Math 211

Lecture #14

October 12, 2000

Nullspace of a Matrix

The nullspace of a matrix A is the set

$$\text{null}(A) = \{\mathbf{x} \,|\, A\mathbf{x} = \mathbf{0}\}\,.$$

- The nullspace of A is the same as the solution set for the homogeneous system $A\mathbf{x} = \mathbf{0}$.
- What are the properties of nullspaces?
- Is there a convenient way to describe them?

Subspaces of $\mathbf{R}^n$

Definition: A nonempty subset V of $\mathbf{R}^n$ that has the properties

1. if $\mathbf{x}$ and $\mathbf{y}$ are vectors in V, $\mathbf{x} + \mathbf{y}$ is in V,
2. if a is a scalar, and $\mathbf{x}$ is in V, then $a\mathbf{x}$ is in V,

is called a subspace of $\mathbf{R}^n$.

- The nullspace of a matrix is a subspace.

Examples of Subspaces

- The nullspace of a matrix is a subspace.
- A line through the origin is a subspace.
  $V = \{t\mathbf{v} \,|\, t \in \mathbf{R}\}$.
- A plane through the origin is a subspace.
  $V = \{a\mathbf{v} + b\mathbf{w} \,|\, a,\, b \in \mathbf{R}\}$.
- $\{\mathbf{0}\}$ and $\mathbf{R}^n$ are subspaces of $\mathbf{R}^n$.

Linear Combinations

Proposition: Any linear combination of vectors in a subspace V is also in V.

- Subspaces of $\mathbf{R}^n$ have the same kind of linear structure as $\mathbf{R}^n$ itself.
- In particular the nullspaces of matrices have the same kind of linear structure as $\mathbf{R}^n$.

Example of a Nullspace

$$A = \begin{pmatrix} 4 & 3 & -1 \\ -3 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

The nullspace of A is

$$\text{null}(A) = \{a\mathbf{v} \,|\, a \in \mathbf{R}\},$$

where $\mathbf{v} = (1,\, -1,\, 1)^T$.

return

Example of a Nullspace

$$B = \begin{pmatrix} 4 & 3 & -1 & 6 \\ -3 & -2 & 1 & -4 \\ 1 & 2 & 1 & 4 \end{pmatrix}$$

- $\text{null}(B) = \{a\mathbf{v} + b\mathbf{w} \,|\, a,\, b \in \mathbf{R}\}$, where $\mathbf{v} = (1,\, -1,\, 1,\, 0)^T$ and $\mathbf{w} = (0,\, -2,\, 0,\, 1)^T$.
- $\text{null}(B)$ consists of all linear combinations of $\mathbf{v}$ and $\mathbf{w}$.

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The Span of a Set of Vectors

In every example the subspace has been the set of all linear combinations of a few vectors.

Definition: The span of a set of vectors is the set of all linear combinations of those vectors.

The span of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ is denoted by

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k).$$

The Span of a Set of Vectors

Proposition: If $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ are all vectors in $\mathbf{R}^n$, then $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is a subspace of $\mathbf{R}^n$.

Examples

Let $\mathbf{v}_1 = (1,\, 2)^T$, $\mathbf{v}_2 = (1,\, 0)^T$, and $\mathbf{v}_3 = (2,\, 0)^T$.

- $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{R}^2$. (Proof?)
- $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_3) = \mathbf{R}^2$. (Proof?)
- $\mathrm{span}(\mathbf{v}_2, \mathbf{v}_3) = \mathrm{span}(\mathbf{v}_2)$. (Proof?)
  - ◇ $\mathrm{span}(\mathbf{v}_2, \mathbf{v}_3) = \{t\mathbf{v}_2 \,|\, t \in \mathbf{R}\}$.
  - ◇ $\mathbf{v}_2$ and $\mathbf{v}_3$ have the same direction.

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Linear Independence

We need a condition that will keep out superfluous vectors from a spanning list. We will work toward a definition.

- Two vectors are linearly dependent if one is a scalar multiple of the other.
  - $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent.
  - $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Linear Independence

- Three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly dependent if one is a linear combination of the other two.
  - ◇ Example: $\mathbf{v}_1 = (1,\ 0,\ 0)^T$, $\mathbf{v}_2 = (0,\ 1,\ 0)^T$, and $\mathbf{v}_3 = (1,\ 2,\ 0)^T$

$$\mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2.$$

  - ◇ Notice that $\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$.

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Linear Independence

- Three vectors are linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.
  - ◇ Non-trivial means that at least one of the coefficients is not 0.
- A set of vectors is linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.

Linear Independence

Definition: The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ are linearly independent if the only linear combination of them which is equal to the zero vector is the one with all of the coefficients equal to 0.

