Math 211

Lecture #20

November 7, 2000

Higher Dimensional Systems

$$\mathbf{x}' = A\mathbf{x}$$

- A is an $n \times n$ real matrix.
- If λ is an eigenvalue and $\mathbf{v} \neq 0$ is an associated eigenvector, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.

Proposition: Suppose that λ_1, . . . , λ_k are distinct eigenvalues of A, and that $\mathbf{v}_1$, . . . , $\mathbf{v}_k$ are associated nonzero eigenvectors. Then $\mathbf{v}_1$, . . . , $\mathbf{v}_k$ are linearly independent.

Return

Theorem: Suppose the $n \times n$ real matrix A has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, and that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are associated nonzero eigenvectors. Then the exponential solutions $\mathbf{x}_i(t) = e^{\lambda_i t}\mathbf{v}_i$, $1 \leq i \leq n$ form a fundamental system of solutions for the system $\mathbf{x}' = A\mathbf{x}$.

- Example

$$A = \begin{pmatrix} 17 & -30 & -8 \\ 16 & -29 & -8 \\ -12 & 24 & 7 \end{pmatrix}$$

Return

Complex Eigenvalues

Suppose A is a real $n \times n$ matrix.

- Suppose λ is a complex eigenvalue and $\mathbf{w}$ is an associated nonzero eigenvector.
- Then $\overline{\lambda}$ is an eigenvalue and $\overline{\mathbf{w}}$ is an associated nonzero eigenvector.
- $\mathbf{z}(t) = e^{\lambda t}\mathbf{w}$ and $\overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}}$ are linearly independent complex valued solutions.
- $\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t))$ and $\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t))$ are linearly independent real valued solutions.

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Example

$$A = \begin{pmatrix} 21 & 10 & 4 \\ -70 & -31 & -10 \\ 30 & 10 & -1 \end{pmatrix}$$

- The theorem applies if some of the eigenvalues are complex and we replace complex conjugate pairs of solutions by their real and imaginary parts.

Theorem

Repeated Eigenvalues – Example 1

$$A = \begin{pmatrix} -5 & -10 & 6 \\ 8 & 19 & -12 \\ 12 & 30 & -19 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- Eigenspace for the eigenvalue $\lambda_1 = -3$ has dimension 1 $\Rightarrow$ one exponential solution

$$\mathbf{x}_1(t) = e^{-3t} \begin{pmatrix} -1/3 \\ 2/3 \\ 1 \end{pmatrix}$$

Repeated Eigenvalues – Example 1

- Eigenspace for the eigenvalue $\lambda_2 = -1$ has dimension 2 ⇒ two linearly independent exponential solutions

$$\mathbf{x}_2(t) = e^{-t}\begin{pmatrix} -5/2 \\ 1 \\ 0 \end{pmatrix} \quad \& \quad \mathbf{x}_3(t) = e^{-t}\begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix}$$

Repeated Eigenvalues – Example 2

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- Eigenspace for the eigenvalue $\lambda_1 = -3$ has dimension 1 ⇒ one exponential solution

$$\mathbf{x}_1(t) = e^{-3t} \begin{pmatrix} -1/2 \\ 3/2 \\ 1 \end{pmatrix}$$

Example 1 Example 1a Example 2a Analysis Return

Repeated Eigenvalues – Example 2

- Eigenspace for the eigenvalue $\lambda_2 = -1$ has dimension 1 ⇒ only one exponential solution

$$\mathbf{x}_2(t) = e^{-t}\begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix}$$

- Need a third solution.

Multiplicities

A an $n \times n$ matrix

- Distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.
- The characteristic polynomial is

$$p(\lambda) = (\lambda - \lambda_1)^{q_1}(\lambda - \lambda_2)^{q_2} \cdot \ldots \cdot (\lambda - \lambda_k)^{q_k}.$$

- The algebraic multiplicity of λ_j is q_j.
- The geometric multiplicity of λ_j is d_j, the dimension of the eigenspace of λ_j.

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Multiplicities (cont.)

We always have:

- $q_1 + q_2 + \cdots + q_k = n$.
- $1 \leq d_j \leq q_j$.
- There are d_j linearly independent exponential solutions corresponding to λ_j.
- If $d_j = q_j$ for all j we have n linearly independent solutions.

Examples

- In both $p(\lambda) = (\lambda + 1)^2(\lambda + 3)$.
- In both $\lambda_2 = -1$ has algebraic multiplicity 2.
- In Ex. 1 $\lambda_2 = -1$ has geom. multiplicity 2.
- In Ex. 2 $\lambda_2 = -1$ has geom. multiplicity 1.
- Problems arise when $d_j < q_j$.

