Math 211

Lecture #13

October 10, 2000

Square Matrices

- There are special kinds:
  - Singular and nonsingular.
  - Invertible and noninvertible.
- What do the terms mean?
- What are the relations bewtween them?

Singular and Nonsingular Matrices

The $n \times n$ matrix A is nonsingular if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any right hand side $\mathbf{b}$.

Proposition: The $n \times n$ matrix A is nonsingular if and only if the simplified matrix (after elimination) has only nonzero entries along the diagonal.

- In reduced row echelon form we get I.

Proposition: If the $n \times n$ matrix A is nonsingular then the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for any right hand side $\mathbf{b}$.

Proposition: The $n \times n$ matrix A is singular if and only if the homogenous equation $A\mathbf{x} = \mathbf{0}$ has a non-zero solution.

Invertible Matrices

An $n \times n$ matrix A is invertible if there is an $n \times n$ matrix B such that $AB = BA = I$. The matrix B is called an inverse of A.

- If B_1 and B_2 are both inverses of A, then

$$B_1 = B_1(AB_2) = (B_1 A)B_2 = B_2$$

- The inverse of A is denoted by A^{-1}.
- Invertible $\Rightarrow$ nonsingular.

Invertible Matrices

Computing the inverse A^{-1}.

- Form the matrix $[A,\ I]$.
- Do elimination until the matrix has the form $[I,\ B]$.
- Then $A^{-1} = B$.
- A matrix is invertible if and only if it is nonsingular.

Solution Set of a Homogeneous System

Our goal is to understand such sets better. In particular we want to know:

- What are the properties of these solution sets?
- Is there a convenient way to describe them?

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Nullspace of a Matrix

The nullspace of a matrix A is the set

$$\{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}.$$

- The nullspace of A is the same as the solution set for the homogeneous system $A\mathbf{x} = \mathbf{0}$.
- The nullspace of A is denoted by $\mathrm{null}(A)$,

Properties of the Nullspace of A

Proposition: Let A be a matrix.

1. If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathrm{null}(A)$, then $\mathbf{x}+\mathbf{y}$ is in $\mathrm{null}(A)$.

2. If a is a scalar and $\mathbf{x}$ is in $\mathrm{null}(A)$, then $a\mathbf{x}$ is in $\mathrm{null}(A)$.

Subspaces of $\mathbf{R}^n$

Definition: A nonempty subset V of $\mathbf{R}^n$ that has the properties

1. if $\mathbf{x}$ and $\mathbf{y}$ are vectors in V, $\mathbf{x}+\mathbf{y}$ is in V,
2. if a is a scalar, and $\mathbf{x}$ is in V, then $a\mathbf{x}$ is in V,

is called a subspace of $\mathbf{R}^n$.

- The nullspace of a matrix is a subspace.

Examples of Subspaces

- The nullspace of a matrix is a subspace.
- A line through the origin is a subspace.
  $V = \{t\mathbf{v} \,|\, t \in \mathbf{R}\}$.
- A plane through the origin is a subspace.
  $V = \{a\mathbf{v} + b\mathbf{w} \,|\, a,\, b \in \mathbf{R}\}$.
- $\{\mathbf{0}\}$ and $\mathbf{R}^n$ are subspaces of $\mathbf{R}^n$.

Linear Combinations

Proposition: Any linear combination of vectors in a subspace V is also in V.

- Subspaces of $\mathbf{R}^n$ have the same kind of linear structure as $\mathbf{R}^n$ itself.
- In particular the nullspaces of matrices have the same kind of linear structure as $\mathbf{R}^n$.

Example

$$A = \begin{pmatrix} 4 & 3 & -1 \\ -3 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

The nullspace of A is

$$\text{null}(A) = \{a\mathbf{v} \,|\, a \in \mathbf{R}\},$$

where $\mathbf{v} = (1, -1, 1)^T$.

Example

$$B = \begin{pmatrix} 4 & 3 & -1 & 6 \\ -3 & -2 & 1 & -4 \\ 1 & 2 & 1 & 4 \end{pmatrix}$$

- $\mathrm{null}(B) = \{a\mathbf{v} + b\mathbf{w} \,|\, a,\ b \in \mathbf{R}\}$, where $\mathbf{v} = (1,\ -1,\ 1,\ 0)^T$ and $\mathbf{w} = (0,\ -2,\ 0,\ 1)^T$.
- $\mathrm{null}(B)$ consists of all linear combinations of $\mathbf{v}$ and $\mathbf{w}$.

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The Span of a Set of Vectors

In every example the subspace has been the set of all linear combinations of a few vectors.

Definition: The span of a set of vectors is the set of all linear combinations of those vectors.

The span of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ is denoted by

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k).$$

The Span of a Set of Vectors

Proposition: If $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ are all vectors in $\mathbf{R}^n$, then $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is a subspace of $\mathbf{R}^n$.