Math 211

Lecture #18

October 31, 2000

Complex Numbers

A complex number is one of the form $z = x + iy$, where x and y are real numbers.

- $i^2 = -1$.
- x is the real part of z; $x = \mathrm{Re} z$.
- y is the imaginary part of z; $y = \mathrm{Im} z$.
  - ◇ The imaginary part is a real number.
- Addition and multiplication.

Complex Conjugate

The conjugate of $z = x + iy$ is $\overline{z} = x - iy$.

- $z = \overline{z} \quad \Leftrightarrow \quad z$ is a real number.
- $x = \mathrm{Re}z = \dfrac{z + \overline{z}}{2}$
- $y = \mathrm{Im}z = \dfrac{z - \overline{z}}{2i}$
- $\overline{z + w} = \overline{z} + \overline{w}; \quad \overline{z - w} = \overline{z} - \overline{w}$
- $\overline{zw} = \overline{z} \cdot \overline{w}; \quad \overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$

return

Absolute Value

The absolute value of $z = x + iy$ is the real number $|z| = \sqrt{x^2 + y^2}$.

- $z \cdot \overline{z} = |z|^2 = x^2 + y^2$.
- $|zw| = |z||w|$
- $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

Quotients

- Reciprocal of $z = x + iy$

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}.$$

$$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

- Quotient z/w

$$\frac{z}{w} = z \cdot \frac{1}{w} = \frac{z\overline{w}}{|w|^2}$$

Geometric Representation

- Complex plane.
- Addition
- Polar representation $z = r[\cos\theta + i\sin\theta]$.
  - ◇ θ is the argument of z: $\tan\theta = y/x$.
  - ◇ $r = |z|$.
- Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.
  - ◇ $z = |z|e^{i\theta}$
- Multiplication

Complex Exponential

For $z = x + iy$ define

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x [\cos y + i \sin y].$$

Properties:

- $e^{z+w} = e^z \cdot e^w; \quad e^{z-w} = e^z \cdot e^{-w} = e^z / e^w$
- $\overline{e^z} = e^{\overline{z}}$
- $|e^z| = e^{\mathrm{Re}\, z}$
- If λ is a complex number, then $\frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t}$

Complex Matrices

Matrices (or vectors) with complex entries inherit many of the properties of complex numbers.

- $M = A + iB$ where $A = \mathrm{Re}M$ and $B = \mathrm{Im}M$ are real matrices.
- $\overline{M} = A - iB; \quad M = \overline{M} \quad \Leftrightarrow \quad M$ is real.
- $\mathrm{Re}M = \frac{1}{2}(M + \overline{M}); \quad \mathrm{Im}M = \frac{1}{2i}(M - \overline{M})$
- $\overline{M + N} = \overline{M} + \overline{N}$
- $\overline{M\mathbf{z}} = \overline{M}\overline{\mathbf{z}}$

Consequences

A a real matrix; complex eigenvalue λ; associated eigenvector $\mathbf{w}$, so $A\mathbf{w} = \lambda\mathbf{w}$

- $\overline{A\mathbf{w}} = \overline{A}\overline{\mathbf{w}} = A\overline{\mathbf{w}}$.
- $\overline{\lambda\mathbf{w}} = \overline{\lambda}\overline{\mathbf{w}}$.
- $\Rightarrow \quad A\overline{\mathbf{w}} = \overline{\lambda}\overline{\mathbf{w}} \quad \Rightarrow \quad \overline{\lambda}$ is an eigenvalue of A with associated eigenvector $\overline{\mathbf{w}}$
- $\lambda \neq \overline{\lambda} \quad \Rightarrow \quad \mathbf{w}$ and $\overline{\mathbf{w}}$ are linearly independent.

return

Solutions

$$\mathbf{z}(t) = e^{\lambda t}\mathbf{w} \quad \text{and} \quad \overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}}.$$

- $\mathbf{z}$ and $\overline{\mathbf{z}}$ are linearly independent complex solutions to $\mathbf{x}' = A\mathbf{x}$.
- $\mathbf{z}(t) = \mathbf{x}(t) + i\mathbf{y}(t) \quad \& \quad \overline{\mathbf{z}}(t) = \mathbf{x}(t) - i\mathbf{y}(t)$
- $\mathbf{x}(t) = \frac{1}{2}(\mathbf{z}(t)+\overline{\mathbf{z}}(t))$ & $\mathbf{y}(t) = \frac{1}{2i}(\mathbf{z}(t)-\overline{\mathbf{z}}(t))$
- $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are real valued solutions.
- $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are linearly independent.

