conditions are already homogeneous, so we do not have to find the steady-state temperature first. Substituting the product solution $u(x,t) = X(x)T(t)$ into the heat equation, and separating variables, we see that the factors must satisfy the differential equations

$$T' + \lambda T = 0 \quad \text{and} \quad X'' + \lambda X = 0,$$

where λ is a constant. The first equation has the general solution

$$T(t) = Ce^{-\lambda t}. \tag{7.21}$$

As usual, we insist that X satisfy the boundary conditions, so we want to solve the Sturm Liouville problem

$$X'' + \lambda X = 0 \quad \text{with } X(0) = X'(L) = 0.$$

We did this is in Example 6.26. The solutions are

$$\lambda_n = \frac{(2n+1)^2\pi^2}{4L^2} \quad \text{and} \quad X_n(x) = \sin\frac{(2n+1)\pi x}{2L}, \quad \text{for } n = 0, 1, 2, \dots .$$

Thus, for every nonnegative integer n we get the product solution

$$u_n(x,t) = e^{-(2n+1)^2\pi^2 t/4L^2} \sin\frac{(2n+1)\pi x}{2L}$$

to the heat equation by using (7.21). This solution also satisfies the boundary conditions $u_n(0,t) = \partial u_n/\partial x(L,t) = 0$.

By the linearity of the heat equation, the function

$$u(x,t) = \sum_{n=0}^{\infty} c_n u_n(x,t) = \sum_{n=0}^{\infty} c_n e^{-(2n+1)^2\pi^2 t/4L^2} \sin\frac{(2n+1)\pi x}{2L} \tag{7.22}$$

is also a solution to the heat equation, provided the series converges. Furthermore, since each of the functions u_n satisfies the homogeneous boundary conditions, so does the linear combination u.

To satisfy the initial condition in (7.20), we must have

$$100x/L = u(x,0) = \sum_{n=0}^{\infty} c_n \sin\frac{(2n+1)\pi x}{2L}.$$

This is the problem we solved in Example 7.17. The coefficients are those in (7.18). Hence, our solution is

$$u(x,t) = \sum_{n=0}^{\infty} (-1)^n \frac{800}{(2n+1)^2\pi^2} e^{-(2n+1)^2\pi^2 t/4L^2} \sin\frac{(2n+1)\pi x}{2L}.$$

The evolution of the temperature $u(x,t)$ is depicted in Figure 2. The initial and steady-state temperatures are plotted in blue. The black curves represent the temperature after increments of 0.1s. Notice how the temperature steadily decreases throughout the rod to the steady-state temperature of 0°. ◆

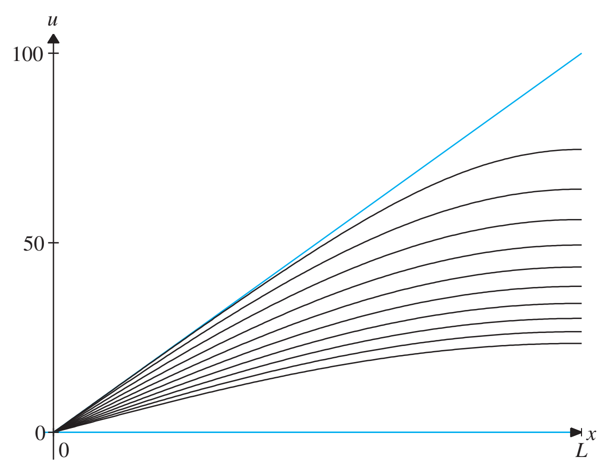


Figure 2 The temperature distribution in Example 7.19.

EXAMPLE 7.23 ◆ Suppose that the rod in Example 7.19 is weakly insulated at $x = L$, and satisfies the Robin condition $u_x(L,t) + \gamma u(L,t) = 0$ posed in Example 6.4. Find the temperature in the rod as a function of t and x.

We will not go into details, since the analysis is very similar to that in the previous example. The only difference is that the Robin boundary condition leads to the Sturm Liouville problem in Example 6.27. The eigenvalues and eigenfunctions are

$$\lambda_n = \frac{\theta_n^2}{L^2} \quad \text{and} \quad \phi_n(x) = \sin\frac{\theta_n x}{L}, \tag{7.24}$$

where θ_n is the nth positive solution of the equation $\tan\theta = -\theta/\gamma L$, which comes from (6.29). If we multiply this equation by $\gamma L\cos\theta$, we see that θ_n is the nth positive solution of the equation

$$\theta\cos\theta + \gamma L\sin\theta = 0. \tag{7.25}$$

Proposition 7.10 assures us that the eigenfunctions in (7.24) are orthogonal on $[0, L]$. Theorem 7.16 assures us that the generalized Fourier series based on these eigenvalues will converge, at least for $0 < x < L$. This series has the form

$$\frac{100x}{L} = \sum_{n=1}^{\infty} c_n \sin\frac{\theta_n x}{L}.$$

The coefficients can be calculated using (7.13). The calculation is similar to the computation in the previous example. If equation (7.25) is used to express everything in terms of $\cos\theta_n$, the result is

$$c_n = -\frac{200(\gamma L + 1)\cos\theta_n}{\theta_n(\gamma L + \cos^2\theta_n)}. \tag{7.26}$$

With $\gamma = L = 1$ the partial sums of order 2, 8, and 20 are shown in Figure 3.

The solution to the initial/boundary value problem is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t}\sin\frac{\theta_n x}{L},$$

where the eigenvalues are given in (7.24), and the coefficients are given in (7.26). The temperature is plotted in black in Figure 4 with temperatures given every 0.05 s. ◆

EXERCISES

In Exercises 1–4 find the generalized Fourier series for the indicated function using the eigenfunctions from Example 6.26 on the interval [0, 1].

1. $f(x) = 1$

2. $f(x) = \sin\pi x$ (*Hint:* Remember the trigonometric identity $\sin\alpha\sin\beta = [\cos(\alpha-\beta) - \cos(\alpha+\beta)]/2$.)

3. $f(x) = 1 - x$

4. $f(x) = \sin^2\pi x$

In Exercises 5–8 find the generalized Fourier series for referenced function using the eigenfunctions from Example 6.27 on the interval [0, 1] with $\gamma = 1$.

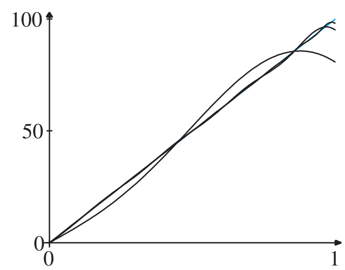


Figure 3 The partial sums of order 2, 8 and 20 for the generalized Fourier series in Example 7.23.

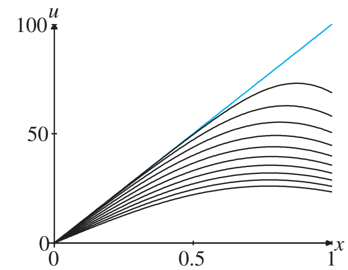


Figure 4 The temperature distribution in Example 7.23.

5. See Exercise 1

6. See Exercise 2

7. See Exercise 3

8. See Exercise 4

In Exercises 9–12 solve the heat equation $u_t = u_{xx}$ on the interval $0 < x < 1$ with the boundary conditions $u(0, t) = u_x(1, t) = 0$, and with $u(x, 0) = f(x)$ for the referenced function f.

9. See Exercise 1

10. See Exercise 2

11. See Exercise 3

12. See Exercise 4

In Exercises 13–16 solve the heat equation $u_t = u_{xx}$ on the interval $0 < x < 1$ with the boundary conditions $u(0, t) = u_x(1, t) + u(1, t) = 0$, and with $u(x, 0) = f(x)$ for the referenced function f.

