Math 322

Project # 2

Due: Under my office door by Monday May 7.

From our lectures we know that proving $L(1,\chi) \neq 0$ for all non-trivial characters on $Z^*(q)$ is the hardest part of proving Dirichlet's Theorem. Remember that

$$L(1,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}.$$

Therefore computing $L(1,\chi)$ amounts to computing series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n}, \tag{1}$$

where the coefficients a_n are periodic with period q, meaning that they satisfy $a_{n+q} = a_n$ for all n. It is remarkable that, at least in theory, the sums of all such series can be found explicitly. In this project it will be your task to develop the method for doing so, and to compute $L(1,\chi)$ in a number of cases.

Almost everything you have to do is covered in Exercises 13 – 16 on pp 278-9 of our textbook. However, there are several errors there, so I will provide complete background here.

All series of the form in equation (1) can be computed in terms of the two functions

$$F(\theta) = \frac{1}{i} \sum_{n \neq 0} \frac{e^{in\theta}}{n} \tag{2}$$

and

$$E(\theta) = \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n}. \tag{3}$$

Your first task is to show that the series in (2) converges for all θ, and that

$$F(\theta) = \begin{cases} -\pi - \theta & \text{if } -\pi \leq \theta < 0, \\ \pi - \theta & \text{if } 0 < \theta \leq \pi. \end{cases} \tag{4}$$

(You have already done this in your homework. See Exercises 7 & 8 in Chapter 2.) Next show that the series in (3) converges for all θ not of the form $2n\pi$ and that the sum is

$$E(\theta) = \frac{1}{2} \log\left(\frac{1}{2 - 2\cos\theta}\right) + \frac{i}{2} F(\theta), \tag{5}$$

where we have expressed $E(\theta)$ in terms of its real and imaginary parts. This is not so easy, since the function E is not bounded, and therefore not Riemann integrable. Consequently, you cannot use any of the standard theorems about convergence of Fourier series. You might start with the series using the Dirichlet criterion and then remind yourself about Abel summability.

Turning to series of the form in (1), where the coefficients are periodic with period q, show that the series converges if $\sum_{n=1}^{q} a_n = 0$. To do this you will need to remind yourself of summation by parts. See Exercise 7 in Chapter 2. (Exercise 13 in the Chapter 8 says that the series converges if and *only if* $\sum_{n=1}^{q} a_n = 0$. If you can prove this I will give you five points extra credit.)

Since $\sum_{n-1}^{q} \chi(n) = 0$ for any non-trivial Dirichlet character mod q, we will assume henceforth that $\sum_{n-1}^{q} a_n = 0$. We will also assume that the $\{a_n\}$ are periodic with period q. Define

$$A(m) = \sum_{n=1}^{q} a_n \omega^{-mn}, \quad \text{where} \quad \omega = e^{2\pi i/q}. \tag{6}$$

Show that $A_q = 0$, and that

$$\sum_{n=1}^{\infty} \frac{a_n}{n} = \frac{1}{q} \sum_{m=1}^{q-1} A(m) E(2\pi m/q). \tag{7}$$

For this you will have to use harmonic analysis on the group $Z(q)$. Notice that equation (7) expresses the infinite sum in terms of a finite sum with known terms, so effectively we have summed the infinite series.

We will say that the sequence of coefficients $\{a_n\}$ is *odd* if $a_{q-n} = -a_q$ for $1 \leq n \leq q$. Here we are setting $a_0 = 0$ so this means that $a_q = 0$ as well. If q is even, what can you say about $a_{q/2}$? If $\{a_n\}$ is odd, show that

$$\begin{aligned} A(q-m) &= -A(m), \quad \text{therefore } \{A(m)\} \text{ is odd,} \\ A(m) &= \sum_{1 \leq n < q/2} a_n (\omega^{-mn} - \omega^{mn}), \\ \sum_{n=1}^{\infty} \frac{a_n}{n} &= \frac{1}{2q} \sum_{m=1}^{q-1} A(m) F(2\pi m/q). \end{aligned} \tag{8}$$

We will say that the sequence of coefficients $\{a_n\}$ is *even* if $a_{q-n} = a_q$ for $1 \leq n \leq q$. Can you find formulas similar to those in (8) for an even sequence of coefficients? Be careful about $q/2$ when q is even.

Now let's turn to the Dirichlet $L-$functions. First look at the characters on $Z^*(3)$. There is one non-trivial character χ. Is it even or odd? Show that

$$L(1, \chi) = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \cdots = \frac{\pi}{3\sqrt{3}}.$$

Next look at the characters on $Z^*(4)$. Again, there is one non-trivial character χ. Is it even or odd? Show that

$$L(1, \chi) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Finally compute $L(1, \chi)$ for all Dirichlet characters on $Z^*(5)$.

Organize your thoughts about this material and present it as you would a chapter in a book. Please use complete sentences, and coherent paragraphs. Number your lemmas, propositions, and theorems, as well as those formulas that you refer to later. If I find it hard to follow your reasoning you will lose points. 20% of your grade will be based on your presentation.