Math 211

Lecture #19

November 2, 2000

Planar System $\mathbf{x}' = A\mathbf{x}$

Equilibrium points for the system

- The set of equilibrium points equals $\mathrm{null}(A)$.
- If A is nonsingular $\mathbf{0}$ is the only equilibrium point.
- Can we list the types of all possible equilibrium points for planar linear systems?
- The complete list is the second project.
- To do this we look at solution curves in the phase plane.

Planar System $\mathbf{x}' = A\mathbf{x}$

- With distinct real eigenvalues.
- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D > 0$.

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} < \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

- Eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. General solution

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

Exponential Solutions

$$\mathbf{x}(t) = Ce^{\lambda t}\mathbf{v}$$

- The solution curve is a straight half-line through $C\mathbf{v}$.
- If $\lambda > 0$ the solution starts at $\mathbf{0}$ for $t = -\infty$, and tends to ∞ as $t \to \infty$. Unstable solution
- If $\lambda < 0$ the solution starts at ∞ for $t = -\infty$, and tends to $\mathbf{0}$ as $t \to \infty$. Stable solution
- Sometimes called half-line solutions.

Saddle Point

- $\lambda_1 < 0 < \lambda_2$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$
- Two stable exponential solutions $(C_2 = 0)$ and two unstable exponential solutions $(C_1 = 0)$.
- As $t \to \infty$, $\mathbf{x}(t) \to \infty$ approaching the half line through $C_2\mathbf{v}_2$.
- As $t \to -\infty$, $\mathbf{x}(t) \to \infty$ approaching the half line through $C_1\mathbf{v}_1$.

Nodal Sink

- $\lambda_1 < \lambda_2 < 0$
- General solution

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

- Stable exponential solutions.
- All solutions $\to \mathbf{0}$ as $t \to \infty$. If $C_2 \neq 0$ tangent to $C_2\mathbf{v}_2$. All solutions are stable.
- All solutions $\to \infty$ as $t \to -\infty$. If $C_1 \neq 0$ parallel to the half line through $C_1\mathbf{v}_1$.

Nodal Source

- $0 < \lambda_1 < \lambda_2$
- General solution

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

- Exponential solutions are unstable.
- All solutions $\to \mathbf{0}$ as $t \to -\infty$. If $C_1 \neq 0$ tangent to $C_1\mathbf{v}_1$. All solutions are unstable.
- All solutions $\to \infty$ as $t \to -\infty$. If $C_2 \neq 0$ parallel to the half line through $C_2\mathbf{v}_2$.

Planar System $\mathbf{x}' = A\mathbf{x}$

- Complex eigenvalues , $\lambda = \alpha + i\beta$ and $\overline{\lambda} = \alpha - i\beta$. $(T^2 - 4D < 0)$
- Eigenvector $\mathbf{w} = \mathbf{v}_1 + i\mathbf{v}_2$ associated to λ.
- General real solution

$$\mathbf{x}(t) = C_1 e^{\alpha t}[\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2] + C_2 e^{\alpha t}[\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2]$$

Center

- $\alpha = \mathrm{Re}(\lambda) = 0$
- General real solution

$$\mathbf{x}(t) = C_1[\cos\beta t \cdot \mathbf{v}_1 - \sin\beta t \cdot \mathbf{v}_2] + C_2[\sin\beta t \cdot \mathbf{v}_1 + \cos\beta t \cdot \mathbf{v}_2]$$

- Every solution is periodic with period $T = 2\pi/\beta$.
- All solution curves are ellipses.

Spiral Sink

- $\alpha = \mathrm{Re}(\lambda) < 0$
- General real solution

$$\mathbf{x}(t) = C_1 e^{\alpha t}[\cos\beta t \cdot \mathbf{v}_1 - \sin\beta t \cdot \mathbf{v}_2] + C_2 e^{\alpha t}[\sin\beta t \cdot \mathbf{v}_1 + \cos\beta t \cdot \mathbf{v}_2]$$

- All solutions spiral into $\mathbf{0}$ as $t \to \infty$.

Spiral Source

- $\alpha = \mathrm{Re}(\lambda) > 0$
- General real solution

$$\begin{aligned}\mathbf{x}(t) = {} & C_1 e^{\alpha t}[\cos\beta t \cdot \mathbf{v}_1 - \sin\beta t \cdot \mathbf{v}_2] \\ & + C_2 e^{\alpha t}[\sin\beta t \cdot \mathbf{v}_1 + \cos\beta t \cdot \mathbf{v}_2]\end{aligned}$$

- All solutions spiral into $\mathbf{0}$ as $t \to -\infty$.

Trace-Determinant Plane

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \quad p(\lambda) = \lambda^2 - T\lambda + D$$

- Eigenvalues $\lambda_1,\ \lambda_2 = \dfrac{T \pm \sqrt{T^2 - 4D}}{2}$.

$$\begin{aligned} p(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2) \\ &= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \end{aligned}$$

- $T = \lambda_1 + \lambda_2$ and $D = \lambda_1\lambda_2$.
- Duality between $(\lambda_1,\ \lambda_2)$ and (T, D).

Trace-Determinant Plane (cont.)

- $T^2 - 4D > 0 \Rightarrow$ real eigenvalues $\lambda_1 \,\&\, \lambda_2$
  - $D = \lambda_1\lambda_2 < 0 \Rightarrow$ Saddle point.
  - $D = \lambda_1\lambda_2 > 0 \Rightarrow$ Eigenvalues have the same sign.
    - $T = \lambda_1 + \lambda_2 > 0$ Nodal source.
    - $T = \lambda_1 + \lambda_2 < 0$ Nodal sink.

Trace-Determinant Plane (cont.)

- $T^2 - 4D < 0 \Rightarrow$ complex eigenvalues

$$\lambda = \alpha + i\beta \quad \text{and} \quad \overline{\lambda} = \alpha - i\beta.$$

  - $T = \lambda + \overline{\lambda} = 2\alpha > 0 \Rightarrow$ Spiral source
  - $T = \lambda + \overline{\lambda} = 2\alpha < 0 \Rightarrow$ Spiral sink
  - $T = \lambda + \overline{\lambda} = 2\alpha = 0 \Rightarrow$ Center

Generic and Nongeneric Equilibrium Points

- Generic types
  - ◇ Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - ◇ All occupy large open subsets of the trace-determinant plane.
- Nongeneric types
  - ◇ Center and all others. Occupy pieces of the boundaries.

Higher Dimensional Systems

$$\mathbf{x}' = A\mathbf{x}$$

- A is an $n \times n$ real matrix.
- If λ is an eigenvalue and $\mathbf{v} \neq 0$ is an associated eigenvector, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.

Proposition: Suppose that $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of A, and that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are associated nonzero eigenvectors. Then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

Theorem: Suppose the $n \times n$ real matrix A has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, and that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are associated nonzero eigenvectors. Then the exponential solutions $\mathbf{x}_i(t) = e^{\lambda_i t}\mathbf{v}_i$, $1 \leq i \leq n$ form a fundamental system of solutions for the system $\mathbf{x}' = A\mathbf{x}$.

- Example

$$A = \begin{pmatrix} 17 & -30 & -8 \\ 16 & -29 & -8 \\ -12 & 24 & 7 \end{pmatrix}$$