Math 211

Lecture #10

September 28, 2000

ode45

- Variable step method.
  - Specify error tolerance instead of step size.
  - MATLAB chooses the step size at each step to achieve the limit on the error.
  - The default tolerance is good enough for this course.
- Syntax

  `[t,y] = ode45(derfile,`$[t_0, t_f], y_0$`);`

Solving Systems

- Example:

$$x' = v$$

$$v' = -9.8 - 0.04v|v|$$

- Change to vector notation. (Use MATLAB vector notation)
  - ◇ `u(1)` $= x$
  - ◇ `u(2)` $= v$

Derivative m-file `ball.m`

```
function upr = ball(t,u)


x = u(1);
v = u(2);
xpr = v;
vpr = -9.8 - 0.04*v*abs(v);
upr = [xpr; vpr];
```

Solving Higher Order Equations

- Reduce to a first order system and solve the system.
- Example: The motion of a pendulum is modeled by $\theta'' = -\frac{g}{L}\sin\theta - D\theta'$.
- Introduce $\omega = \theta'$. Notice

$$\omega' = -\frac{g}{L}\sin\theta - D\theta'.$$

Equivalent First Order System

$$\theta' = \omega$$
$$\omega' = -\frac{g}{L}\sin\theta - D\omega$$

- Change to vector notation. (Use MATLAB vector notation)
  - ◇ `u(1)` $= \theta$
  - ◇ `u(2)` $= \omega$

Derivative m-file pend.m

```
function upr = pend(t,u)


L= 1;
global D
th = u(1);
om = u(2);
thpr = om;
ompr = -(9.8/L)*sin(th) - D*om;
upr = [thpr; ompr];
```

Solving the Pendulum Equation

Syntax

```
>> global D % declare the global
>> D = 0;    % variable
>> [t,u]=ode45('pend',[0,20],[pi/2;0]);
>> plot(t,u) % to see both θ and ω
>> plot(t,u(:,1)) % to see only θ
>> plot(u(:,1),u(:,2)) % phase plane
>> plot3(u(:,1),u(:,2),t) % 3D plot
```

Systems of Linear Equations

Solve

$$3x - 4y + 5z = 3$$

$$-x + 2y - 2z = -2$$

- Find *all* solutions.
- Find a method which works for all systems, not matter how large.

Solve

$$3x - 4y + 5z = 3$$

$$-x + 2y - 2z = -2$$

Introduce the vectors

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$ and the matrix

$$C = \begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix}$$

The system can be written as $C\mathbf{x} = \mathbf{b}$.

return

Vectors

- A vector is a list of numbers
- 2-vectors, 3-vectors, n-vectors
  - ◇ A vector has length and direction
  - ◇ Parallel vectors are equal
- Transpose of a vector.

Addition of Vectors

- Algebraic view of addition
- Geometric view of addition
- Addition of more than two vectors

Multiplication by a scalar

- Algebraic view
- Geometric view

Linear Combinations of Vectors

- Vectors $\mathbf{x} = (2, -3)^T$ and $\mathbf{y} = (-1, 2)^T$.
- Any vector of the form $a\mathbf{x} + b\mathbf{y}$ is a linear combination of $\mathbf{x}$ and $\mathbf{y}$.
- $3\mathbf{x} + 2\mathbf{y} = (4, -5)^T$.
- Any 2-vector is a linear combination of $\mathbf{x}$ and $\mathbf{y}$.
- Linear combinations of more vectors.

Matrices

- A matrix is a rectangular array of numbers.
- Example

$$\begin{pmatrix} -1 & 0 & 2 & 6 \\ 0 & 3 & -4 & 10 \\ 3 & 3 & 2 & -5 \end{pmatrix}$$

- Size $=$ (3,4); 3 rows & 4 columns.

System of Equations

$$3x - 4y + 5z = 3$$

$$-x + 2y - 2z = -2$$

As a vector

$$\begin{pmatrix} 3x - 4y + 5z \\ -x + 2y - 2z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

or

$$x \begin{pmatrix} 3 \\ -1 \end{pmatrix} + y \begin{pmatrix} -4 \\ 2 \end{pmatrix} + z \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

System of Equations

Notice that

$$\begin{pmatrix} 3 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} -4 \\ 2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

are column vectors in the matrix

$$C = \begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix}$$

Product of a Matrix with a Vector

Define the product of a matrix A and a vector $\mathbf{x}$ to be the linear combination of the columns of A with the elements of $\mathbf{x}$ as coefficients.

$$\begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= x \begin{pmatrix} 3 \\ -1 \end{pmatrix} + y \begin{pmatrix} -4 \\ 2 \end{pmatrix} + z \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

Product of a Matrix with a Vector

Thus the system of equations becomes

$$\begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

or

$$C\mathbf{x} = \mathbf{b}$$

Product of a Matrix with a Vector

The product of a matrix A and a vector $\mathbf{x}$ is the linear combination of the columns of A with the elements of $\mathbf{x}$ as coefficients.

- Computing the product $A\mathbf{x}$.

Properties of the Product

- $A(a\mathbf{x}) = a(A\mathbf{x})$
- $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$
- $A(a\mathbf{x} + b\mathbf{y}) = aA\mathbf{x} + bA\mathbf{y}$

Product of Two Matrices

Suppose A is $n \times p$ and B is $p \times q$.
Write B in terms of its column vectors

$$B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \dots \ \mathbf{b}_q]$$

Define the product AB by

$$AB = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \dots \ A\mathbf{b}_q]$$

Properties of the Product

A, B, and C matrices

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- However $AB \neq BA$ in general