Math 211

Lecture #17

October 26, 2000

Solving $\mathbf{x}' = A\mathbf{x}$

- Homogeneous linear system.
- Assume the system has constant coefficients, so A is a constant matrix.
- Example $A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$
- or
$$\begin{aligned} x_1' &= -4x_1 + 2x_2 \\ x_2' &= -3x_1 + x_2 \end{aligned}$$

Solution Strategy

Definition: A set of n linear independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ is called a fundamental set of solutions.

- The obvious strategy for completely solving the system is to look for n linearly independent solutions — a fundamental set of solutions.

Structure of the Solution Space

Theorem: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_n(t)$ are linearly independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ on the interval I. Then every solution to the system is a linear combination of $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_n(t)$. That is, if $\mathbf{x}(t)$ is a solution, then there are constants C_1, C_2, . . . , and C_n such that

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) + \cdots + C_n\mathbf{x}_n(t).$$

Linear Independence

Definition: A set of k solutions to the linear system $\mathbf{x}' = A\mathbf{x}$ is linearly independent if they are linearly independent at one value of t.

- Proposition $\Rightarrow$ the solutions are linearly independent for all values of t.

return

D $= 1$

- One equation: $x' = ax$
  - ◇ a is a constant.
- Solution: $x(t) = Ce^{at}$
- Solutions are exponentials. Can we find exponential solutions to a system of equations?

Exponential Solutions to $\mathbf{x}' = A\mathbf{x}$

- Can we find a solution of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, where $\mathbf{v}$ is a vector with constant entries?
- $\mathbf{x}' = \lambda e^{\lambda t}\mathbf{v}$
- $A\mathbf{x} = e^{\lambda t}A\mathbf{v}$
- $\mathbf{x}' = A\mathbf{x} \quad \Leftrightarrow \quad A\mathbf{v} = \lambda\mathbf{v}.$
- If $A\mathbf{v} = \lambda\mathbf{v}$ then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Can we find λ and $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$?

return

Eigenvalues & Eigenvectors

Definition: λ is an eigenvalue of A if there is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

If λ is an eigenvalue of A, then any vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ is called an eigenvector associated with λ.

- If λ is an eigenvalue of A, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}$ for any associated eigenvector $\mathbf{v}$.
- How do we find eigenvalues and eigenvectors?

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Finding Eigenvalues

λ is an eigenvalue of A if there is a vector $\mathbf{v} \neq \mathbf{0}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

$\Leftrightarrow \mathbf{v} \neq \mathbf{0}$ and

$$\begin{aligned} \mathbf{0} &= A\mathbf{v} - \lambda\mathbf{v} \\ &= A\mathbf{v} - \lambda I\mathbf{v} \\ &= (A - \lambda I)\mathbf{v} \end{aligned}$$

$\Leftrightarrow A - \lambda I$ has a nontrivial nullspace.

$\Leftrightarrow \det(A - \lambda I) = 0.$

Example

$$
\begin{aligned}
A &= \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} \\
A - \lambda I &= \begin{pmatrix} -4-\lambda & 2 \\ -3 & 1-\lambda \end{pmatrix} \\
\det(A-\lambda I) &= (-4-\lambda)(1-\lambda)+6 \\
&= \lambda^2 + 3\lambda + 2 \\
&= (\lambda+1)(\lambda+2)
\end{aligned}
$$

- A has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

return

Characteristic Polynomial of A

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{pmatrix}$$

return

Characteristic Polynomial of A

- The characterisitic polynomial of A is $p(\lambda) = \det(A - \lambda I)$.
- If A is an $n \times n$ matrix $p(\lambda)$ is a polynomial of degree n.
- Each root of $p(\lambda) = 0$ is an eigenvalue of A.
- We find the eigenvalues of A by finding the roots of the charactristic equation $p(\lambda) = 0$.
- Usually $p(\lambda) = 0$ has n roots. Usually A has n eigenvalues.

Finding Eigenvectors

$\mathbf{v}$ is an eigenvector associated with λ if

$$
\begin{aligned}
& A\mathbf{v} = \lambda\mathbf{v}. \\
\Leftrightarrow & (A - \lambda I)\mathbf{v} = \mathbf{0}. \\
\Leftrightarrow & \mathbf{v} \in \operatorname{null}(A - \lambda I).
\end{aligned}
$$

- The set of all eigenvectors associated to the eigenvalue λ is equal to the nullspace of $A - \lambda I$. It is therefore a subspace of $\mathbf{R}^n$, called the eigenspace of λ.

Example

$A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

- $\lambda_1 = -1$

  $$A - \lambda_1 I = \begin{pmatrix} -4+1 & 2 \\ -3 & 1+1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix}$$

  $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector;

  $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 = e^{-t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is a solution.

Example (cont.)

- $\lambda_2 = -2$

  $$A - \lambda_2 I = \begin{pmatrix} -4+2 & 2 \\ -3 & 1+2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix}$$

  $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector;

  $\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2 = e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a solution.

Example (cont.)

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

has solutions

$$\mathbf{x}_1(t) = e^{-t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$x_1(0) = \mathbf{v}_1$ and $\mathbf{x}_2(0) = \mathbf{v}_2$ are linearly independent. They form a fundamental set of solutions.

Example (cont.)

