Math 211

Lecture #16

October 24, 2000

General System in Higher D

$$
\begin{aligned}
x_1' &= f_1(t, x_1, x_2, \ldots, x_n) \\
x_2' &= f_2(t, x_1, x_2, \ldots, x_n) \\
\vdots &= \qquad \vdots \\
x_n' &= f_n(t, x_1, x_2, \ldots, x_n)
\end{aligned}
$$

Vector Notation

Set

$$\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$$

$$\mathbf{f}(t, \mathbf{x}) = (f_1(t, \mathbf{x}), f_2(t, \mathbf{x}), \ldots, f_n(t, \mathbf{x}))^T.$$

- The general system can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}).$$

Existence & Uniqueness

General System $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$

- $\mathbf{x}$ in an open set $U \subset \mathbf{R}^n$
- t in an interval $I = (a, b)$
  $R = I \times U = \{(t, x) \,|\, t \in I \quad \text{and} \quad \mathbf{x} \in U\}$.

Theorem: Suppose that $\mathbf{f}(t, \mathbf{x})$ is continuous in R, and that all first partials of $\mathbf{f}$ are also continuous in R. Then given any $t_0 \in I$ and $\mathbf{x}_0 \in U$ there is a unique solution to the initial value problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \text{with} \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

defined on an interval containing t_0. The solution exists at least until the solution curve $t \to (t, \mathbf{x}(t))$ leaves R.

Autonomous Systems

System of the form

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}).$$

- $\mathbf{R}^n$ is called phase space.
  - If $n = 2$ this is the phase plane.
  - If $n = 1$ this is the phase line.

Uniqueness in Phase Space

Two solution curves in phase space for an autonomous system cannot meet at a point unless the solution curves coincide.

- If $n = 2$, two solution curves in the phase plane cannot cross, or even touch.
- If the system is not autonomous, solution curves in the phase plane can cross.

Equilibrium Points & Solutions

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}).$$

- $\mathbf{x}_0$ is an equilibrium point if $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$.
- $\mathbf{x}(t) = \mathbf{x}_0$ is the corresponding equilibrium solution.
- Nullclines.

Example

$$x' = x^2 - y$$

$$y' = x - xy$$

- x-nullcline: $x^2 - y = 0$.
- y-nullcline: $x(1 - y) = 0$.
- 3 equilibrium points: $(0, 0)$, $(1, 1)$, and $(-1, 1)$.

Linear Systems

A system is linear if the unknown functions appear linearly in the right-hand sides.

- Appear linearly means that there are no products, powers, or higher order functions.
- Examples

Planar Linear Systems

A planar linear system is one of the form

$$x' = a(t)x + b(t)y + f(t)$$

$$y' = c(t)x + d(t)y + g(t)$$

- The coefficients can depend on t.

General Linear Systems

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1 \\
x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2 \\
\vdots &= \qquad \vdots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n
\end{aligned}
$$

- The coefficients can depend on t.

General Linear System

- Set $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$
- $\mathbf{f} = (f_1, f_2, \ldots, f_n)^T$
- $$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$
- The system becomes $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$.

Existence & Uniqueness

Theorem: Suppose $A = A(t)$ is a matrix valued function and $\mathbf{f}(t)$ are defined and continuous in an interval $I = (\alpha, \beta)$. Then for any t_0 in I and any $\mathbf{x}_0$ in $\mathbf{R}^n$, the initial value problem

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f} \quad \text{with} \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

has a unique solution defined for all t in I.

Homogeneous Systems

$$\mathbf{x}' = A\mathbf{x}$$

Proposition: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_k(t)$ are solutions to the homogeneous system, and c_1, c_2, . . . , and c_k are scalars. Then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_k\mathbf{x}_k(t)$$

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.

Example

Let

$$A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}.$$

The system $\mathbf{x}' = A\mathbf{x}$ has solutions

$$\mathbf{x}_1(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Proposition $\Rightarrow \mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)$ is a solution for any constants C_1 and C_2.

Does every solution of $\mathbf{y}' = A\mathbf{y}$ have this form?

- Can we find C_1 and C_2 so that

$$\mathbf{y}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)?$$

- We can find C_1 and C_2 so that

$$\mathbf{y}(0) = C_1\mathbf{x}_1(0) + C_2\mathbf{x}_2(0).$$

- Uniqueness theorem $\Rightarrow$

$$\mathbf{y}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \quad \text{for all } t.$$

Key Point in the Argument

- Need $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ to be linearly independent.
- Makes it possible to solve the equation

$$\mathbf{y}_0 = C_1\mathbf{x}_1(0) + C_2\mathbf{x}_2(0)$$

  for any $\mathbf{y}_0 = \mathbf{y}(0)$.
- Uniqueness does the rest.
- We only needed $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ to be linearly independent at one point.

Proposition: $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_k(t)$ solutions to the homogeneous system $\mathbf{x}' = A\mathbf{x}$ on the interval I.

1. If $\mathbf{x}_1(t_0)$, $\mathbf{x}_2(t_0)$, . . . , and $\mathbf{x}_k(t_0)$ are linearly independent for some $t_0 \in I$, then they are linearly independent for all $t \in I$.

2. If $\mathbf{x}_1(t_0)$, $\mathbf{x}_2(t_0)$, . . . , and $\mathbf{x}_k(t_0)$ are linearly dependent for some $t_0 \in I$, then they are linearly dependent for all $t \in I$.

Linear Independence

Definition: A set of k solutions to the linear system $\mathbf{x}' = A\mathbf{x}$ is linearly independent if they are linearly independent at one value of t.

- Proposition $\Rightarrow$ the solutions are linearly independent for all values of t.

Structure of the Solution Space

Theorem: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_n(t)$ are linearly independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ on the interval I. Then every solution to the system is a linear combination of $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_n(t)$. That is, if $\mathbf{x}(t)$ is a solution, then there are constants C_1, C_2, . . . , and C_n such that

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) + \cdots + C_n\mathbf{x}_n(t).$$

Solution Strategy

Definition: A set of n linear independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ is called a fundamental set of solutions.

- The obvious strategy for completely solving the system is to look for n linearly independent solutions — a fundamental set of solutions.

Example: $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\mathbf{x}_1(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

are a fundamental set of solutions.

Example: $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$$

$$\mathbf{x}_1(t) = e^{t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

are a fundamental set of solutions.