13. See Exercise 5

14. See Exercise 6

15. See Exercise 7

16. See Exercise 8

17. Find the steady-state temperature u in a square plate of side length 1, where $u(x, 0) = T_1$, $u(x, 1) = T_2$, $u(0, y) = 0$, and $u_x(1, y) = 0$. (*Hint:* Look back at the methods used in Section 4.)

13.8 Temperatures in a Ball—Legendre Polynomials

We solved the problem of finding the steady-state temperature at points inside a disk or a rectangle in Sections 4 and 5. Now we want to do the same thing for the ball.

The ball of radius a is defined to be

$$B = \left\{(x, y, z) \,\middle|\, x^2 + y^2 + z^2 < a^2\right\}.$$

The boundary of the ball is the sphere

$$S = \left\{(x, y, z) \,\middle|\, x^2 + y^2 + z^2 = a^2\right\}.$$

According to (4.3), we want to find the function $u(x, y, z, t)$ on B that satisfies

$$\begin{aligned} \nabla^2 u(x, y, z) &= 0, \quad \text{for } (x, y, z) \in B, \\ u(x, y, z) &= f(x, y, z), \quad \text{for } (x, y, z) \in S, \end{aligned} \tag{8.1}$$

where f is a given function defined on the sphere S.

Since we have spherical symmetry, it is best to use spherical coordinates r, θ, and ϕ, which we discussed in Section 5. They are related to Cartesian coordinates by

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi. \tag{8.2}$$

The expression for the Laplacian in spherical coordinates is

$$\nabla^2 u = \frac{1}{r^2}(r^2 u_r)_r + \frac{1}{r^2\sin\phi}(\sin\phi \cdot u_\phi)_\phi + \frac{1}{r^2\sin^2\phi}u_{\theta\theta}.$$

In spherical coordinates the ball B and its boundary sphere S are described by

$$\begin{aligned} B &= \left\{(r, \theta, \phi) \,\middle|\, 0 \le r < a,\ -\pi < \theta \le \pi,\ 0 \le \phi \le \pi\right\}, \quad \text{and} \\ S &= \left\{(a, \theta, \phi) \,\middle|\, -\pi < \theta \le \pi,\ 0 \le \phi \le \pi\right\}. \end{aligned}$$

Since the boundary temperature is defined on S, where $r = a$, in spherical coordinates it is given by

$$F(\theta, \phi) = f(a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi).$$

Thus the boundary condition in (8.1) can be written as $u(a, \theta, \phi) = F(\theta, \phi)$.

We will solve the problem in the easier case when the temperature u is axially symmetric, meaning that it depends only on the radius r and the polar angle ϕ, and not on the variable θ. Then the boundary value problem in (8.1) becomes

$$\begin{aligned} \frac{1}{r^2}(r^2 u_r)_r + \frac{1}{r^2\sin\phi}(\sin\phi \cdot u_\phi)_\phi &= 0, \quad \text{for } 0 \le r < a \text{ and } 0 \le \phi \le \pi, \\ u(a, \phi) &= F(\phi), \quad \text{for } 0 \le \phi \le \pi. \end{aligned} \tag{8.3}$$

For (x, y, z) in the boundary of the ball, we have $z = a\cos\phi$, and $\phi = \cos^{-1}(z/a)$. Thus the boundary temperature will be axially symmetric if and only if the function $f(x, y, z)$ depends only on z. Then we have $F(\phi) = f(a\cos\phi)$.

We look for product solutions of the form $u(r, \phi) = R(r) \cdot T(\phi)$. For such functions the differential equation in (8.3) becomes

$$\frac{(r^2 R')' \cdot T}{r^2} + \frac{(\sin\phi \cdot T')' \cdot R}{r^2\sin\phi} = 0.$$

After separating the variables, we see that there is a constant λ such that

$$(r^2 R')' = \lambda R \quad \text{and} \quad -(\sin\phi \cdot T')' = \lambda\sin\phi \cdot T. \tag{8.4}$$

A singular Sturm Liouville problem

We will solve the second equation in (8.4) first. To be specific, we want to solve

$$-(\sin\phi \cdot T')' = \lambda \sin\phi \cdot T, \quad \text{for } 0 \le \phi \le \pi. \tag{8.5}$$

Notice that the points in the ball where $\phi = 0$ are on the positive z-axis, and $\phi = \pi$ corresponds to the negative z-axis. The product function u must be well behaved[14] on the entire sphere, including along the z-axis. Therefore, the factor $T(\phi)$ must be well behaved at $\phi = 0, \pi$.

The differential equation in (8.5) becomes more familiar when we make the substitution $s = \cos\phi$. Then $\sin^2\phi = 1 - \cos^2\phi = 1 - s^2$, and by the chain rule,

$$\frac{d}{d\phi} = \frac{ds}{d\phi}\frac{d}{ds} = -\sin\phi\frac{d}{ds}.$$

Hence,

$$-\frac{d}{d\phi}\left(\sin\phi\frac{dT}{d\phi}\right) = -\sin\phi\frac{d}{ds}\left[\sin^2\phi\frac{dT}{ds}\right] = -\sin\phi\frac{d}{ds}\left[(1-s^2)\frac{dT}{ds}\right].$$

Therefore, the differential equation in (8.5) becomes

$$LT = -\frac{d}{ds}\left[(1-s^2)\frac{dT}{ds}\right] = \lambda T, \quad \text{for } -1 \le s \le 1.$$

Notice that since $\phi \in [0, \pi]$, $s = \cos\phi \in [-1, 1]$. The operator L in this equation is formally self-adjoint, but while the coefficient $p(s) = 1 - s^2$ is positive on $(-1, 1)$, it vanishes at both endpoints. Thus the operator L is singular at both endpoints.

Although L is singular, from Proposition 6.12 we see that

$$(Lf, g) = (f, Lg) + (1-s^2)(fg' - f'g)\Big|_{-1}^{1} = (f, Lg),$$

for any functions f and g that have continuous second derivatives on $[-1, 1]$. Thus, no explicit boundary conditions are needed to make the operator L self-adjoint, although we do need that f and g and their first derivatives are continuous on the closed interval $[-1, 1]$. Since both the physics and the mathematics agree, we are led to pose the problem to find numbers λ and functions T such that

$$\begin{gathered} -\left((1-s^2)T'\right)' = \lambda T, \quad \text{for } -1 \le s \le 1, \\ \text{with } T \text{ and } T' \text{ continuous on } [-1, 1]. \end{gathered} \tag{8.6}$$

Although this is a self-adjoint Sturm Liouville problem, it is singular, so we cannot blindly apply the results of Sections 6 and 7. Indeed, the theory of singular Sturm Liouville problems leads to a variety of new phenomena. For example, Theorem 6.20, which states that the eigenvalues of a nonsingular Sturm Liouville problem form a sequence that converges to ∞, is not true in general for singular problems. In fact, it can happen that every positive real number is an eigenvalue. Singular Sturm

[14] The expression *well behaved* means bounded, continuous, or continuously differentiable. However, it is best kept a little vague.