Example

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$$

- $p(\lambda) = \lambda^2 - 2\lambda + 5.$
- Eigenvalue: $\lambda = 1 + 2i$
- Eigenvector: $\mathbf{w} = \begin{pmatrix} 3 - i \\ 1 \end{pmatrix}$

return

Complex Solutions

$$\mathbf{z}(t) = e^{\lambda t}\mathbf{w} = e^{(1+2i)t}\begin{pmatrix} 3-i \\ 1 \end{pmatrix}$$

$$\overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}} = e^{(1-2i)t}\begin{pmatrix} 3+i \\ 1 \end{pmatrix}$$

Real Solutions

$$\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t)) = e^t \begin{pmatrix} 3\cos 2t + \sin 2t \\ \cos 2t \end{pmatrix}$$

$$\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t)) = e^t \begin{pmatrix} 3\sin 2t - \cos 2t \\ \sin 2t \end{pmatrix}$$

return

Complex

Initial Value Problem

Solve

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$$

with the intial condition

$$\mathbf{x}(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

Initial Value Problem

Solution is

$$\begin{aligned}\mathbf{u}(t) &= 3e^t \begin{pmatrix} 3\cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} \\ &\quad + 4e^t \begin{pmatrix} 3\sin 2t - \cos 2t \\ \sin 2t \end{pmatrix} \\ &= e^t \begin{pmatrix} 5\cos 2t + 15\sin 2t \\ 3\cos 2t + 4\sin 2t \end{pmatrix}\end{aligned}$$

Summary

Suppose A is a real 2×2 matrix with

- complex conjugate eigenvalues λ and $\overline{\lambda}$, and
- associated nonzero eigenvectors $\mathbf{w}$ and $\overline{\mathbf{w}}$.

Then

- $\mathbf{z}(t) = e^{\lambda t}\mathbf{w}$ and $\overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}}$ form a complex valued fundamental set of solutions, and
- $\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t))$ and $\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t))$ form a real valued fundamental set of solutions.

return

Double Real Root

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} = \frac{T}{2}.$$

- $T^2 - 4D = 0$
- Eigenspace has dimension 2: $\Rightarrow\ A = \lambda I$.
- Every vector is an eigenvector. Every solution has the form

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}.$$

return

Double Real Root

- Eigenspace has dimension 1.
- Standard procedure gives only one solution. If $\mathbf{v}_1 \neq 0$ is an eigenvector, then

  $$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1$$

  is a solution.
- The solution to the initial value problem $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$ is

  $$\mathbf{x}(t) = e^{\lambda t}[I + t(A - \lambda I)]\mathbf{x}_0$$

Example

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix}$$

- $p(\lambda) = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2; \quad \lambda = -2$
- $A - \lambda I = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}; \quad \mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$
- One solution

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1 = e^{-2t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

return

Example (cont.)

$$I + t(A - \lambda I) = \begin{pmatrix} 1 + 3t & 9t \\ -t & 1 - 3t \end{pmatrix}$$

- Solution with initial value $\mathbf{x}_0$ is

$$\mathbf{x}(t) = e^{-2t} \begin{pmatrix} 1 + 3t & 9t \\ -t & 1 - 3t \end{pmatrix} \mathbf{x}_0.$$

- Easy fundamental set of solutions

$$\mathbf{y}_1(t) = e^{-2t} \begin{pmatrix} 1 + 3t \\ -t \end{pmatrix} \ \& \ \mathbf{y}_2(t) = e^{-2t} \begin{pmatrix} 9t \\ 1 - 3t \end{pmatrix}.$$

Example (cont.)

- Fundamental set including $\mathbf{x}_1(t)$
  - ◇ Find $\mathbf{v}_2$ with $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$.

    $$\mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$
  - ◇

    $$\begin{aligned} \mathbf{x}_2(t) &= e^{\lambda t}[\mathbf{v}_2 + t\mathbf{v}_1] \\ &= e^{-2t}\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \end{pmatrix}\right) \end{aligned}$$

Summary

Suppose A is a real 2×2 matrix with

- a double real eigenvalue λ, and
- the dimension of the eigenspace is one.

If $\mathbf{v}_1$ is an eigenvector, and if $\mathbf{v}_2$ satisfies $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$, then

- $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda t}[\mathbf{v}_2 + t\mathbf{v}_1]$ form a fundamental set of solutions.
- the general solution is $\mathbf{x}(t) = e^{\lambda t}[(C_1 + C_2 t)\mathbf{v}_1 + C_2\mathbf{v}_2]$.