# Math 211

Lecture #20

November 7, 2000

# Higher Dimensional Systems

$$\mathbf{x}' = A\mathbf{x}$$

- $A$ is an $n \times n$ real matrix.
- If $\lambda$ is an eigenvalue and $\mathbf{v} \neq 0$ is an associated eigenvector, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.

**Proposition:** Suppose that $\lambda_1$, ..., $\lambda_k$ are distinct eigenvalues of $A$, and that $\mathbf{v}_1$, ..., $\mathbf{v}_k$ are associated nonzero eigenvectors. Then $\mathbf{v}_1$, ..., $\mathbf{v}_k$ are linearly independent.

Return

**Theorem:** Suppose the $n \times n$ real matrix $A$ has $n$ distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, and that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are associated nonzero eigenvectors. Then the exponential solutions $\mathbf{x}_i(t) = e^{\lambda_i t}\mathbf{v}_i$, $1 \le i \le n$ form a fundamental system of solutions for the system $\mathbf{x}' = A\mathbf{x}$.

- Example

$$A = \begin{pmatrix} 17 & -30 & -8 \\ 16 & -29 & -8 \\ -12 & 24 & 7 \end{pmatrix}$$

Return

## Complex Eigenvalues

Suppose $A$ is a real $n \times n$ matrix.

- Suppose $\lambda$ is a complex eigenvalue and $\mathbf{w}$ is an associated nonzero eigenvector.
- Then $\overline{\lambda}$ is an eigenvalue and $\overline{\mathbf{w}}$ is an associated nonzero eigenvector.
- $\mathbf{z}(t) = e^{\lambda t}\mathbf{w}$ and $\overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}}$ are linearly independent complex valued solutions.
- $\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t))$ and $\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t))$ are linearly independent real valued solutions.

Return

## Example

$$A = \begin{pmatrix} 21 & 10 & 4 \\ -70 & -31 & -10 \\ 30 & 10 & -1 \end{pmatrix}$$

- The theorem applies if some of the eigenvalues are complex and we replace complex conjugate pairs of solutions by their real and imaginary parts.

Theorem

# Repeated Eigenvalues – Example 1

$$A = \begin{pmatrix} -5 & -10 & 6 \\ 8 & 19 & -12 \\ 12 & 30 & -19 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- Eigenspace for the eigenvalue $\lambda_1 = -3$ has dimension 1 $\Rightarrow$ one exponential solution

$$\mathbf{x}_1(t) = e^{-3t} \begin{pmatrix} -1/3 \\ 2/3 \\ 1 \end{pmatrix}$$

Example 1a Example 2 Example 2a Analysis Return

# Repeated Eigenvalues – Example 1

- Eigenspace for the eigenvalue $\lambda_2 = -1$ has dimension 2 $\Rightarrow$ two linearly independent exponential solutions

$$\mathbf{x}_2(t) = e^{-t} \begin{pmatrix} -5/2 \\ 1 \\ 0 \end{pmatrix} \quad \& \quad \mathbf{x}_3(t) = e^{-t} \begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix}$$

Example 1 Example 2 Example 2a Analysis Return

# Repeated Eigenvalues – Example 2

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- Eigenspace for the eigenvalue $\lambda_1 = -3$ has dimension 1 $\Rightarrow$ one exponential solution

$$\mathbf{x}_1(t) = e^{-3t} \begin{pmatrix} -1/2 \\ 3/2 \\ 1 \end{pmatrix}$$

Example 1 Example 1a Example 2a Analysis Return

# Repeated Eigenvalues – Example 2

- Eigenspace for the eigenvalue $\lambda_2 = -1$ has dimension 1 ⇒ only one exponential solution

$$\mathbf{x}_2(t) = e^{-t}\begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix}$$

- Need a third solution.