Liouville problems are best analyzed on an ad hoc basis. When we do this for the problem in (8.6), we discover that all of the results in Sections 6 and 7 remain true.[15]

In particular, the proof of Proposition 6.24 can be easily modified for this case, and we see that all of the eigenvalues are nonnegative. Hence we can write $\lambda = n(n+1)$ where n is a nonnegative real number. Writing out the differential equation in (8.6), we get

$$(1-s^2)T'' - 2sT' + n(n+1)T = 0.$$

This will be recognized as Legendre's equation, which we studied in Section 11.3. By (8.6), we need solutions that are bounded at both endpoints. It is a fact, not easily proven, that we get solutions to Legendre's equation that are bounded at both endpoints only if n is a nonnegative integer. Furthermore, the only solution that is bounded at both endpoints is $P_n(s)$, the Legendre polynomial of degree n (see Exercise 23 in Section 11.6 for partial results in this direction). Thus, the solution to the Sturm Liouville problem in (8.6) is

$$\lambda_n = n(n+1) \quad \text{and} \quad P_n(s), \quad \text{for } n = 0, 1, 2, \ldots. \tag{8.7}$$

From Proposition 7.10, we see that two Legendre polynomials of different degrees are orthogonal. Since the weight in equation (8.6) is $w(s) = 1$, we have

$$(P_j, P_k) = \int_{-1}^{1} P_j(s)P_k(s)\,ds = 0, \quad \text{if } j \neq k.$$

We state without proof that

$$(P_n, P_n) = \int_{-1}^{1} P_n^2(s)\,ds = \frac{2}{2n+1}.$$

According to Theorem 7.16, if g is a piecewise continuous function on $[-1, 1]$, then it has an associated ***Legendre series***

$$g(s) \sim \sum_{n=0}^{\infty} c_n P_n(s), \quad \text{with} \quad c_n = \frac{(g, P_n)}{(P_n, P_n)} = \frac{2n+1}{2}\int_{-1}^{1} g(s)P_n(s)\,ds. \tag{8.8}$$

Solution to the boundary value problem

If we substitute $s = \cos\phi$ into (8.7), we see that the solutions to the second equation in (8.4) are

$$\lambda_n = n(n+1) \quad \text{and} \quad T_n(\phi) = P_n(\cos\phi), \quad \text{for } n = 0, 1, 2, \ldots,$$

where $P_n(s)$ is the Legendre polynomial of degree n. With $\lambda_n = n(n+1)$, the first equation in (8.4) becomes

$$r^2R'' + 2rR' - n(n+1)R = 0.$$

[15] See *Linear Differential Operators* by M. A. Naimark.

This will be recognized as a special case of Euler's equation (see Section 11.3). The only solution that is bounded near $r = 0$ is $R(r) = r^n$. Hence the product solutions are of the form

$$r^n P_n(\cos\phi), \quad \text{for } n = 0,\ 1,\ 2, \ldots,$$

and we look for a solution of the form

$$u(r, \phi) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\phi). \tag{8.9}$$

By the linearity of the Laplacian, if the series converges, this function is a solution to Laplace's equation. Thus we need only show that we can find the coefficients c_n so that the boundary condition in (8.3),

$$u(a, \phi) = \sum_{n=0}^{\infty} c_n a^n P_n(\cos\phi) = F(\phi),$$

is satisfied. If we again set $s = \cos\phi$, then $F(\phi) = f(a\cos\phi) = f(as)$, so we want

$$f(as) = \sum_{n=0}^{\infty} c_n a^n P_n(s). \tag{8.10}$$

This is just the Legendre series for $f(as)$, so by (8.8) we need

$$c_n a^n = \frac{2n+1}{2} \int_{-1}^{1} f(as) P_n(s)\, ds. \tag{8.11}$$

EXAMPLE 8.12 ◆ Find the steady-state temperature in a ball of radius $a = 1$, when the boundary is kept at the temperature $f(z) = 1 - z^2$.

Since the boundary temperature depends only on z, it is axially symmetric. Since $f(s) = 1 - s^2$ is a polynomial of degree 2, we expect that it is a linear combination of the first three Legendre polynomials, $P_0(s) = 1$, $P_1(s) = s$, and $P_2(s) = (3s^2 - 1)/2$. We easily see that in this case (8.10) becomes $f(s) = 2[P_0(s) - P_2(s)]/3$. Then, using (8.9), we see that the solution is

$$\begin{aligned} u(r, \phi) &= \frac{2}{3}[P_0(\cos\phi) - r^2 P_2(\cos\phi)] \\ &= \frac{2}{3}\left[1 - r^2 \frac{3\cos^2\phi - 1}{2}\right] \\ &= \frac{2 + r^2}{3} - r^2\cos^2\phi. \end{aligned}$$

We can express the temperature in Cartesian coordinates using (8.2). In fact, this is quite easy, since $r^2 = x^2 + y^2 + z^2$, and $r\cos\phi = z$. We see that the steady-state temperature is given by

$$u(x, y, z) = \frac{2 + x^2 + y^2 - 2z^2}{3}.$$

◆

EXERCISES

1. Find the steady-state temperature in a ball, assuming that the surface of the ball is kept at a uniform temperature of T.

2. Find the steady-state temperature in a ball of radius $a = 1$, assuming that the surface of the ball is kept at the temperature $f(z) = 1 - z$.

3. Find the steady-state temperature in a ball of radius $a = 1$, assuming that the surface of the ball is kept at the temperature $f(z) = z^3$.

4. Find the steady-state temperature in a ball of radius $a = 1$, assuming that the surface of the ball is kept at the temperature $f(z) = z^4$.

In Section 11.3 we presented the identity $xP_n'(x) - nP_n(x) = P_{n-1}'(x)$, and stated that $P_{2n+1}(0) = 0$, while $P_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}(n!)^2}$. You will find these facts useful in Exercises 5-8.

5. Suppose a ball of radius 1 is exactly half immersed into ice, so that the bottom half of the surface is at 0°C, while the upper half is kept at 10°C. Find the first three nonzero terms in the series expansion (8.9) of the steady-state temperature in the ball.

6. Find the complete Legendre series for the temperature in Exercise 5.

7. Suppose the surface of the ball of radius $a = 1$ is kept at the temperature

$$f(z) = \begin{cases} z & \text{if } 0 \le z \le 1, \\ 0 & \text{if } -1 \le z < 0. \end{cases}$$

Find the first three nonsero terms in the series expansion (8.9) of the steady-state temperature in the ball.

8. Find the complete Legendre series for the temperature in Exercise 7.

9. Withous doing any series computations, what is the temperature at the center of the ball in Exercise 5?

10. Suppose we have a spherical shell with inner radius 1 and outer radius 2. Suppose that the inner boundary is kept at 0°, and the outer boundary at 10°. Find the steady-state temperature throughout the shell.

13.9 The Heat and Wave Equations in Higher Dimension

We have successfully used the method of separation of variables to solve the heat equation and the wave equation when there is only one space variable. The method is also applicable when there are several space variables. In theory, the method is the same as it is with one space variable. In place of the Sturm Liouville problem that comes up naturally in one space variable, there is the more general eigenvalue problem for the Laplacian. When the space domain is a rectangle or a sphere, the geometric symmetry allows us to solve this problem using our favorite method of separation of variables.

Heat transfer on a rectangle

As an example of the method, let's consider the rectangle D of width a and height b which we first discussed in Section 3. The rectangle is illustrated in Figure 1, together with the initial and boundary conditions to be satisfied by the temperature $u(t, x, y)$.

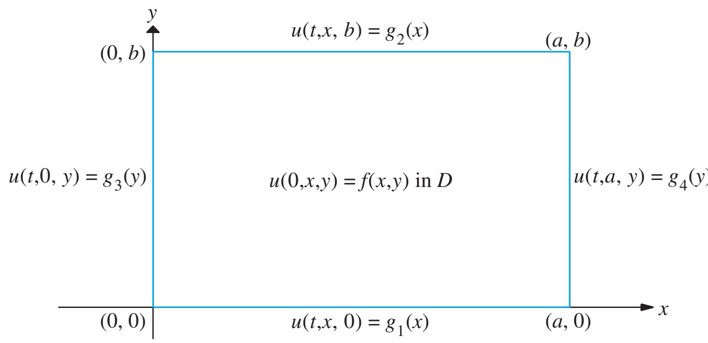


Figure 1 The Dirichlet problem for the rectangle D.

