# Math 211

Lecture #12

October 5, 2000

# Solving Systems of Equations

- We want to find a way to find the solution set of any system.
- We will build towards the method by looking at a series of examples.

# Example

$$x + y = 3$$
$$2x - 3y = 1$$

Solve the first equation for $x$ and substitute into second equation. We get the system

$$x + y = 3$$
$$-5y = -5$$

- The two systems have the same solutions.
- The second is very easy to solve.

# Example (continued)

$$x + y = 3$$

$$2x - 3y = 1$$

Add $-2$ times the first equation to the second equation. We get the system

$$x + y = 3$$

$$-5y = -5$$

- Solve by backsolving: $y = 1$, then $x = 2$.

# Example — Matrix Notation

$$\begin{aligned} x + y &= 3 \\ 2x - 3y &= 1 \end{aligned} \qquad \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

- Form the augmented matrix

$$M = \begin{pmatrix} 1 & 1 & 3 \\ 2 & -3 & 1 \end{pmatrix}$$

- Each row in $M$ contains all of the information about one of the equations in the system.

# Example (continued)

$$\begin{aligned} x + y &= 3 \\ 2x - 3y &= 1 \end{aligned} \Rightarrow \begin{pmatrix} 1 & 1 & 3 \\ 2 & -3 & 1 \end{pmatrix}$$

- Add $-2$ times the first row to the second row, eliminating a coefficient.

$$\begin{pmatrix} 1 & 1 & 3 \\ 0 & -5 & -5 \end{pmatrix} \Rightarrow \begin{aligned} x + y &= 3 \\ -5y &= -5 \end{aligned}$$

- Backsolve.

# Method of Solution

- Write down the augmented matrix.
- Eliminate as many coefficients as possible.
  - ◇ This is not well defined yet.
- Write down the simplified system.
- Solve the simplified system by backsolving.

# Example

$$\begin{aligned} x + y - z &= 3 \\ 2x - 3y + 4z &= 1 \end{aligned} \Rightarrow \begin{pmatrix} 1 & 1 & -1 & 3 \\ 2 & -3 & 4 & 1 \end{pmatrix}$$

- Add $-2$ times the first row to the second row, eliminating a coefficient.

$$\begin{pmatrix} 1 & 1 & -1 & 3 \\ 0 & -5 & 6 & -5 \end{pmatrix} \Rightarrow \begin{aligned} x + y - z &= 3 \\ -5y + 6z &= -5 \end{aligned}$$

- Backsolve

# Example (continued)

$$x + y - z = 3$$

$$-5y + 6z = -5$$

- $z$ is a free variable. Set $z = t$.
- Solve for $y = 1 + 6t/5$.
- Solve for

$$x = 3 - y + z = 3 - (1 + 6t/5) + t = 2 - t/5$$

# Example (continued)

Solutions are the vectors

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 - t/5 \\ 1 + 6t/5 \\ t \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1/5 \\ 6/5 \\ 1 \end{pmatrix}$$

- The solution set is a line in $\mathbf{R}^3$.

# Elimination

We only use operations on the equations which will lead to systems of equations with the same solutions.

- Add a multiple of one equation to another.
- Interchange two equations.
- Multiply an equation by a non-zero number.

# Elimination (continued)

The corresponding operations on the rows of the augmented matrix are called row operations.

- Add a multiple of one row to another.
- Interchange two rows.
- Multiply a row by a non-zero number.

# The Goal of Elimination

- How simple can we make it?

$$\begin{pmatrix} P & * & * & * & * & * & * & * \\ 0 & P & * & * & * & * & * & * \\ 0 & 0 & 0 & P & * & * & * & * \\ 0 & 0 & 0 & 0 & P & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & P & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & P \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- $P$ is a nonzero number, $*$ is any number.

return

# Row Echelon Form

- The pivot of a row is the first non-zero element from the left.
- A matrix is in row echelon form if every pivot lies strictly to the right of those in rows above.