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# Multiplicities

$A$ an $n \times n$ matrix

- Distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.
- The characteristic polynomial is

$$p(\lambda) = (\lambda - \lambda_1)^{q_1}(\lambda - \lambda_2)^{q_2} \cdot \ldots \cdot (\lambda - \lambda_k)^{q_k}.$$

- The algebraic multiplicity of $\lambda_j$ is $q_j$.
- The geometric multiplicity of $\lambda_j$ is $d_j$, the dimension of the eigenspace of $\lambda_j$.

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# Multiplicities (cont.)

We always have:

- $q_1 + q_2 + \cdots + q_k = n$.
- $1 \leq d_j \leq q_j$.
- There are $d_j$ linearly independent exponential solutions corresponding to $\lambda_j$.
- If $d_j = q_j$ for all $j$ we have $n$ linearly independent solutions.

## Examples

- In both $p(\lambda) = (\lambda + 1)^2(\lambda + 3)$.
- In both $\lambda_2 = -1$ has algebraic multiplicity 2.
- In Ex. 1 $\lambda_2 = -1$ has geom. multiplicity 2.
- In Ex. 2 $\lambda_2 = -1$ has geom. multiplicity 1.
- Problems arise when $d_j < q_j$.

Return Example 1 Example 1a Example 2 Example 2a Multiplicities.a

## New Approach

- $D = 1$ $x' = ax$
  - ◇ Solution $x(t) = Ce^{at}$.
- $D > 1$ $\mathbf{x}' = A\mathbf{x}$
  - ◇ Tried $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$.
  - ◇ Why not $\mathbf{x}(t) = e^{tA}\mathbf{v}$?
- But what is $e^{tA}$?

## Exponential of a Matrix

**Definition:** The exponential of the $n \times n$ matrix $A$ is the $n \times n$ matrix

$$\begin{aligned} e^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \\ &= \sum_0^\infty \frac{1}{n!}A^n. \end{aligned}$$

# Examples

- $A = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$

$$e^A = \begin{pmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{pmatrix}.$$

- $e^{\lambda I} = e^{\lambda} I.$
- $e^{0I} = I.$

Return

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# Properties 1

- $A$ commutes with $e^A$.
- If $A$ and $B$ commute, then $e^{A+B} = e^A \cdot e^B$.
- The inverse of $e^A$ is $e^{-A}$.
- $\frac{d}{dt} e^{tA} = A e^{tA}$.

Return

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# Properties 2

- The solution to the intial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}$$

is

$$\mathbf{x}(t) = e^{tA}\mathbf{v}.$$

Return

# The Key Idea

Let $\lambda$ be a number (an eigenvalue), and $A$ an $n \times n$ matrix.

- $A = \lambda I + (A - \lambda I)$ ; $\lambda I$ & $A - \lambda I$ commute.

$$\begin{aligned} e^{tA} &= e^{t[\lambda I + (A-\lambda I)]} \\ &= e^{t\lambda I} \cdot e^{t(A-\lambda I)} \\ &= e^{\lambda t} \cdot e^{t(A-\lambda I)} \\ &= e^{\lambda t} \cdot [I + t(A-\lambda I) + \frac{t^2}{2!}(A-\lambda I)^2 + \cdots] \end{aligned}$$

Examples Prop 1 Return

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# Eigenvector

Let $\lambda$ be an eigenvalue and $\mathbf{v}$ an associated eigenvector. $\Rightarrow$ $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

$$\begin{aligned} e^{tA}\mathbf{v} &= e^{\lambda t} \cdot e^{t(A-\lambda I)}\mathbf{v} \\ &= e^{\lambda t}[I + t(A-\lambda I) + \frac{t^2}{2!}(A-\lambda I)^2 + \cdots]\mathbf{v} \\ &= e^{\lambda t}[\mathbf{v} + t(A-\lambda I)\mathbf{v} + \frac{t^2}{2!}(A-\lambda I)^2\mathbf{v} + \cdots] \\ &= e^{\lambda t}\mathbf{v} \end{aligned}$$

Key Idea Return

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# Matrices with One Eigenvalue

$A$ an $n \times n$ matrix with characteristic polynomial $p(\lambda) = (\lambda - \lambda_1)^n$.

