

# Math 211

Lecture #19

November 2, 2000

## Planar System $\mathbf{x}' = A\mathbf{x}$

Equilibrium points for the system

- The set of equilibrium points equals  $\text{null}(A)$ .
- If  $A$  is nonsingular  $\mathbf{0}$  is the only equilibrium point.
- Can we list the types of all possible equilibrium points for planar linear systems?
- The complete list is the second project.
- To do this we look at solution curves in the phase plane.

## Planar System $\mathbf{x}' = A\mathbf{x}$

- With distinct real eigenvalues.
- $p(\lambda) = \lambda^2 - T\lambda + D$  with  $T^2 - 4D > 0$ .

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} < \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

- Eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ . General solution

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2$$

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## Exponential Solutions

$$\mathbf{x}(t) = Ce^{\lambda t}\mathbf{v}$$

- The solution curve is a straight half-line through  $C\mathbf{v}$ .
- If  $\lambda > 0$  the solution starts at  $\mathbf{0}$  for  $t = -\infty$ , and tends to  $\infty$  as  $t \rightarrow \infty$ . Unstable solution
- If  $\lambda < 0$  the solution starts at  $\infty$  for  $t = -\infty$ , and tends to  $\mathbf{0}$  as  $t \rightarrow \infty$ . Stable solution
- Sometimes called half-line solutions.

Real eigenvalues

## Saddle Point

- $\lambda_1 < 0 < \lambda_2$
- General solution  $\mathbf{x}(t) = C_1e^{\lambda_1 t}\mathbf{v}_1 + C_2e^{\lambda_2 t}\mathbf{v}_2$
- Two stable exponential solutions ( $C_2 = 0$ ) and two unstable exponential solutions ( $C_1 = 0$ ).
- As  $t \rightarrow \infty$ ,  $\mathbf{x}(t) \rightarrow \infty$  approaching the half line through  $C_2\mathbf{v}_2$ .
- As  $t \rightarrow -\infty$ ,  $\mathbf{x}(t) \rightarrow \infty$  approaching the half line through  $C_1\mathbf{v}_1$ .

## Nodal Sink

- $\lambda_1 < \lambda_2 < 0$
- General solution
 
$$\mathbf{x}(t) = C_1e^{\lambda_1 t}\mathbf{v}_1 + C_2e^{\lambda_2 t}\mathbf{v}_2$$
- Stable exponential solutions.
- All solutions  $\rightarrow \mathbf{0}$  as  $t \rightarrow \infty$ . If  $C_2 \neq 0$  tangent to  $C_2\mathbf{v}_2$ . All solutions are stable.
- All solutions  $\rightarrow \infty$  as  $t \rightarrow -\infty$ . If  $C_1 \neq 0$  parallel to the half line through  $C_1\mathbf{v}_1$ .

## Nodal Source

- $0 < \lambda_1 < \lambda_2$
- General solution

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2$$

- Exponential solutions are unstable.
- All solutions  $\rightarrow \mathbf{0}$  as  $t \rightarrow -\infty$ . If  $C_1 \neq 0$  tangent to  $C_1 \mathbf{v}_1$ . All solutions are unstable.
- All solutions  $\rightarrow \infty$  as  $t \rightarrow -\infty$ . If  $C_2 \neq 0$  parallel to the half line through  $C_2 \mathbf{v}_2$ .

## Planar System $\mathbf{x}' = A\mathbf{x}$

- Complex eigenvalues,  $\lambda = \alpha + i\beta$  and  $\bar{\lambda} = \alpha - i\beta$ . ( $T^2 - 4D < 0$ )
- Eigenvector  $\mathbf{w} = \mathbf{v}_1 + i\mathbf{v}_2$  associated to  $\lambda$ .
- General real solution

$$\begin{aligned} \mathbf{x}(t) = & C_1 e^{\alpha t} [\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2] \\ & + C_2 e^{\alpha t} [\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2] \end{aligned}$$

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## Center

- $\alpha = \text{Re}(\lambda) = 0$
- General real solution

$$\begin{aligned} \mathbf{x}(t) = & C_1 [\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2] \\ & + C_2 [\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2] \end{aligned}$$

- Every solution is periodic with period  $T = 2\pi/\beta$ .
- All solution curves are ellipses.

