Math 211

Lecture #15

October 19, 2000

Definition of Determinant

The general $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$\det(A) = \sum_{\sigma} (-1)^{\sigma} a_{1\sigma_1} \cdot a_{2\sigma_2} \cdot \ldots \cdot a_{n\sigma_n}$$

return

Definition of Determinant

- Each summand of the definition is the product of n entries, one from each row, and one from each column.
- The definition is the sum of $n!$ terms. It does not provide an effective way to compute the determinant.
- One exception: The determinant of an upper triangular matrix is the product of the diagonal terms.

Definition

4

Row Operations and Determinants

If B is obtained from A by

- adding a multiple of one row to another,
$$\det(B) = \det(A).$$
- interchanging two rows,
$$\det(B) = -\det(A).$$
- multiplying a row by $c \neq 0$,
$$\det(B) = c\det(A).$$

return

5

Example

$$A = \begin{pmatrix} -5 & 0 & 0 \\ -25 & -4 & -14 \\ 65 & 7 & 17 \end{pmatrix}$$

$$\det(A) = -150$$

6

Properties

- If A has two equal rows, then $\det(A) = 0$.
- $\det(A^T) = \det(A)$.
- If A has two equal columns, then $\det(A) = 0$.

Row operations

Column Operations and Determinants

If B is obtained from A by

- adding a multiple of one column to another,
  $$\det(B) = \det(A).$$
- interchanging two columns,
  $$\det(B) = -\det(A).$$
- multiplying a column by $c \neq 0$,
  $$\det(B) = c\det(A).$$

return

Expansion by a Row

Definition: The ij-minor of an $n \times n$ matrix A is the $(n-1) \times (n-1)$ matrix A_{ij} obtained from A by deleting the i^{th} row and the j^{th} column.

With this definition we can prove that

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- This is called expansion by the i^{th} row.

Definition return

Expansion by a Column

We can also expand by a column.

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- This is called expansion by the j^{th} column.

Definition Expansion by row return

Example

$$A = \begin{pmatrix} -5 & -6 & 0 \\ 3 & 4 & 0 \\ -8 & -16 & 9 \end{pmatrix}$$

$$\begin{aligned} \det(A) &= 9 \cdot \det \begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix} \\ &= 9 \cdot (-2) \\ &= -18 \end{aligned}$$

Example

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -2 \\ -2 & -1 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix}$$

$$\det(A) = 1.$$

Example

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 \\ 12 & -6 & 0 & 5 \\ 32 & -15 & -3 & 13 \\ 18 & -10 & -1 & 8 \end{pmatrix}$$

$$\det(A) = -1.$$

Determinants and Bases

Proposition: A collection of n vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$,$\mathbf{v}_n$ in $\mathbf{R}^n$ is a basis for $\mathbf{R}^n$ if and only if

$$\det([\mathbf{v}_1\,\mathbf{v}_2\,\ldots\,\mathbf{v}_n]) \neq 0.$$

Systems of Differential Equations

Example: SIR model of the spread of infectious disease. Assume:

- The disease is of short duration and rarely fatal.
- The disease spreads through human contact.
- Recovered indiviuals are immune.

return

SIR Model

- Three subpopulations; susceptible, $S(t)$, infecteds, $I(t)$, and recovered, $R(t)$

$$S' = -aSI$$
$$I' = aSI - bI$$
$$R' = bI.$$

- $N = S + I + R$ is constant.

Disease
return

SIR Model

- New variables $s = S/N$, $i = I/N$, and $r = R/N$.

$$s' = -Asi$$
$$i' = Asi - bi$$

- $A = Na$.
- $s + i + r = 1$, so $r = 1 - s - i$.
- MATLAB& `pplane5`

SIR

General System in 2D

$$x' = f(t, x, y)$$
$$y' = g(t, x, y)$$

- Example:

$$x' = y$$
$$y' = -x$$

- Solution: $x(t) = \sin t \,\&\, y(t) = \cos t$
  - ◇ Verify by direct substitution.

return

General System in Higher D

$$\begin{aligned} x_1' &= f_1(t, x_1, x_2, \ldots, x_n) \\ x_2' &= f_2(t, x_1, x_2, \ldots, x_n) \\ \vdots &= \quad \vdots \\ x_n' &= f_n(t, x_1, x_2, \ldots, x_n) \end{aligned}$$

return

19

Vector Notation

Set $u_1(t) = x(t)$ & $u_2(t) = y(t)$.

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \Leftrightarrow \mathbf{u}' = \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}$$

Set $\mathbf{F}(t, \mathbf{u}) = (f(t, u_1, u_2), g(t, u_1, u_2))^T$

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \end{aligned} \Leftrightarrow \mathbf{u}' = \mathbf{F}(t, \mathbf{u})$$

2D

20

Vector Notation

Set

$$\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$$
$$\mathbf{f}(t, \mathbf{x}) = (f_1(t, \mathbf{x}), f_2(t, \mathbf{x}), \ldots, f_n(t, \mathbf{x}))^T.$$

- The general system can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}).$$

General System

21

Initial Value Problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

- Each component of $\mathbf{x}(t_0)$ must be specified.
- Example

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \quad \text{with} \quad \begin{aligned} x(0) &= 2 \\ y(0) &= 13 \end{aligned}$$

22

Reduction of Higher Order Equation to a System

For any higher order equation there is a first order system which is equivlaent to it, in the sense that solutions of the system lead easily to solutions of the equation, and vice versa.

23

Example of Reduction

- Third-order equation: $y''' + 2yy' = 3\cos t$
- Set $x_1 = y$, $x_2 = y'$, and $x_3 = y''$.
- Then

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 3\cos t - 2x_1x_2 \end{aligned}$$

24

Geometric Interpretation of Solutions

- `pplane5`
- Component plot
- Parametric plot
- Phase plane
- 3-D plot
- Composite plot