Notice that the boundary value of the temperature is described differently on each edge of the rectangle. To simplify our notation, we will define the function $g(x, y)$ on the boundary of the disk, ∂D, by

$$g(x, y) = \begin{cases} g_1(x), & \text{if } y = 0, \\ g_2(x), & \text{if } y = b, \\ g_3(y), & \text{if } x = 0, \\ g_4(y), & \text{if } x = a. \end{cases}$$

Then the initial/boundary value problem for the heat equation in the rectangle is

$$\begin{aligned} u_t(t, x, y) &= k\nabla^2 u(t, x, y), && \text{for } (x, y) \in D \text{ and } t > 0, \\ u(t, x, y) &= g(x, y), && \text{for } (x, y) \in \partial D \text{ and } t > 0, \\ u(0, x, y) &= f(x, y), && \text{for } (x, y) \in D. \end{aligned} \tag{9.1}$$

Let's suppose that the initial temperature is constant throughout D:

$$u(0, x, y) = f(x, y) = T_1, \quad \text{for } 0 < x < a \text{ and } 0 < y < b. \tag{9.2}$$

Let's also suppose that beginning at time $t = 0$ the boundary of the rectangle is submitted to a source of heat at the constant temperature T_2. Hence the boundary condition is

$$u(t, x, y) = g(x, y) = T_2, \quad \text{for } (x, y) \in \partial D \text{ and } t > 0. \tag{9.3}$$

We want to discover how the temperature in D varies as t increases.

Reduction to homogeneous boundary conditions

Our first step is to reduce the problem to one with homogeneous boundary conditions, as we did in Section 2. We do this by finding the steady-state solution u_s that solves the boundary value problem

$$\begin{aligned} \nabla^2 u_s(x, y) &= 0, \quad \text{for } (x, y) \in D, \\ u_s(x, y) &= g(x, y) = T_2, \quad \text{for } (x, y) \in \partial D, \end{aligned} \tag{9.4}$$

for Laplace's equation. We showed how to solve this problem for the rectangle D in Section 4. However, in our case the boundary temperature is constant, so we are led to expect that $u_s(x, y) = T_2$. Substituting into (9.4) verifies that this is correct.

Having found the steady-state temperature, it remains to find $v = u - u_s$. By combining the information in (9.1) and (9.4), we see that v must solve the homogeneous initial/boundary value problem

$$\begin{aligned} v_t(t, x, y) &= k\nabla^2 v(t, x, y), \quad \text{for } x, y \in D \text{ and } t > 0. \\ v(t, x, y) &= 0, \quad \text{for } (x, y) \in \partial D \text{ and } t > 0. \\ v(0, x, y) &= F(x, y) = f(x, y) - u_s(x, y), \quad \text{for } (x, y) \in D. \end{aligned} \tag{9.5}$$

In the case at hand, $F(x, y) = T_1 - T_2$. To solve the problem in (9.5), we use the method of separation of variables. It will be useful to compare what we do here with the method used in Section 2.

Step 1: Separate the PDE into an ODE in t and a PDE in x y. When we insert the product $v = T(t)\phi(x, y)$ into the heat equation $v_t = k\nabla^2 v$, we obtain $T'(t)\phi(x, y) = kT(t)\nabla^2\phi(x, y)$. Separating the variable t from the pair of variables x and y, we get the two differential equations

$$T' + \lambda k T = 0 \quad \text{and} \quad -\nabla^2\phi = \lambda\phi, \tag{9.6}$$

where λ is a constant. Notice the similarity with equation (2.8). The first equation has the solution

$$T(t) = Ce^{-\lambda k t}. \tag{9.7}$$

It is the second equation that requires our attention. This time it is a partial differential equation.

Step 2: Solve the eigenvalue problem for the Laplacian. We will insist that the product solution $v(t, x, y) = T(t)\phi(x, y)$ satisfy the homogeneous boundary condition coming from (9.5). Since this condition affects only the factor ϕ, the problem to be solved is finding λ and ϕ such that

$$-\nabla^2\phi = \lambda\phi, \quad \text{with } \phi(x, y) = 0 \text{ for } (x, y) \in \partial D. \tag{9.8}$$

This is called an ***eigenvalue problem*** for the Laplacian. Using the same terminology we used for the Sturm Liouville problem in dimension $n = 1$, the number λ is called an ***eigenvalue***, and the function ϕ is an ***associated eigenfunction***. In our case we have the Dirichlet boundary condition $\phi(x, y) = 0$ for $(x, y) \in \partial D$, but we could have Neumann or Robin conditions.

We look for product solutions $\phi(x, y) = X(x)Y(y)$. The differential equation becomes $-X''(x)Y(y) - X(x)Y''(y) = \lambda X(x)Y(y)$. When we separate variables, we see that there must be constants μ and ν such that

$$-X'' = \mu X \quad \text{and} \quad -Y'' = \nu Y, \quad \text{with} \quad \mu + \nu = \lambda. \tag{9.9}$$

Next, look at the boundary condition. For example, we require that $\phi(0, y) = X(0)Y(y) = 0$ for $0 < y < b$. This means that we must have $X(0) = 0$. In the same way, we see that $X(a) = 0$, and $Y(0) = Y(b) = 0$. Together with the differential equations in (9.9), we see that we have Sturm Liouville problems for both X and Y. It is essentially the same problem for both, and it is the problem we solved in Section 2, ending with equation (2.12). According to (2.12), we have solutions

$$\begin{aligned} \mu_i &= \frac{i^2\pi^2}{a^2} \quad \text{and} \quad X_i(x) = \sin\frac{i\pi x}{a} \\ \nu_j &= \frac{j^2\pi^2}{b^2} \quad \text{and} \quad Y_j(y) = \sin\frac{j\pi y}{b}, \end{aligned}$$

for $i, j = 1, 2, 3, \ldots$. To sum up, the eigenvalue problem for the rectangle D has solution

$$\lambda_{i,j} = \frac{i^2\pi^2}{a^2} + \frac{j^2\pi^2}{b^2} \quad \text{and} \quad \phi_{i,j}(x, y) = \sin\frac{i\pi x}{a}\cdot\sin\frac{j\pi y}{b}, \tag{9.10}$$

for $i, j = 1, 2, 3, \ldots$.

Step 3: Solving the initial/boundary value problem. The finish of the process is very much like it was in dimension $n = 1$. Notice that for each pair of positive integers i and j we have the solution $T_{i,j}(t) = e^{-\lambda_{i,j}kt}$ from (9.7). The product $T_{i,j}(t)\phi_{i,j}(x, y) = e^{-\lambda_{i,j}kt}\phi_{i,j}(x, y)$ is a solution to the heat equation and the homogeneous boundary conditions. Using the linearity of the heat equation, and assuming that there are no convergence problems, we see that any function of the form

$$v(t, x, y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{i,j}e^{-\lambda_{i,j}kt}\phi_{i,j}(x, y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{i,j}e^{-\lambda_{i,j}kt}\sin\frac{i\pi x}{a}\cdot\sin\frac{j\pi y}{b}$$

will be a solution to the heat equation and will also satisfy the boundary conditions. In order that the initial condition be solved, we set $t = 0$ to get

$$\begin{aligned} v(0, x, y) = F(x, y) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{i,j}\phi_{i,j}(x, y) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{i,j}\sin\frac{i\pi x}{a}\cdot\sin\frac{j\pi y}{b}, \end{aligned} \tag{9.11}$$

for $0 \le x \le a$ and $0 \le y \le b$. Equation (9.11) is a Fourier series in both x and y simultaneously. Using the orthogonality relations for Fourier series, we see that

$$\begin{aligned}\int_D \phi_{i,j}\,\phi_{i',j'}\,dx\,dy &= \int_0^a \int_0^b \sin\frac{i\pi x}{a}\sin\frac{j\pi y}{b}\sin\frac{i'\pi x}{a}\sin\frac{j'\pi y}{b}\,dx\,dy\\ &= \int_0^a \sin\frac{i\pi x}{a}\sin\frac{i'\pi x}{a}\,dx\int_0^b \sin\frac{j\pi y}{b}\sin\frac{j'\pi y}{b}\,dy \\ &= \begin{cases}\dfrac{ab}{4}, & \text{if } i = i' \text{ and } j = j',\\ 0, & \text{otherwise.}\end{cases}\end{aligned} \tag{9.12}$$