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

- The general solution is the set of all linear combinations:

$$\begin{aligned} \mathbf{x}(t) &= C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \\ &= C_1 e^{-t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2C_1e^{-t} + C_2e^{-2t} \\ 3C_1e^{-t} + C_2e^{-2t} \end{pmatrix} \end{aligned}$$

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Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A
  - ◇ the roots of $\det(A - \lambda I) = 0$
- For each eigenvalue λ find the eigenspace
  - ◇ $= \operatorname{null}(A - \lambda I)$
- If λ is an eigenvalue and $\mathbf{v}$ is an associated eigenvector, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Show that n of these are linearly independent.

Cases

- Distinct real eigenvalues.
  - In this case the method works as described.
- Complex eigenvalues.
  - The method yields complex solutions.
- Repeated eigenvalues.
  - The method does not always give enough solutions.

Planar System $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

In nonvector form

$$x_1' = a_{11}x_1 + a_{12}x_2$$

$$x_2' = a_{21}x_1 + a_{22}x_2$$

Characteristic Polynomial

$$
\begin{aligned}
p(\lambda) &= \det(A - \lambda I) \\
&= \det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} \\
&= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\
&= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})
\end{aligned}
$$

Cont.

- Set $D = \det(A) = a_{11}a_{22} - a_{12}a_{21}$
- The trace of A is $\mathrm{tr}(A) = a_{11} + a_{22}$.
  Set $T = \mathrm{tr}(A)$.
- Then

$$p(\lambda) = \lambda^2 - T\lambda + D.$$

Eigenvalues of A

- Roots of $p(\lambda) = \lambda^2 - T\lambda + D = 0.$

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

- Three cases:
  - ◇ 2 real roots if $T^2 - 4D > 0$
  - ◇ 2 complex conjugate roots if $T^2 - 4D < 0$
  - ◇ Double real root if $T^2 - 4D = 0$

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Eigenvectors

The problem of determining that solutions are linearly independent is eased by the following.

Proposition: $\lambda_1 \neq \lambda_2$ eigenvalues of A. $\mathbf{v}_1 \neq 0$ and $\mathbf{v}_2 \neq 0$ eigenvectors associated with λ_1 and λ_2, resp. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

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Two Distinct Real Eigenvalues

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2}, \qquad \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

- $T^2 - 4D > 0$ so $\lambda_1 < \lambda_2$.
- Associated nonzero eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.
- Solutions $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2$.
- $\lambda_1 \neq \lambda_2 \quad \Rightarrow \quad \mathbf{x}_1(0) = \mathbf{v}_1$ and $\mathbf{x}_2(0) = \mathbf{v}_2$ are linearly independent.

Two Distinct Real Eigenvalues

If A is a 2×2 matrix with

- two real eigenvalues $\lambda_1 \neq \lambda_2$, and
- associated nonzero eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$

the general solution is

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2.$$

Example

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -6 & -8 \\ 4 & 6 \end{pmatrix}$$

- $p(\lambda) = \lambda^2 - 4$. Eigenvalues: -2 and 2.
- Eigenvectors: $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$
- Fundamental set of solutions:

$$\mathbf{x}_1(t) = e^{-2t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

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Complex Eigenvalues

$$\lambda = \frac{T + i\sqrt{4D - T^2}}{2}, \qquad \bar{\lambda} = \frac{T - i\sqrt{4D - T^2}}{2}$$

Example: $\begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$

- $p(\lambda) = \lambda^2 - 2\lambda + 5.$
- Eigenvalue: $\lambda = 1 + 2i$
- Eigenvector: $\mathbf{w} = \begin{pmatrix} 3 - i \\ 1 \end{pmatrix}$

Example

- Solution

$$\begin{aligned} \mathbf{z}(t) &= e^{\lambda t}\mathbf{w} \\ &= e^{(1+2i)t}\begin{pmatrix} 3-i \\ 1 \end{pmatrix} \end{aligned}$$

- What is $e^{(1+2i)t}$?

Euler's Formula

- Define

$$e^{iy} = \cos y + i \sin y$$

- Define

$$\begin{aligned} e^{x+iy} &= e^{x} e^{iy} \\ &= e^{x}[\cos y + i \sin y]. \end{aligned}$$

- Then

$$e^{(1+2i)t} = e^{t}[\cos 2t + i \sin 2t]$$

Example

- Solution

$$\begin{aligned}\mathbf{z}(t) &= e^{\lambda t}\mathbf{w}\\ &= e^{(1+2i)t}\begin{pmatrix} 3-i \\ 1 \end{pmatrix}\end{aligned}$$

- Solution is complex valued.
- Solution corresponding to $\overline{\lambda}$ is

$$\overline{\mathbf{z}}(t) = e^{(1-2i)t}\begin{pmatrix} 3+i \\ 1 \end{pmatrix}$$

Complex Conjugate Eigenvalues

If A is a 2×2 matrix with

- complex eigenvalues λ and $\overline{\lambda}$, and
- associated nonzero eigenvectors $\mathbf{w}$ and $\overline{\mathbf{w}}$

the general complex solution is

$$C_1 e^{\lambda t}\mathbf{w} + C_2 e^{\overline{\lambda} t}\overline{\mathbf{w}}.$$

- We want real solutions.

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Real Solutions

We have solutions

$$\mathbf{z}(t) = \mathbf{x}(t) + i\mathbf{y}(t) \quad \text{and} \quad \overline{\mathbf{z}}(t) = \mathbf{x}(t) - i\mathbf{y}(t)$$

Thus

$$\begin{aligned} \mathbf{x}(t) &= \frac{1}{2}(\mathbf{z}(t) + \overline{\mathbf{z}}(t)) \\ \mathbf{y}(t) &= \frac{1}{2i}(\mathbf{z}(t) - \overline{\mathbf{z}}(t)) \end{aligned}$$

are also solutions, and they are real valued.

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