# Reduced Row Echelon Form

- All pivots are equal to 1 and all other entries in a pivot column are 0.

$$\begin{pmatrix} 1 & 0 & * & 0 & 0 & * & 0 & 0 \\ 0 & 1 & * & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

# Example

$$
\begin{aligned}
3x_2 - 4x_3 &= -7 \\
-x_1 + 2x_2 &= -3 \\
3x_1 + 2x_2 + x_3 &= 2
\end{aligned}
\Rightarrow
\begin{pmatrix}
0 & 3 & -4 & -7 \\
-1 & 2 & 0 & -3 \\
3 & 2 & 1 & 2
\end{pmatrix}
$$

Elimination

$$
\begin{pmatrix}
-1 & 2 & 0 & -3 \\
0 & 3 & -4 & -7 \\
0 & 0 & 1 & 1
\end{pmatrix}
\Rightarrow
\begin{aligned}
-x_1 + 2x_2 &= -3 \\
3x_2 - 4x_3 &= -7 \\
x_3 &= 1
\end{aligned}
$$

Backsolve: $x_3 = 1$, $x_2 = -1$, and $x_1 = 1$.

# Example

$$A = \begin{pmatrix} 1 & 2 & 5 & -1 \\ 1 & 2 & -3 & 8 \\ 3 & 6 & 7 & 6 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -2 \\ -12 \\ -16 \end{pmatrix}$$

System $A\mathbf{x} = \mathbf{b}$

The augmented matrix is

$$M = [A,\ \mathbf{b}]$$

Elimination using MATLAB.

# Example (After elimination)

$$\begin{pmatrix} 1 & 2 & 5 & -1 & -2 \\ 0 & 0 & -8 & 9 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- There are pivots in columns 1 & 3. These are pivots columns. The corresponding variables $x_1$ and $x_3$ are called pivot variables.
- The other columns are called free columns.The corresponding variables $x_2$ and $x_4$ are called free variables.

# Example (Backsolving)

$$
\begin{aligned}
x_1 + 2x_2 + 5x_3 - x_4 &= -2 \\
-8x_3 + 9x_4 &= -10
\end{aligned}
$$

- The free variables may be assigned arbitrary values: $x_2 = s$ and $x_4 = t$.

return

# Example (Backsolving)

- Solve for the pivot variables.

$$x_3 = (10 + 9x_4)/8 = 5/4 + 9t/8$$

$$\begin{aligned} x_1 &= -2 - x_2 - 5x_3 + x_4 \\ &= -2 - s - 5(5/4 + 9t/8) + t \\ &= -33/4 - s - 37t/8 \end{aligned}$$

# Example (Solution Set)

- The solutions are

$$\mathbf{x} = \begin{pmatrix} -33/4 \\ 0 \\ 5/4 \\ 0 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -37/8 \\ 0 \\ 9/8 \\ 0 \end{pmatrix}$$

- The solution set is a plane in $\mathbf{R}^4$.

# Method of Solution for $A\mathbf{x} = \mathbf{b}$

- Use the augmented matrix $M = [A, \ \mathbf{b}]$.
- Eliminate as many coefficients as possible.
  - ◇ Row operations $\Rightarrow$ row echelon form.
- Write down the simplified system.
- Backsolve.
  - ◇ Assign arbitrary values to the free variables.
  - ◇ Solve for the pivot variables.

# Consistent Systems

- A system is consistent if and only if the simplified version (after elimination) is consistent.
- This is true if and only if the last column (after elimination) does not contain a pivot.

# Homogeneous Systems

Example

$$A = \begin{pmatrix} -5 & -4 & -2 \\ -6 & -6 & -2 \\ 30 & 27 & 11 \end{pmatrix} \Rightarrow \begin{pmatrix} -5 & -4 & -2 & 0 \\ -6 & -6 & -2 & 0 \\ 30 & 27 & 11 & 0 \end{pmatrix}$$

- During elimination the column of zeros is unchanged.
- It is unnecessary to augment a homogenous system.

# Structure of the Solution Set

**Theorem:** Let $\mathbf{x}_p$ be a particular solution to $A\mathbf{x}_p = \mathbf{b}$.

1. If $A\mathbf{x}_h = \mathbf{0}$ then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ also satisfies $A\mathbf{x} = \mathbf{b}$.
2. If $A\mathbf{x} = \mathbf{b}$, then there is a vector$\mathbf{x}_h$ such that $A\mathbf{x}_h = \mathbf{0}$ and $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$.

- Solution set for $A\mathbf{x} = \mathbf{b}$ is known if we know one particular solution $\mathbf{x}_p$ and the solution set for the homogeneous system $A\mathbf{x}_h = \mathbf{0}$.