Therefore, if we multiply the series in (9.11) by $\phi_{i',j'}$ and integrate over the rectangle D, we get

$$\int_D F\,\phi_{i',j'}\,dx\,dy = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{i,j}\int_D \phi_{i,j}\,\phi_{i',j'}\,dx\,dy = \frac{ab}{4}c_{i',j'}.$$

Consequently, the coefficients are given by

$$c_{i,j} = \frac{4}{ab}\int_D F(x,y)\,\phi_{i,j}(x,y)\,dx\,dy. \tag{9.13}$$

In our case, we have $F(x, y) = T_1 - T_2$, so

$$\begin{aligned} c_{i,j} &= \frac{4}{ab}(T_1 - T_2)\int_0^a \sin\frac{i\pi x}{a}\,dx \int_0^b \sin\frac{j\pi y}{b}\,dy\\ &= (T_1 - T_2)\frac{4}{ij\pi^2}[1 - \cos i\pi][1 - \cos j\pi].\end{aligned}$$

Hence, $c_{i,j} = 0$ unless both i and j are odd, and

$$c_{2i+1,2j+1} = \frac{16(T_1 - T_2)}{(2i+1)(2j+1)\pi^2}.$$

Thus, the solution to the homogeneous initial/boundary value problem is

$$v(t, x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{16(T_1 - T_2)}{\pi^2(2i+1)(2j+1)}e^{-\lambda_{2i+1,2j+1}kt}\phi_{2i+1,2j+1}(x, y).$$

The solution to the original problem is

$$\begin{aligned} u(t, x, y) &= u_s(x, y) + v(t, x, y)\\ &= T_2 + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{16(T_1 - T_2)}{\pi^2(2i+1)(2j+1)}e^{-\lambda_{2i+1,2j+1}kt}\phi_{2i+1,2j+1}(x, y).\end{aligned}$$

Vibrations of a rectangular drum

Without much more work we can analyze the modes of vibration of a rectangular drum. The displacement $u(t, x, y)$ of the drum is governed by the wave equation $u_{tt} = c^2\nabla^2 u$. The edge of the drum is fixed, so it satisfies the homogeneous boundary condition $u(t, x, y) = 0$ for $(x, y) \in \partial D$. The drum has an initial displacement $f_0(x, y)$ and initial velocity $f_1(x, y)$. Hence the displacement of the drum satisfies the initial/boundary value problem

$$\begin{aligned} u_{tt}(t, x, y) &= c^2\nabla^2 u(t, x, y), \quad \text{for } (x, y) \in D \text{ and } t > 0, \\ u(t, x, y) &= 0, \quad \text{for } (x, y) \in \partial D \text{ and } t > 0, \\ u(0, x, y) &= f_0(x, y), \quad \text{for } (x, y) \in D, \\ u_t(0, x, y) &= f_1(x, y), \quad \text{for } (x, y) \in D. \end{aligned} \tag{9.14}$$

We look for product solutions to the wave equation, so we set $u(t, x, y) = T(t)\phi(x, y)$ and substitute into the wave equation, getting

$$T''(t)\phi(x, y) = c^2 T(t)\nabla^2\phi(x, y).$$

Arguing as we have before, we get the two differential equations

$$T'' + \lambda c^2 T = 0 \quad \text{and} \quad -\nabla^2\phi = \lambda\phi, \tag{9.15}$$

where λ is a constant. Since are looking for solutions to the second equation that vanish on the boundary of D, we have once more the eigenvalue problem in (9.8), and the solutions are those in (9.10).

It remains to solve the first equation in (9.15) with $\lambda = \lambda_{i,j}$. If we set

$$\omega_{i,j} = c\sqrt{\lambda_{i,j}} = c\pi\sqrt{\frac{i^2}{a^2} + \frac{j^2}{b^2}}, \tag{9.16}$$

the equation becomes $T'' + \omega_{i,j}^2 T = 0$. This equation has the fundamental set of solutions $\sin\omega_{i,j}t$ and $\cos\omega_{i,j}t$. Hence for the eigenvalue $\lambda_{i,j}$ we have two linearly independent product solutions

$$\sin(\omega_{i,j}t)\cdot\phi_{i,j}(x, y) \quad \text{and} \quad \cos(\omega_{i,j}t)\cdot\phi_{i,j}(x, y), \tag{9.17}$$

where $\phi_{i,j}(x, y)$ is the solution found in (9.10). Every solution to the initial/boundary value problem in (9.14) is an infinite series in these product solutions. Hence if u is a solution we have

$$u(t, x, y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}[a_{i,j}\cos(\omega_{i,j}t) + b_{i,j}\sin(\omega_{i,j}t)]\phi_{i,j}(x, y).$$

Evaluating u and u_t at $t = 0$, we see that

$$\begin{aligned} f_0(x, y) = u(0, x, y) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j}\phi_{i,j}(x, y), \quad \text{and} \\ f_1(x, y) = u_t(0, x, y) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \omega_{i,j}b_{i,j}\phi_{i,j}(x, y). \end{aligned} \tag{9.18}$$

These are double Fourier series like that in (9.11), so the coefficients can be evaluated using (9.13). We get

$$a_{i,j} = \frac{4}{ab}\int_D f_0(x,y)\,\phi_{i,j}(x,y)\,dx\,dy \quad \text{and}$$
$$b_{i,j} = \frac{4}{ab\omega_{i,j}}\int_D f_1(x,y)\,\phi_{i,j}(x,y)\,dx\,dy.$$

Notice that the product solutions in (9.17) are periodic in time with frequency $\omega_{i,j}$ given in (9.16). Unlike the case of the vibrating string, these frequencies are not integer multiples of the lowest frequency $\omega_{1,1}$. Consequently the vibrations of a rectangular drum will not have the fine musical qualities of a violin.

EXERCISES

in Exercises 1– 6 we will further explore heat transfer with in a square plate D of side length 1. Suppose first that the plate has three sides which are kept at 0°, while the fourth side is insulated. Then the boundary conditions for the temperature can be written as

$$u(t,x,0) = u(t,x,1) = 0, \quad \text{and} \quad u(t,0,y) = u_x(t,1,y) = 0. \tag{9.19}$$

Notice that the steady-state temperature in the plate is 0°. The temperature in D satisfies the heat equation $u_t = k\nabla^2 u$.

1. Suppose that $u(t,x,y) = T(t)\phi(x,y)$ is a product solution of the heat equation, together with the boundary conditions in 9.19. Show that there is a constant λ such that

(a) T satisfies the equation $T' + \lambda T = 0$.

(b) ϕ satisfies $-\nabla^2\phi = \lambda\phi$, together with the boundary conditions

$$\phi(x,0) = \phi(x,1) = 0, \quad \text{and} \quad \phi(0,y) = \phi_x(1,y) = 0. \tag{9.20}$$

2. Solve the eigenvalue problem for the Laplacian in part (b) of Exercise 1, and show that the solutions are

$$\lambda_{p,q} = \frac{(2p+1)^2\pi^2}{4} + q^2\pi^2 \quad \text{with}$$
$$\phi_{p,q}(x,y) = \sin\left(\frac{2p+1)\pi x}{2}\right)\sin q\pi y,$$

for $p \geq 0$ and $q \geq 1$.

