

# Math 211

Lecture #18

October 31, 2000

## Complex Numbers

A complex number is one of the form  $z = x + iy$ , where  $x$  and  $y$  are real numbers.

- $i^2 = -1$ .
- $x$  is the real part of  $z$ ;  $x = \operatorname{Re}z$ .
- $y$  is the imaginary part of  $z$ ;  $y = \operatorname{Im}z$ .
  - ◊ The imaginary part is a real number.
- Addition and multiplication.

## Complex Conjugate

The conjugate of  $z = x + iy$  is  $\bar{z} = x - iy$ .

- $z = \bar{z} \iff z$  is a real number.
- $x = \operatorname{Re}z = \frac{z + \bar{z}}{2}$
- $y = \operatorname{Im}z = \frac{z - \bar{z}}{2i}$
- $\overline{z + w} = \bar{z} + \bar{w}$ ;  $\overline{z - w} = \bar{z} - \bar{w}$
- $\overline{zw} = \bar{z} \cdot \bar{w}$ ;  $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

return

## Absolute Value

The absolute value of  $z = x + iy$  is the real number  $|z| = \sqrt{x^2 + y^2}$ .

- $z \cdot \bar{z} = |z|^2 = x^2 + y^2$ .
- $|zw| = |z||w|$
- $\left| \frac{z}{w} \right| = \frac{|z|}{|w|}$

## Quotients

- Reciprocal of  $z = x + iy$

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}.$$

$$\frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}$$

- Quotient  $z/w$

$$\frac{z}{w} = z \cdot \frac{1}{w} = \frac{z\bar{w}}{|w|^2}$$

## Geometric Representation

- Complex plane.
- Addition
- Polar representation  $z = r[\cos \theta + i \sin \theta]$ .
  - ◊  $\theta$  is the argument of  $z$ :  $\tan \theta = y/x$ .
  - ◊  $r = |z|$ .
- Euler's formula  $e^{i\theta} = \cos \theta + i \sin \theta$ .
  - ◊  $z = |z|e^{i\theta}$
- Multiplication

## Complex Exponential

For  $z = x + iy$  define

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x [\cos y + i \sin y].$$

Properties:

- $e^{z+w} = e^z \cdot e^w$ ;  $e^{z-w} = e^z \cdot e^{-w} = e^z / e^w$
- $\overline{e^z} = e^{\bar{z}}$
- $|e^z| = e^{\operatorname{Re}z}$
- If  $\lambda$  is a complex number, then  $\frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t}$

## Complex Matrices

Matrices (or vectors) with complex entries inherit many of the properties of complex numbers.

- $M = A + iB$  where  $A = \operatorname{Re}M$  and  $B = \operatorname{Im}M$  are real matrices.
- $\overline{\overline{M}} = M$ ;  $M = \overline{M} \iff M$  is real.
- $\operatorname{Re}M = \frac{1}{2}(M + \overline{M})$ ;  $\operatorname{Im}M = \frac{1}{2i}(M - \overline{M})$
- $\overline{M + N} = \overline{M} + \overline{N}$
- $\overline{Mz} = \overline{M}\overline{z}$

Conjugate

return

## Consequences

$A$  a real matrix; complex eigenvalue  $\lambda$ ;  
associated eigenvector  $\mathbf{w}$ , so  $A\mathbf{w} = \lambda\mathbf{w}$

- $\overline{A\mathbf{w}} = \overline{\lambda\mathbf{w}} = \overline{\lambda}\overline{\mathbf{w}}$ .
- $\overline{\lambda\mathbf{w}} = \overline{\lambda}\overline{\mathbf{w}}$ .
- $\Rightarrow A\overline{\mathbf{w}} = \overline{\lambda}\overline{\mathbf{w}} \Rightarrow \overline{\lambda}$  is an eigenvalue of  $A$  with associated eigenvector  $\overline{\mathbf{w}}$
- $\lambda \neq \overline{\lambda} \Rightarrow \mathbf{w}$  and  $\overline{\mathbf{w}}$  are linearly independent.

Complex Matrices

return

## Solutions

$$\mathbf{z}(t) = e^{\lambda t} \mathbf{w} \quad \text{and} \quad \bar{\mathbf{z}}(t) = e^{\bar{\lambda} t} \bar{\mathbf{w}}.$$

- $\mathbf{z}$  and  $\bar{\mathbf{z}}$  are linearly independent complex solutions to  $\mathbf{x}' = A\mathbf{x}$ .
- $\mathbf{z}(t) = \mathbf{x}(t) + iy(t)$  &  $\bar{\mathbf{z}}(t) = \mathbf{x}(t) - iy(t)$
- $\mathbf{x}(t) = \frac{1}{2}(\mathbf{z}(t) + \bar{\mathbf{z}}(t))$  &  $\mathbf{y}(t) = \frac{1}{2i}(\mathbf{z}(t) - \bar{\mathbf{z}}(t))$
- $\mathbf{x}(t)$  and  $\mathbf{y}(t)$  are real valued solutions.
- $\mathbf{x}(t)$  and  $\mathbf{y}(t)$  are linearly independent.