New Approach

- $D = 1$ $x' = ax$
  - ◇ Solution $x(t) = Ce^{at}$.
- $D > 1$ $\mathbf{x}' = A\mathbf{x}$
  - ◇ Tried $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$.
  - ◇ Why not $\mathbf{x}(t) = e^{tA}\mathbf{v}$?
- But what is e^{tA}?

Exponential of a Matrix

Definition: The exponential of the $n \times n$ matrix A is the $n \times n$ matrix

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$

$$= \sum_{0}^{\infty} \frac{1}{n!} A^n.$$

Examples

- $A = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$

$$e^A = \begin{pmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{pmatrix}.$$

- $e^{\lambda I} = e^{\lambda} I.$
- $e^{0I} = I.$

Properties 1

- A commutes with e^A.
- If A and B commute, then $e^{A+B} = e^A \cdot e^B$.
- The inverse of e^A is e^{-A}.
- $\dfrac{d}{dt} e^{tA} = Ae^{tA}$.

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Properties 2

- The solution to the intial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}$$

is

$$\mathbf{x}(t) = e^{tA}\mathbf{v}.$$

The Key Idea

Let λ be a number (an eigenvalue), and A an $n \times n$ matrix.

- $A = \lambda I + (A - \lambda I)$; λI & $A - \lambda I$ commute.

$$\begin{aligned} e^{tA} &= e^{t[\lambda I + (A - \lambda I)]} \\ &= e^{t\lambda I} \cdot e^{t(A - \lambda I)} \\ &= e^{\lambda t} \cdot e^{t(A - \lambda I)} \\ &= e^{\lambda t} \cdot \left[I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots \right] \end{aligned}$$

Eigenvector

Let λ be an eigenvalue and $\mathbf{v}$ an associated eigenvector. $\Rightarrow$ $(A - \lambda I)\mathbf{v} = \mathbf{0}$.

$$\begin{aligned} e^{tA}\mathbf{v} &= e^{\lambda t} \cdot e^{t(A-\lambda I)}\mathbf{v} \\ &= e^{\lambda t}[I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots]\mathbf{v} \\ &= e^{\lambda t}[\mathbf{v} + t(A - \lambda I)\mathbf{v} + \frac{t^2}{2!}(A - \lambda I)^2\mathbf{v} + \cdots] \\ &= e^{\lambda t}\mathbf{v} \end{aligned}$$

Matrices with One Eigenvalue

A an $n \times n$ matrix with characteristic polynomial $p(\lambda) = (\lambda - \lambda_1)^n$.

- Cayley-Hamilton Theorem: If $p(\lambda)$ is the characteristic polynomial of the matrix A then $p(A) = 0I$.
- In our case $(A - \lambda_1 I)^n = 0I$.

Matrices with One Eigenvalue (cont.)

$$\begin{aligned} e^{tA} = e^{\lambda_1 t} \cdot [I + t(A - \lambda_1 I) \\ + \frac{t^2}{2!}(A - \lambda_1 I)^2 + \cdots \\ + \frac{t^{n-1}}{(n-1)!}(A - \lambda_1 I)^{n-1}] \end{aligned}$$

Key Idea Return

Example 1

$$A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 2)^2$.

$$A + 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \qquad (A + 2I)^2 = 0I$$

$$\begin{aligned} e^{tA} &= e^{-2t}[I + t(A + 2I)] \\ &= e^{-2t} \begin{pmatrix} 1 - t & t \\ -t & 1 + t \end{pmatrix}. \end{aligned}$$

Example 2

$$A = \begin{pmatrix} 0 & -9 & 27 \\ -2 & 3 & -18 \\ -1 & 3 & -12 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)^3. \quad (A + 3I)^2 = 0I.$

$$\begin{aligned} e^{tA} &= e^{-3t}[I + t(A + 3I)] \\ &= e^{-3t} \begin{pmatrix} 1 + 3t & -9t & 27t \\ -2t & 1 + 6t & -18t \\ -t & 3t & 1 - 9t \end{pmatrix}. \end{aligned}$$

Example 3 (a)

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix}$$

- Earlier example.
- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- Distinct eigenvalues $\lambda_1 = -3$ & $\lambda_2 = -1$
- Different from previous two examples.

Example 3 (b)

- Eigenspace for the eigenvalue $\lambda_1 = -3$ has dimension 1 $\Rightarrow$ one exponential solution

$$\begin{aligned}\mathbf{x}_1(t) &= e^{\lambda_1 t}\mathbf{v}_1 \\ &= e^{-3t}\begin{pmatrix} -1/2 \\ 3/2 \\ 1 \end{pmatrix}\end{aligned}$$

Return

Example 3 (c)

- Eigenspace for the eigenvalue $\lambda_2 = -1$ has dimension 1 ⇒ only one exponential solution

$$\begin{aligned} \mathbf{x}_2(t) &= e^{\lambda_2 t}\mathbf{v}_2 \\ &= e^{-t}\begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix} \end{aligned}$$

- However, $\mathrm{null}((A - \lambda_2 I)^2)$ has dimension 2.