- Cayley-Hamilton Theorem: If $p(\lambda)$ is the characteristic polynomial of the matrix $A$ then $p(A) = 0I$.
- In our case $(A - \lambda_1 I)^n = 0I$.

Key Idea Return

# Matrices with One Eigenvalue (cont.)

$$\begin{aligned} e^{tA} = e^{\lambda_1 t} \cdot [I &+ t(A - \lambda_1 I) \\ &+ \frac{t^2}{2!}(A - \lambda_1 I)^2 + \cdots \\ &+ \frac{t^{n-1}}{(n-1)!}(A - \lambda_1 I)^{n-1}] \end{aligned}$$

Key Idea Return

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# Example 1

$$A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 2)^2$.

$$A + 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \qquad (A + 2I)^2 = 0I$$

$$\begin{aligned} e^{tA} &= e^{-2t}[I + t(A + 2I)] \\ &= e^{-2t} \begin{pmatrix} 1 - t & t \\ -t & 1 + t \end{pmatrix}. \end{aligned}$$

Key Idea Solution 1 E Return

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# Example 2

$$A = \begin{pmatrix} 0 & -9 & 27 \\ -2 & 3 & -18 \\ -1 & 3 & -12 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)^3$. $(A + 3I)^2 = 0I$.

$$\begin{aligned} e^{tA} &= e^{-3t}[I + t(A + 3I)] \\ &= e^{-3t} \begin{pmatrix} 1 + 3t & -9t & 27t \\ -2t & 1 + 6t & -18t \\ -t & 3t & 1 - 9t \end{pmatrix}. \end{aligned}$$

1 E Key Idea Solution Return

# Example 3 (a)

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix}$$

- Earlier example.
- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- Distinct eigenvalues $\lambda_1 = -3$ & $\lambda_2 = -1$
- Different from previous two examples.

Example

Return

# Example 3 (b)

- Eigenspace for the eigenvalue $\lambda_1 = -3$ has dimension 1 $\Rightarrow$ one exponential solution

$$\begin{aligned} \mathbf{x}_1(t) &= e^{\lambda_1 t}\mathbf{v}_1 \\ &= e^{-3t}\begin{pmatrix} -1/2 \\ 3/2 \\ 1 \end{pmatrix} \end{aligned}$$

Return

# Example 3 (c)

- Eigenspace for the eigenvalue $\lambda_2 = -1$ has dimension 1 $\Rightarrow$ only one exponential solution

$$\begin{aligned} \mathbf{x}_2(t) &= e^{\lambda_2 t}\mathbf{v}_2 \\ &= e^{-t}\begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

- However, $\mathrm{null}((A - \lambda_2 I)^2)$ has dimension 2.

Return

# Example 3 (d)

- If $\mathbf{v} \in \mathrm{null}((A - \lambda_2 I)^2)$ then

$$\begin{aligned} e^{tA}\mathbf{v} &= e^{\lambda_2 t}[I + t(A - \lambda_2 I) \\ &\quad + \frac{t^2}{2!}(A - \lambda_2 I)^2 + \cdots ]\mathbf{v} \\ &= e^{\lambda_2 t}[\mathbf{v} + t(A - \lambda_2 I)\mathbf{v} \\ &\quad + \frac{t^2}{2!}(A - \lambda_2 I)^2\mathbf{v} + \cdots ] \\ &= e^{\lambda_2 t}[\mathbf{v} + t(A - \lambda_2 I)\mathbf{v}]. \end{aligned}$$

# Example 3 (e)

- $\mathrm{null}(A + I)^2$ has basis

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Third solution:

$$\mathbf{x}_3(t) = e^{tA}\mathbf{v}_3 = e^{-t}[\mathbf{v}_3 + t(A + I)\mathbf{v}_3]$$