3. Show that

$$\int_D \phi_{p,q}(x,y)\,\phi_{p',q'}(x,y)\,dx\,dy = \begin{cases} 1/4, & \text{if } p = p' \text{ and } q = q' \\ 0, & \text{otherwise.} \end{cases}$$

4. Suppose that the initial temperature is $u(0, x, y) = f(x, y)$. Show that the temperature is given by

$$u(t, x, y) = \sum_{p=0}^{\infty} \sum_{q=1}^{\infty} c_{p,q} e^{\lambda_{p,q} kt} \phi_{p,q}(x, y),$$

where the coefficients are given by

$$c_{p,q} = 4 \int_D f(x, y) \phi_{p,q}(x, y)\, dx\, dy.$$

5. How must the solution in Exercise 4 be modified if the boundary conditions are changed to

$$u(t, x, 0) = u(t, x, 1) = T_1, \quad \text{and} \quad u(t, 0, y) = u_x(t, 1, y) = 0?$$

6. How must the solution in Exercise 4 be modified if the plate is insulated on two opposite edges, so that the boundary conditions are changed to

$$u(t, x, 0) = u(t, x, 1) = 0, \quad \text{and} \quad u_x(t, 0, y) = u_x(t, 1, y) = 0?$$

7. Suppose we have a square drum with side length π, and suppose that it is plucked in the middle and then released. Then its initial displacement is given by $f(x, y) = \min\{x, y, \pi - x, \pi - y\}$, while its initial velocity is 0. The graph of f is a four-sided pyramid with height $\pi/2$. Use the techniques of this section to compute the displacement as a function of both time and space. This seemingly daunting task is made easier if you follow these steps.

(a) Show that
$$f(x, y) = [F(x - y) - F(x + y)]/2,$$
where $F(z)$ is the periodic extension of $\pi - |z|$ from the interval $[-\pi, \pi]$ to the reals. (*Hint:* Just check the cases.)

(b) Compute the Fourier series for F on the interval $[-\pi, \pi]$.

(c) Use the formula in 7a and the addition formula for the cosine to complete the computation of the double Fourier series for f. You will notice that the series has the form $f(x, y) = \sum_{p=1}^{\infty} a_p \sin px \sin py$. Comparing this with the series that appear in 9.18, we see that the coefficients of all of the off-diagonal terms are equal to 0.

(d) Find the displacement $u(t, x, y)$ in the way described in this section. Is the vibration of the drum with these initial conditions periodic in time?

8. Using the terminology in Exercise 7, show that the function

$$\begin{aligned} u(t, x, y) = \frac{1}{4}[&F(x - y + \sqrt{2}ct) + F(x - y - \sqrt{2}ct) \\ &- F(x + y + \sqrt{2}ct) - F(x + y - \sqrt{2}ct)] \end{aligned}$$

is a solution to the wave equation and also satisfies the initial and boundary conditions in Exercise 7.

13.10 Domains with Circular Symmetry—Bessel Functions

In this section we will analyze the vibrations of a circular drum. Let D be the disk of radius a, which we describe as

$$D = \left\{(x, y) \in \mathbf{R}^2 \middle| x^2 + y^2 < a^2\right\}.$$

The displacement $u(t, x, y)$ of the circular drum on D satisfies the initial/boundary value problem in (9.14). Separation of variables leads us once again to the two differential equations in (9.15). Hence we are led to an eigenvalue problem for the disk D, which is to find all numbers λ and functions ϕ such that

$$\begin{aligned} -\nabla^2\phi(x, y) &= \lambda\phi(x, y), \quad \text{for } (x, y) \in D, \text{ and} \\ \phi(x, y) &= 0, \quad \text{for } (x, y) \in \partial D. \end{aligned} \tag{10.1}$$

As we did in Section 5, we will use polar coordinates (see equation (5.2)) to solve the problem in (10.1). In these coordinates the eigenvalue problem in (10.1) becomes

$$\begin{aligned} -\left[\phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}u_{\theta\theta}\right](r, \theta) &= \lambda\phi(r, \theta), \quad \text{for } r < a, \\ \phi(a, \theta) &= 0, \quad \text{for } 0 \le \theta \le 2\pi. \end{aligned} \tag{10.2}$$

When we substitute a product function of the form $\phi(r, \theta) = R(r)U(\theta)$ into the differential equation in (10.2), we get

$$R_{rr}(r)U(\theta) + \frac{1}{r}R_r(r)U(\theta) + \frac{1}{r^2}R(r)U_{\theta\theta}(\theta) + \lambda R(r)U(\theta) = 0.$$

To separate variables, we multiply by r^2/RU, obtaining

$$\frac{r^2R_{rr} + rR_r + \lambda r^2 R}{R} + \frac{U_{\theta\theta}}{U} = 0.$$

This sum of a function of r and a function of θ can be equal to 0 only if each is constant. Hence there is a constant μ such that

$$r^2R_{rr} + rR_r + \lambda r^2 R - \mu R = 0 \quad \text{and} \quad U_{\theta\theta} + \mu U = 0. \tag{10.3}$$

We will solve the second equation in (10.3) first. Remember that θ represents the polar angle in the disk, so the solution U must be periodic with period 2π in θ. We examined the resulting Sturm Liouville problem in Example 6.17 in Section 6, and found that we must have $\mu = n^2$, where n is a nonnegative integer, and that the eigenfunctions are

$$\begin{aligned} &1, \quad \text{for } n = 0 \\ \sin n\theta, \quad &\text{and} \quad \cos n\theta, \quad \text{for } n \ge 1. \end{aligned} \tag{10.4}$$

Bessel functions

Substituting $\mu = n^2$ into the first equation in (10.3) and then rearranging it, we get the equation

$$r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} - n^2 R = -\lambda r^2 R. \tag{10.5}$$

After dividing by r and multiplying by -1, it becomes

$$-(rR')' + \frac{n^2}{r} R = \lambda r R. \tag{10.6}$$

The operator L defined by $LR = -(rR')' + n^2 R/r$ that appears in (10.6) is formally self-adjoint. However, the coefficient $p(r) = r$ vanishes at $r = 0$, and the coefficient $q(r) = n^2/r$ has an infinite discontinuity there. Hence, the operator is singular at $r = 0$.

As we did for the Legendre equation in Section 8, we will require that the eigenfunctions are continuous at $r = 0$ together with their first derivatives. At the other boundary point $r = a$, the boundary condition comes from (10.1). Our Sturm Liouville problem is

$$\begin{gathered} -(rR')' + \frac{n^2}{r} R = \lambda r R, \quad \text{for } 0 < r < a, \\ R \text{ and } R' \text{ are continuous at } r = 0, \\ R(a) = 0. \end{gathered} \tag{10.7}$$

Notice that the weight function is $w(r) = r$.

Once again we are fortunate. Even though the Sturm Liouville problem is singular, it has all of the properties of nonsingular problems that we described in Sections 6 and 7. In particular, Proposition 6.24 remains true, and we see that all of the eigenvalues are positive. Hence, we can write $\lambda = \nu^2$, where $\nu > 0$. If we make the change of variables $s = \nu r$ in the differential equation in (10.5) and rearrange it, it becomes

$$s^2 \frac{d^2 R}{ds^2} + s \frac{dR}{ds} + [s^2 - n^2]R = 0.$$

(See Exercise 9 in Section 11.7.) This is Bessel's equation of order n. In Section 11.7 we discovered that a fundamental set of solutions is the pair $J_n(s)$ and $Y_n(s)$. Therefore, the general solution to the differential equation in (10.7) is $R(r) = AJ_n(\nu r) + BY_n(\nu r)$. However, since $Y_n(\nu r)$ has an infinite singularity at $r = 0$, it does not satisfy the boundary condition at $r = 0$ in (10.7). Therefore, $B = 0$. Taking $A = 1$, we have $R(r) = J_n(\nu r)$.