Return

Example 3 (d)

- If $\mathbf{v} \in \mathrm{null}((A - \lambda_2 I)^2)$ then

$$\begin{aligned} e^{tA}\mathbf{v} &= e^{\lambda_2 t}[I + t(A - \lambda_2 I) \\ &\qquad + \frac{t^2}{2!}(A - \lambda_2 I)^2 + \cdots]\mathbf{v} \\ &= e^{\lambda_2 t}[\mathbf{v} + t(A - \lambda_2 I)\mathbf{v} \\ &\qquad + \frac{t^2}{2!}(A - \lambda_2 I)^2\mathbf{v} + \cdots] \\ &= e^{\lambda_2 t}[\mathbf{v} + t(A - \lambda_2 I)\mathbf{v}]. \end{aligned}$$

Example 3 (e)

- $\text{null}(A+I)^2$ has basis

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Third solution:

$$\mathbf{x}_3(t) = e^{tA}\mathbf{v}_3 = e^{-t}[\mathbf{v}_3 + t(A+I)\mathbf{v}_3]$$

Example 3 (f)

$$\begin{aligned}\mathbf{x}_3(t) &= e^{-t}[\mathbf{v}_3 + t(A+I)\mathbf{v}_3] \\ &= e^{-t}[\mathbf{v}_3 - 4t\mathbf{v}_2] \\ &= e^{-t}\begin{pmatrix} 1+2t \\ -4t \\ -4t \end{pmatrix}.\end{aligned}$$

Summary

- In Examples 1 & 2 the matrix has one eigenvalue.
    - ◇ The series for $e^{t(A-\lambda I)}$ truncated to a finite sum.
- In Example 3 the matrix had two eigenvalues.
    - ◇ The series for $e^{t(A-\lambda_2 I)}$ does not truncate.
    - ◇ The series for $e^{t(A-\lambda_2 I)}\mathbf{v}$ does truncate if $(A-\lambda_2 I)^2\mathbf{v}=\mathbf{0}$.

Generalized Eigenvectors

Definition: If λ is an eigenvalue of A and $(A - \lambda I)^p \mathbf{v} = \mathbf{0}$ for some integer $p \geq 1$, then $\mathbf{v}$ is called a generalized eigenvector associated with λ.

- The series for $e^{t(A-\lambda I)}\mathbf{v}$ truncates to a finite sum if $\mathbf{v}$ is a generalized eigenvector associated with λ.
- We can compute $e^{tA}\mathbf{v}$.

Generalized Eigenvectors

Theorem: If λ is an eigenvalue of A with algebraic multiplicity q, then there is an integer $p \leq q$ such that $\mathrm{null}((A - \lambda I)^p)$ has dimension q.

- For each generalized eigenvector $\mathbf{v}$ we can compute $e^{tA}\mathbf{v}$.
- We can find q linearly independent solutions this way.

Procedure (a)

To find q linearly independent solutions associated with an eigenvalue λ of algebraic multiplicity q.

- Find the smallest integer p such that $\mathrm{null}((A - \lambda I)^p)$ has dimension q.
- Find a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q$ of $\mathrm{null}((A - \lambda I)^p)$.

Procedure (b)

- For $j = 1, 2, \ldots, q$

$$\begin{aligned} \mathbf{x}_j(t) &= e^{tA}\mathbf{v}_j \\ &= e^{\lambda t}[\mathbf{v}_j + t(A - \lambda I)\mathbf{v}_j \\ &\quad + \frac{t^2}{2!}(A - \lambda I)^2\mathbf{v}_j + \cdots \\ &\quad + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}\mathbf{v}_j] \end{aligned}$$

Example

Procedure (c)

If λ is complex of algebraic multiplicity q. Then $\overline{\lambda}$ also has multiplicity q.

- Find the smallest integer p such that $\mathrm{null}((A - \lambda I)^p)$ has dimension q.
- Find a basis $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q$ of $\mathrm{null}((A - \lambda I)^p)$.

Procedure (d)

- For $j = 1, 2, \ldots, q$

$$\begin{aligned}\mathbf{z}_j(t) = e^{\lambda t}[\mathbf{w}_j &+ t(A - \lambda I)\mathbf{w}_j \\ &+ \frac{t^2}{2!}(A - \lambda I)^2\mathbf{w}_j + \cdots \\ &+ \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}\mathbf{w}_j]\end{aligned}$$

- For $j = 1, 2, \ldots, q$ set $\mathbf{x}_j(t) = \mathrm{Re}(\mathbf{z}_j(t))$ and $\mathbf{y}_j(t) = \mathrm{Im}(\mathbf{z}_j(t))$.