# Example 3 (f)

$$\begin{aligned} \mathbf{x}_3(t) &= e^{-t}[\mathbf{v}_3 + t(A + I)\mathbf{v}_3] \\ &= e^{-t}[\mathbf{v}_3 - 4t\mathbf{v}_2] \\ &= e^{-t}\begin{pmatrix} 1 + 2t \\ -4t \\ -4t \end{pmatrix}. \end{aligned}$$

# Summary

- In Examples 1 & 2 the matrix has one eigenvalue.
    - ◇ The series for $e^{t(A-\lambda I)}$ truncated to a finite sum.
- In Example 3 the matrix had two eigenvalues.
    - ◇ The series for $e^{t(A-\lambda_2 I)}$ does not truncate.
    - ◇ The series for $e^{t(A-\lambda_2 I)}\mathbf{v}$ does truncate if $(A-\lambda_2 I)^2\mathbf{v} = \mathbf{0}$.

Key Idea 1 2 3(a) 3(d) Return

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# Generalized Eigenvectors

**Definition:** If $\lambda$ is an eigenvalue of $A$ and $(A-\lambda I)^p\mathbf{v} = \mathbf{0}$ for some integer $p \geq 1$, then $\mathbf{v}$ is called a generalized eigenvector associated with $\lambda$.

- The series for $e^{t(A-\lambda I)}\mathbf{v}$ truncates to a finite sum if $\mathbf{v}$ is a generalized eigenvector associated with $\lambda$.
- We can compute $e^{tA}\mathbf{v}$.

Key Idea Summary

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# Generalized Eigenvectors

**Theorem:** If $\lambda$ is an eigenvalue of $A$ with algebraic multiplicity $q$, then there is an integer $p \leq q$ such that $\mathrm{null}((A-\lambda I)^p)$ has dimension $q$.

- For each generalized eigenvector $\mathbf{v}$ we can compute $e^{tA}\mathbf{v}$.
- We can find $q$ linearly independent solutions this way.

Key Idea

# Procedure (a)

To find $q$ linearly independent solutions associated with an eigenvalue $\lambda$ of algebraic multiplicity $q$.

- Find the smallest integer $p$ such that $\mathrm{null}((A-\lambda I)^p)$ has dimension q.
- Find a basis $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_q$ of $\mathrm{null}((A-\lambda I)^p)$.

# Procedure (b)

- For $j = 1, 2, \ldots, q$

$$\begin{aligned} \mathbf{x}_j(t) &= e^{tA}\mathbf{v}_j \\ &= e^{\lambda t}[\mathbf{v}_j + t(A-\lambda I)\mathbf{v}_j \\ &\quad + \frac{t^2}{2!}(A-\lambda I)^2\mathbf{v}_j + \cdots \\ &\quad + \frac{t^{p-1}}{(p-1)!}(A-\lambda I)^{p-1}\mathbf{v}_j] \end{aligned}$$

Key Idea

# Example

# Procedure (c)

If $\lambda$ is complex of algebraic multiplicity $q$. Then $\overline{\lambda}$ also has multiplicity $q$.

- Find the smallest integer $p$ such that $\mathrm{null}((A-\lambda I)^p)$ has dimension q.
- Find a basis $\mathbf{w}_1,\ \mathbf{w}_2,\ \ldots,\ \mathbf{w}_q$ of $\mathrm{null}((A-\lambda I)^p)$.

# Procedure (d)

- For $j = 1, 2, \ldots, q$

$$\begin{aligned}\mathbf{z}_j(t) = e^{\lambda t}[\mathbf{w}_j &+ t(A-\lambda I)\mathbf{w}_j \\ &+ \frac{t^2}{2!}(A-\lambda I)^2\mathbf{w}_j + \cdots \\ &+ \frac{t^{p-1}}{(p-1)!}(A-\lambda I)^{p-1}\mathbf{w}_j]\end{aligned}$$

- For $j = 1, 2, \ldots, q$ set $\mathbf{x}_j(t) = \mathrm{Re}(\mathbf{z}_j(t))$ and $\mathbf{y}_j(t) = \mathrm{Im}(\mathbf{z}_j(t))$.

Key Idea