[Complex eigenvalues](#)

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## Spiral Sink

- $\alpha = \operatorname{Re}(\lambda) < 0$
- General real solution

$$\mathbf{x}(t) = C_1 e^{\alpha t} [\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2] \\ + C_2 e^{\alpha t} [\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2]$$

- All solutions spiral into  $\mathbf{0}$  as  $t \rightarrow \infty$ .

Complex eigenvalues

Center

## Spiral Source

- $\alpha = \operatorname{Re}(\lambda) > 0$
- General real solution

$$\mathbf{x}(t) = C_1 e^{\alpha t} [\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2] \\ + C_2 e^{\alpha t} [\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2]$$

- All solutions spiral into  $\mathbf{0}$  as  $t \rightarrow -\infty$ .

Complex eigenvalues

Center

## Trace-Determinant Plane

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \quad p(\lambda) = \lambda^2 - T\lambda + D$$

- Eigenvalues  $\lambda_1, \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2}$ .

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \\ = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

- $T = \lambda_1 + \lambda_2$  and  $D = \lambda_1\lambda_2$ .
- Duality between  $(\lambda_1, \lambda_2)$  and  $(T, D)$ .

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### Trace-Determinant Plane (cont.)

- $T^2 - 4D > 0 \Rightarrow$  real eigenvalues  $\lambda_1$  &  $\lambda_2$ 
  - ◊  $D = \lambda_1\lambda_2 < 0 \Rightarrow$  Saddle point.
  - ◊  $D = \lambda_1\lambda_2 > 0 \Rightarrow$  Eigenvalues have the same sign.
    - ★  $T = \lambda_1 + \lambda_2 > 0$  Nodal source.
    - ★  $T = \lambda_1 + \lambda_2 < 0$  Nodal sink.

### Trace-Determinant Plane (cont.)

- $T^2 - 4D < 0 \Rightarrow$  complex eigenvalues
  - $\lambda = \alpha + i\beta$  and  $\bar{\lambda} = \alpha - i\beta$ .
  - ◊  $T = \lambda + \bar{\lambda} = 2\alpha > 0 \Rightarrow$  Spiral source
  - ◊  $T = \lambda + \bar{\lambda} = 2\alpha < 0 \Rightarrow$  Spiral sink
  - ◊  $T = \lambda + \bar{\lambda} = 2\alpha = 0 \Rightarrow$  Center

### Generic and Nongeneric Equilibrium Points

- Generic types
  - ◊ Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - ◊ All occupy large open subsets of the trace-determinant plane.
- Nongeneric types
  - ◊ Center and all others. Occupy pieces of the boundaries.

## Higher Dimensional Systems

$$\mathbf{x}' = A\mathbf{x}$$

- $A$  is an  $n \times n$  real matrix.
- If  $\lambda$  is an eigenvalue and  $\mathbf{v} \neq 0$  is an associated eigenvector, then  $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$  is a solution.

**Proposition:** Suppose that  $\lambda_1, \dots, \lambda_k$  are distinct eigenvalues of  $A$ , and that  $\mathbf{v}_1, \dots, \mathbf{v}_k$  are associated nonzero eigenvectors. Then  $\mathbf{v}_1, \dots, \mathbf{v}_k$  are linearly independent.

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**Theorem:** Suppose the  $n \times n$  real matrix  $A$  has  $n$  distinct eigenvalues  $\lambda_1, \dots, \lambda_n$ , and that  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are associated nonzero eigenvectors. Then the exponential solutions  $\mathbf{x}_i(t) = e^{\lambda_i t}\mathbf{v}_i$ ,  $1 \leq i \leq n$  form a fundamental system of solutions for the system  $\mathbf{x}' = A\mathbf{x}$ .

- Example

$$A = \begin{pmatrix} 17 & -30 & -8 \\ 16 & -29 & -8 \\ -12 & 24 & 7 \end{pmatrix}$$

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