- In symbols,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

$$\Rightarrow c_1 = c_2 = \cdots = c_k = 0.$$

Basis of a Subspace

Definition: A set of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ form a basis of a subspace V if

1. $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$
2. $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ are linearly independent.

Basis of a Subspace

- The vector $\mathbf{v} = (1,\ -1,\ 1)^T$ is a basis for $\text{null}(A)$.
  - ◇ $\text{null}(A)$ is the subspace of $\mathbf{R}^3$ with basis $\mathbf{v}$.
- The vectors $\mathbf{v} = (1,\ -1,\ 1,\ 0)^T$ and $\mathbf{w} = (0,\ -2,\ 0,\ 1)^T$ form a basis for $\text{null}(B)$.
  - ◇ $\text{null}(B)$ is the subspace of $\mathbf{R}^4$ with basis $\{\mathbf{v}, \mathbf{w}\}$.

Basis of a Subspace

Proposition: Let V be a subspace of $\mathbf{R}^n$.

1. If $V \neq \{\mathbf{0}\}$, then V has a basis.
2. Every basis of V has the same number of elements.

Definition: The dimension of a subspace V is the number of elements in a basis of V.

Linear Independence?

How do we decide if a set of vectors is linearly independent?

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 6 \end{pmatrix}$$

$$-3\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}.$$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \; \mathbf{v}_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \; \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 3 \end{pmatrix}$$

- These vectors are linearly independent.

Proposition: Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ are vectors in $\mathbf{R}^n$. Set $V = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k]$.

1. If $\text{null}(V) = \{\mathbf{0}\}$, then $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ are linearly independent.

2. If $\mathbf{c} = (c_1, c_2, \ldots, c_k)^T$ is a nonzero vector in $\text{null}(V)$, then

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0},$$

so the vectors are linearly dependent.

Solutions Sets and Nullspaces

Theorem: The solution set of the inhomogeneous system $A\mathbf{x} = \mathbf{b}$ has the form

$$\{\mathbf{x} = \mathbf{x}_p + \mathbf{v} \,|\, \mathbf{v} \in \text{null}(A)\},$$

where $\mathbf{x}_p$ is any particular solution of the inhomogeneous system.

- Restatement of previous result.

Determinants in 2D

- How do we decide if a matrix A is nonsingular?
- A is nonsingular if and only if $\texttt{rref}(A) = I$.
- $$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

  is nonsingular if and only if $ad - bc \neq 0$.

  ◇ if and only if $\det(A) = ad - bc \neq 0$

Determinants in 3D

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det(A) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32}.$$

- A is nonsingular if and only if $\det(A) \neq 0$.

Determinants

Theorem: The $n \times n$ matrix A is nonsingular if and only if $\det(A) \neq 0$.

Corollary: If A is an $n \times n$ matrix, then $\mathrm{null}(A)$ contains a nonzero vector if and only if $\det(A) = 0$.

- The corollary contains the most important fact about determinants for ODEs.

Typical Term

- Typical term $a_{13}a_{21}a_{32}$
- First subscripts are 1, 2, & 3. True for all terms.
- Second subscripts are 3, 1, & 2. Different order for each terms, but always a list of 1, 2, & 3.

Permutations

- A permutation is a list of the numbers 1, 2, . . . , n in any order.
  - ◇ Example $\sigma = (3,\ 1,\ 2)$.
  - ◇ Typical term is $a_{1\sigma_1}a_{2\sigma_2}a_{3\sigma_3}$
- All permutations of 3 numbers appear in the determinant of a 3×3 matrix (sometimes with minus signs).

Sign of a Permutation

Transform a permutation into $(1, 2, \ldots, n)$ using interchanges.

- $(3, 1, 2) \to (1, 3, 2) \to (1, 2, 3)$
- This can be done in many ways, but for a given permutation, the number of interchanges is always odd or always even.
- A permutation is odd if the number is odd, and even if the number is even.

Sign of a Permutation

Define

$$(-1)^\sigma = \begin{cases} 1 & \text{if } \sigma \text{ is even,} \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Definition of Determinant

The general $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$\det(A) = \sum_{\sigma} (-1)^{\sigma} a_{1\sigma_1} \cdot a_{2\sigma_2} \cdot \ldots \cdot a_{n\sigma_n}$$

Definition of Determinant

- Each summand of the definition is the product of n entries, one from each row, and one from each column.
- The definition is the sum of $n!$ terms. It does not provide an effective way to compute the determinant.
- One exception: The determinant of an upper triagular matrix is the product of the diagonal terms.

Row Operations and Determinants

If B is obtained from A by

- adding a multiple of one row to another,
$$\det(B) = \det(A).$$
- interchanging two rows,
$$\det(B) = -\det(A).$$
- multiplying a row by $c \neq 0$,
$$\det(B) = c\det(A).$$