It remains to satisfy the boundary condition $J_n(\nu a) = 0$. We discussed the zeros of the Bessel functions in Section 11.7. There are infinitely many of them. If $\alpha_{n,k}$ is the kth zero of J_n, then we need $\nu = \nu_{n,k} = \alpha_{n,k}/a$. Consequently, the solutions to the Sturm Liouville problem in (10.7) are

$$\lambda_k = \frac{\alpha_{n,k}^2}{a^2} \quad \text{and} \quad R_k(r) = J_n(\alpha_{n,k} r/a), \quad \text{for } k = 1, 2, \ldots. \tag{10.8}$$

From Proposition 7.10, we see that the functions R_k are orthogonal with respect to the weight $w(r) = r$. This means that

$$(R_k, R_j)_r = \int_0^a J_n(\alpha_{n,k}r/a)\, J_n(\alpha_{n,j}r/a)\, r\, dr = 0, \quad \text{if } j \neq k.$$

A rather difficult computation shows that

$$(R_k, R_k)_r = \int_0^a J_n(\alpha_{n,k}r/a)^2\, r\, dr = \frac{a^2}{2} J_{n+1}^2(\alpha_{n,k}).$$

If f is a piecewise continuous function on $[0, a]$, then its associated ***Fourier-Bessel series*** is

$$f(r) \sim \sum_{k=1}^{\infty} c_k J_n(\alpha_{n,k}r/a), \tag{10.9}$$

where the coefficients are given by

$$c_k = \frac{(f, R_k)_r}{(R_k, R_k)_r} = \frac{2}{a^2 J_{n+1}^2(\alpha_{n,k})} \int_0^a f(r)\, J_n(\alpha_{n,k}r/a)\, r\, dr. \tag{10.10}$$

The integral in (10.10) is difficult to compute, even for the simplest functions f. Not infrequently it is necessary to compute the integral approximately for small values of k.

Solution to the eigenvalue problem on the disk

Bringing together the results in (10.4) and (10.8), we see that the solutions to the eigenvalue problem in (10.1) for the Laplacian on the disk are

$$\begin{aligned}
&\lambda_{0,k} = \frac{\alpha_{0,k}^2}{a^2} \quad \text{with} \quad \phi_{0,k}(r,\theta) = J_0(\alpha_{0,k}r/a),\\
&\qquad\text{for } n = 0 \text{ and } k = 1, 2, 3, \ldots\\
&\lambda_{n,k} = \frac{\alpha_{n,k}^2}{a^2} \quad \text{with} \quad \begin{cases} \phi_{n,k}(r,\theta) = \cos n\theta \cdot J_n(\alpha_{n,k}r/a) & \text{and}\\ \psi_{n,k}(r,\theta) = \sin n\theta \cdot J_n(\alpha_{n,k}r/a), \end{cases}\\
&\qquad\text{for } n = 1, 2, 3, \ldots \text{ and } k = 1, 2, 3, \ldots .
\end{aligned} \tag{10.11}$$

By integrating using polar coordinates, and using the orthogonality relations for the Bessel functions and the trigonometric functions, we see that the eigenfunctions $\phi_{n,k}$ and $\psi_{n,k}$ satisfy the orthogonality relations

$$\begin{aligned}
&\int_D \phi_{n,k}\phi_{n',k'}\, dx\, dy = \begin{cases} \pi a^2 J_{n+1}^2(\alpha_{n,k})/2, & \text{if } n' = n \text{ and } k' = k.\\ 0, & \text{otherwise,} \end{cases}\\
&\int_D \psi_{n,k}\psi_{n',k'}\, dx\, dy = \begin{cases} \pi a^2 J_{n+1}^2(\alpha_{n,k})/2, & \text{if } n' = n \text{ and } k' = k.\\ 0, & \text{otherwise,} \end{cases}\\
&\int_D \phi_{n,k}\psi_{n',k'}\, dx\, dy = 0, \quad \text{in all cases.}
\end{aligned} \tag{10.12}$$

The solution of the wave equation

For the time dependence of the product solution to the wave equation, we must solve the first equation in (9.15). With $\omega_{n,k}^2 = c^2\lambda_{n,k} = (c\alpha_{n,k}/a)^2$, this equation becomes $T'' + \omega_{n,k}^2 T = 0$. The solutions are $\cos(\omega_{n,k}t)$ and $\sin(\omega_{n,k}t)$.Thus the product solutions of the wave equation are of the form

$$\begin{aligned}&\cos(\omega_{n,k}t)\cdot\phi_{n,k}(r,\theta), \quad \cos(\omega_{n,k}t)\cdot\psi_{n,k}(r,\theta),\\ &\sin(\omega_{n,k}t)\cdot\phi_{n,k}(r,\theta), \quad \text{and} \quad \sin(\omega_{n,k}t)\cdot\psi_{n,k}(r,\theta),\end{aligned} \tag{10.13}$$

for all appropriate choices of the indices. By linearity, any function of the form

$$\begin{aligned}u(t,r,\theta) = &\sum_{k=1}^{\infty} J_0\left(\frac{\alpha_{0,k}r}{a}\right)\left[A_{0,k}\cos\frac{c\alpha_{0,k}t}{a} + B_{0,k}\sin\frac{c\alpha_{0,k}t}{a}\right]\\ &+ \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} J_n\left(\frac{\alpha_{n,k}r}{a}\right)\left[A_{n,k}\cos n\theta + B_{n,k}\sin n\theta\right]\cos\frac{c\alpha_{n,k}t}{a}\\ &+ \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} J_n\left(\frac{\alpha_{n,k}r}{a}\right)\left[C_{n,k}\cos n\theta + D_{n,k}\sin n\theta\right]\sin\frac{c\alpha_{n,k}t}{a}\end{aligned} \tag{10.14}$$

is formally a solution to the wave equation on the disk that satisfies Dirichlet boundary conditions.

The coefficients are evaluated using the initial conditions $u(0,r,\theta) = f_0(r,\theta)$ and $u_t(0,r,\theta) = f_1(r,\theta)$. Evaluating (10.14) at $t = 0$, we see that

$$\begin{aligned}f_0(r,\theta) = &\sum_{k=1}^{\infty} A_{0,k}J_0\left(\frac{\alpha_{0,k}r}{a}\right)\\ &+ \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} J_n\left(\frac{\alpha_{n,k}r}{a}\right)\left[A_{n,k}\cos n\theta + B_{n,k}\sin n\theta\right].\end{aligned}$$

The coefficients can be found in the usual way, using the orthogonality relations in (10.12). We get

$$\begin{aligned}A_{n,k} &= \frac{2}{\pi a^2 J_{n+1}(\alpha_{n,k})}\int_0^a\int_0^{2\pi} f_0(r,\theta)\,J_n\left(\frac{\alpha_{n,k}r}{a}\right)\cos n\theta\, r\,dr\,d\theta, \quad \text{and}\\ B_{n,k} &= \frac{2}{\pi a^2 J_{n+1}^2(\alpha_{n,k})}\int_0^a\int_0^{2\pi} f_0(r,\theta)\,J_n\left(\frac{\alpha_{n,k}r}{a}\right)\sin n\theta\, r\,dr\,d\theta.\end{aligned} \tag{10.15}$$

The remaining coefficients in (10.14) can be evaluated in the same way using the initial velocity. They are

$$\begin{aligned}C_{n,k} &= \frac{2}{c\pi a\alpha_{n,k}J_{n+1}^2(\alpha_{n,k})}\int_0^a\int_0^{2\pi} f_1(r,\theta)\,J_n\left(\frac{\alpha_{n,k}r}{a}\right)\cos n\theta\, r\,dr\,d\theta, \quad \text{and}\\ D_{n,k} &= \frac{2}{c\pi a\alpha_{n,k}J_{n+1}^2(\alpha_{n,k})}\int_0^a\int_0^{2\pi} f_1(r,\theta)\,J_n\left(\frac{\alpha_{n,k}r}{a}\right)\sin n\theta\, r\,dr\,d\theta.\end{aligned} \tag{10.16}$$

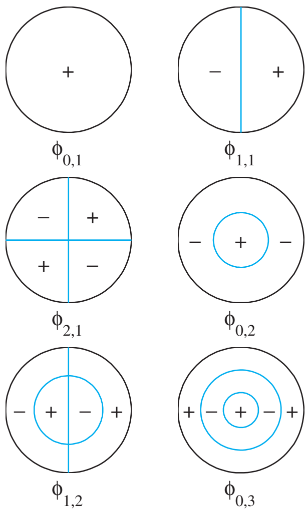


Figure 1 The nodal sets for some fundamental modes.

