Math 211

Lecture #11

October 3, 2000

Geometry of Solution Sets

- The solution set is set of *all* solutions to a system of linear equations.
  - What kinds of sets can be solution sets?
  - Can a circle be a solution set?
- We will examine all possibilities in 2 and 3 dimensions.
- Geometry will tell us the answer.

One Equation in Two Variables

Example: $2x - 3y = 1$

- Solution set is a line in the plane.
- Solve for y : $y = (-1 + 2x)/3$

The Solution Set

The solution set consists of all vectors of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ (-1+2x)/3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/3 \end{pmatrix} + \begin{pmatrix} x \\ 2x/3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/3 \end{pmatrix} + x \begin{pmatrix} 1 \\ 2/3 \end{pmatrix}$$

- x is a free parameter.

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Parametric Equation for a Line

$$\mathbf{u} = \mathbf{u}_0 + x\mathbf{v}$$

- In our case $\mathbf{u}_0 = (0,\ -1/3)^T$ and $\mathbf{v} = (1,\ 2/3)^T$
- The vector $\mathbf{u}_0$ locates one point on the line.
- The vector $\mathbf{v}$ gives the direction of the line.
- The number x tells how far the point $\mathbf{u}$ is from $\mathbf{u}_0$.

Two Equations in Two Variables

Example: $2x - 3y = 1$ and $x + y = 3$

In matrix form

$$\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

- Two equations — two lines
- Three possibilities
  - ◇ In this case the lines intersect in one point $(2,\ 1)^T$.

Two Equations in Two Variables

Three possibilities:

- Two lines intersect in one point.
- The two lines are the same line, and intersect in a line.
- The two lines are parallel, and the intersection is empty.
  - ◇ Such equations are inconsistent.

Solution Sets in Dimension 2

Four possibilities:

- The empty set.
- A single point.
- A line.
- All of $\mathbf{R}^2$.
  - Only if all coefficients are equal to 0.
- Can a circle be a solution set?

One Equation in Three Variables

Example: $2x - 3y + 4z = 1$

- Solution set is a plane in 3-space.
- Solve for z : $z = (1 - 2x + 3y)/4$.

The Solution Set

The solution set consists of all vectors of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ (1-2x+3y)/4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1/4 \end{pmatrix} + \begin{pmatrix} x \\ y \\ -x/2+3y/4 \end{pmatrix}$$

The Solution Set

The solution set consists of all vectors of the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1/4 \end{pmatrix} + x \begin{pmatrix} 1 \\ 0 \\ -1/2 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 3/4 \end{pmatrix}$$

- x and y are free parameters.

Parametric Equation for Plane

$$\mathbf{u} = \mathbf{u}_0 + x\mathbf{v} + y\mathbf{w}$$

- In our case $\mathbf{u}_0 = (0,\ 0,\ 1/4)^T$, $\mathbf{v} = (1,\ 0,\ -1/2)^T$, and $\mathbf{w} = (0,\ 1,\ 3/4)^T$
- The vector $\mathbf{u}_0$ locates one point on the plane.
- The vectors $\mathbf{v}$ and $\mathbf{w}$ give two different directions in the plane.
- $\mathbf{u}$ differs from $\mathbf{u}_0$ by the linear combination of $\mathbf{v}$ and $\mathbf{w}$ with coefficients x & y.

back

Two Equations in Three Variables

Example: $2x - 3y + 4z = 1$ and $x + y - z = 3$

In matrix form

$$\begin{pmatrix} 2 & -3 & 4 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

- Two equations — two planes
- Three possibilities — $\emptyset$, a line, or a plane.

In this case the two planes intersect in a line.

- Solve for z & y in terms of x:
  $y = 13 - 6x \quad$ and $\quad z = 10 - 5x$
- Thus the solutions are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 13 - 6x \\ 10 - 5x \end{pmatrix} = \begin{pmatrix} 0 \\ 13 \\ 10 \end{pmatrix} + x \begin{pmatrix} 1 \\ -6 \\ -5 \end{pmatrix}$$

Three Equations in Three Variables

Example: $2x - 3y + 4z = 1$, $x + y - z = 3$, and $3x - y + 3z = 5$

In matrix form

$$\begin{pmatrix} 2 & -3 & 4 \\ 1 & 1 & -1 \\ 3 & -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

- Three equations — three planes
- 4 possibilities — $\emptyset$, a point, a line, or a plane.
  - ◇ In this case a point: $(2,\ 1,\ 0)^T$

Solution Sets in Dimension 3

Five possibilities:

- The empty set.
- A single point.
- A line.
- A plane.
- All of $\mathbf{R}^3$.
  - Only if all coefficients are equal to 0.

Solution Sets in Higher Dimension

By analogy with dimensions 2 &3, we expect

- The solution set could be $\emptyset$ or a point.
- If a solution set contains 2 points, then it contains the line through them.
- If a solution set contains 3 points not on the same line, then it contains the plane though them.

Solution Sets of Homogeneous Equations

Example: $2x - 3y + 4z = 0$, $x + y - z = 0$, and $3x - y + 3z = 0$

$\mathbf{0}$ is the vector with all entries $=$0.

A homogeneous system is one of the form

$$A\mathbf{x} = \mathbf{0}.$$

A homogeneous system always has $\mathbf{0}$ as a solution. Hence the solution set of a homogeneous system is never the empty set.