Complex Matrices

Consequences

## Example

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$$

- $p(\lambda) = \lambda^2 - 2\lambda + 5$ .
- Eigenvalue:  $\lambda = 1 + 2i$
- Eigenvector:  $\mathbf{w} = \begin{pmatrix} 3 - i \\ 1 \end{pmatrix}$

return

## Complex Solutions

$$\mathbf{z}(t) = e^{\lambda t} \mathbf{w} = e^{(1+2i)t} \begin{pmatrix} 3 - i \\ 1 \end{pmatrix}$$

$$\bar{\mathbf{z}}(t) = e^{\bar{\lambda} t} \bar{\mathbf{w}} = e^{(1-2i)t} \begin{pmatrix} 3 + i \\ 1 \end{pmatrix}$$

System

return

## Real Solutions

$$\mathbf{x}(t) = \operatorname{Re}(\mathbf{z}(t)) = e^t \begin{pmatrix} 3 \cos 2t + \sin 2t \\ \cos 2t \end{pmatrix}$$

$$\mathbf{y}(t) = \operatorname{Im}(\mathbf{z}(t)) = e^t \begin{pmatrix} 3 \sin 2t - \cos 2t \\ \sin 2t \end{pmatrix}$$

return

Complex

## Initial Value Problem

Solve

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$$

with the initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

## Initial Value Problem

Solution is

$$\begin{aligned} \mathbf{u}(t) &= 3e^t \begin{pmatrix} 3 \cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} \\ &\quad + 4e^t \begin{pmatrix} 3 \sin 2t - \cos 2t \\ \sin 2t \end{pmatrix} \\ &= e^t \begin{pmatrix} 5 \cos 2t + 15 \sin 2t \\ 3 \cos 2t + 4 \sin 2t \end{pmatrix} \end{aligned}$$

Real solutions

## Summary

Suppose  $A$  is a real  $2 \times 2$  matrix with

- complex conjugate eigenvalues  $\lambda$  and  $\bar{\lambda}$ , and
- associated nonzero eigenvectors  $\mathbf{w}$  and  $\bar{\mathbf{w}}$ .

Then

- $\mathbf{z}(t) = e^{\lambda t} \mathbf{w}$  and  $\bar{\mathbf{z}}(t) = e^{\bar{\lambda} t} \bar{\mathbf{w}}$  form a complex valued fundamental set of solutions, and
- $\mathbf{x}(t) = \operatorname{Re}(\mathbf{z}(t))$  and  $\mathbf{y}(t) = \operatorname{Im}(\mathbf{z}(t))$  form a real valued fundamental set of solutions.

return

## Double Real Root

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} = \frac{T}{2}.$$

- $T^2 - 4D = 0$
- Eigenspace has dimension 2:  $\Rightarrow A = \lambda I$ .
- Every vector is an eigenvector. Every solution has the form

$$\mathbf{x}(t) = e^{\lambda t} \mathbf{v}.$$

return

## Double Real Root

- Eigenspace has dimension 1.
- Standard procedure gives only one solution. If  $\mathbf{v}_1 \neq 0$  is an eigenvector, then

$$\mathbf{x}_1(t) = e^{\lambda t} \mathbf{v}_1$$

is a solution.

- The solution to the initial value problem  $\mathbf{x}' = A\mathbf{x}$  with  $\mathbf{x}(0) = \mathbf{x}_0$  is

$$\mathbf{x}(t) = e^{\lambda t} [I + t(A - \lambda I)] \mathbf{x}_0$$

Degenerate

return

### Example

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix}$$

- $p(\lambda) = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2$ ;  $\lambda = -2$
- $A - \lambda I = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}$ ;  $\mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$
- One solution

$$\mathbf{x}_1(t) = e^{\lambda t} \mathbf{v}_1 = e^{-2t} \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

return

### Example (cont.)

$$I + t(A - \lambda I) = \begin{pmatrix} 1 + 3t & 9t \\ -t & 1 - 3t \end{pmatrix}$$

- Solution with initial value  $\mathbf{x}_0$  is

$$\mathbf{x}(t) = e^{-2t} \begin{pmatrix} 1 + 3t & 9t \\ -t & 1 - 3t \end{pmatrix} \mathbf{x}_0.$$

- Easy fundamental set of solutions

$$\mathbf{y}_1(t) = e^{-2t} \begin{pmatrix} 1 + 3t \\ -t \end{pmatrix} \quad \& \quad \mathbf{y}_2(t) = e^{-2t} \begin{pmatrix} 9t \\ 1 - 3t \end{pmatrix}.$$

Solution

Example

### Example (cont.)

- Fundamental set including  $\mathbf{x}_1(t)$ 
  - ◊ Find  $\mathbf{v}_2$  with  $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$ .

$$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

◊

$$\begin{aligned} \mathbf{x}_2(t) &= e^{\lambda t} [\mathbf{v}_2 + t\mathbf{v}_1] \\ &= e^{-2t} \left( \begin{pmatrix} -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right) \end{aligned}$$

Solution

Example

## Summary

Suppose  $A$  is a real  $2 \times 2$  matrix with

- a double real eigenvalue  $\lambda$ , and
- the dimension of the eigenspace is one.

If  $\mathbf{v}_1$  is an eigenvector, and if  $\mathbf{v}_2$  satisfies

$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$ , then

- $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1$  and  $\mathbf{x}_2(t) = e^{\lambda t}[\mathbf{v}_2 + t\mathbf{v}_1]$  form a fundamental set of solutions.
- the general solution is  $\mathbf{x}(t) = e^{\lambda t}[(C_1 + C_2 t)\mathbf{v}_1 + C_2\mathbf{v}_2]$ .