The fundamental modes of vibration of a drum

Notice that the product solutions in (10.12) represent vibrations of the drum with frequency $\omega_{n,k} = c\alpha_{n,k}/a$, and with an amplitude that varies over the drum like the functions $\phi_{n,k}$ and $\psi_{n,k}$. For this reason, the functions $\phi_{n,k}$ and $\psi_{n,k}$ are referred to as the ***fundamental modes*** of vibration for the drum.

The frequencies are proportional to the zeros of the Bessel's functions. According to Table 1 in Section 11.7, the four smallest zeros are $\alpha_{0,1} = 2.4048$, $\alpha_{1,1} = 3.8317$, $\alpha_{2,1} = 5.1356$, and $\alpha_{0,2} = 5.5201$. We see that the frequencies are clearly not integer multiples of a lowest, fundamental frequency, which is the case for the vibrating string. This explains the quite different sounds of a kettle drum and a violin.

The ***nodal set*** of a fundamental mode is the set where it vanishes. During a vibration in a fundamental mode, the points in the nodal set do not move. Since $\phi_{0,1}(r,\theta) = J_0(\alpha_{0,1}r/a)$ is not equal to 0 for $r < a$, its nodal set is empty. Similarly, $\phi_{1,1} = \cos\theta\, J_1(\alpha_{1,1}r/a) = 0$ only where $\cos\theta = 0$, so its nodal set is the y-axis. The nodal sets for several fundamental modes are shown in Figure 1. The $+$ and $-$ signs indicate regions where the drum head has opposite displacement during the oscillation. As n and k get large, the motion of the drum in a fundamental mode can get quite complicated.

If you strike a kettle drum in the middle, seemingly a natural place to do so, you will get a mixture of all of the frequencies as shown in (10.14). The result is a sound that is really awful. Naturally, professional tympanists avoid this. Instead, they carefully strike the drum near the edge. The result is that the lowest frequency is eliminated from the mixture. In fact, a professional tympanist gets a sound that is almost a pure $\phi_{1,1}$ mode.

EXERCISES

1. Verify the orthogonality relations in (10.12).

2. Verify the formulas in (10.15) and (10.16).

3. Plot the nodal sets for the fundamental modes $\phi_{1,3}$, $\phi_{3,2}$, and $\phi_{2,4}$.

4. Suppose that the initial displacement of the drum is a function $u(0,r,\theta) = f(r)$, that is independent of the angle θ, and the initial velocity is 0. How does the series for the solution in (10.14) simplify?

5. Suppose that the initial temperature in a disk D of radius a is $u(0,r,\theta) = f(r)$, where $f(r)$ is a function of the radius r only. It is safe to assume that $u = u(t,r)$ is also independent of the angle θ.

(a) Show that $\nabla^2 u = u_{rr} + u_r/r$.

(b) Assuming that the temperature vanishes on the boundary of the disk, the initial/boundary value problem in polar coordinates is

$$u_t = k\left[u_{rr} + \frac{1}{r}u_r\right], \quad \text{for } 0 \le r < a \text{ and } t > 0,$$
$$u(t,a) = 0, \quad \text{for } t > 0,$$
$$u(0,r) = f(r), \quad \text{for } 0 \le r < a.$$

Find the product solutions that satisfy the Dirichlet boundary condition.

(c) Find a series expansion for the temperature $u(t, r, \theta)$.

6. Find a series expansion for the solution to the initial/boundary value problem

$$\begin{aligned} u_t(t, x, y) &= k\nabla^2 u(t, x, y), \quad \text{for } (x, y) \in D \text{ and } t > 0, \\ u(t, x, y) &= 0, \quad \text{for } (x, y) \in \partial D \text{ and } t > 0, \\ u(t, x, y) &= f(x, y), \quad \text{for } (x, y) \in D, \end{aligned}$$

where D is the disk of radius a.

7. Consider the initial/boundary value problem

$$\begin{aligned} u_t(t, x, y) &= k\nabla^2 u(t, x, y), \quad \text{for } (x, y) \in D \text{ and } t > 0, \\ \frac{\partial u}{\partial \mathbf{n}}(t, x, y) &= 0, \quad \text{for } (x, y) \in \partial D \text{ and } t > 0, \\ u(t, x, y) &= f(x, y), \quad \text{for } (x, y) \in D, \end{aligned}$$

where D is the disk of radius 1. The problem models the temperature in a circular plate when the boundary is insulated.

(a) What is the eigenvalue problem for the disk that arises when you solve this problem by separation of variables?

(b) Restate the eigenvalue problem in part (a) in polar coordinates.

(c) What are the eigenvalues?

8. Consider the cylinder described in cylindrical coordinates by

$$C = \{(r, \theta, z) \mid 0 \le r < a, 0 \le \theta \le 2\pi, \text{ and } 0 < z < L\}.$$

(See section 13.5 for a discussion of the Laplacian in cylindrical coordinates.)

(a) Suppose $u(r, \theta, z) = \phi(r, \theta)Z(z)$ is a product solution of Laplace's equation in C. Show that there is a constant λ such that

$$-\nabla^2\phi = \lambda\phi, \quad \text{and} \quad Z'' = \lambda Z.$$

(b) Find the product solutions to Laplace's equation that vanish on the curved portion of the boundary of the cylinder, so that $\phi(a, \theta) = 0$ for $0 \le \theta \le 2\pi$.

(c) Find a series solution to the boundary value problem

$$\begin{aligned} \nabla^2 u(r, \theta, z) &= 0, \quad \text{for } (r, \theta, z) \in C, \\ u(r, \theta, 0) &= f(r, \theta), \quad \text{for } 0 \le r < a \text{ and } 0 \le \theta \le 2\pi, \\ u(r, \theta, L) &= 0, \quad \text{for } 0 \le r < a \text{ and } 0 \le \theta \le 2\pi, \\ u(a, \theta, z) &= 0, \quad \text{for } 0 \le \theta \le 2\pi \text{ and } 0 < z < L, \end{aligned}$$

for a steady-state temperature in C.

9. Find the solutions to the eigenvalue problem for the Laplacian on the cylinder C. This means we want to find numbers λ and functions ϕ such that

$$-\nabla^2\phi = \lambda\phi \quad \text{in } C, \text{ and} \quad \phi = 0 \quad \text{on the boundary of } C.$$

(a) First suppose that $\phi(r, \theta, z) = A(r, \theta)B(z)$ and show that there are numbers μ and ν such that $\mu + \nu = \lambda$ for which

$$-\nabla^2 A = \mu A \quad \text{for } (r, \theta) \in D$$
$$A(a, \theta) = 0 \quad \text{for } (r, \theta) \in \partial D,$$

and

$$-B'' = \nu B \quad \text{for } 0 < z < L$$
$$B(0) = B(L) = 0.$$

(b) Solve the two eigenvalue problems in part (a) to find the eigenvalues and eigenfunctions for the Laplacian on C.