Math 211

Lecture #21

November 9, 2000

The Key Idea

Let λ be a number (an eigenvalue), and A an $n \times n$ matrix.

- $A = \lambda I + (A - \lambda I)$; λI & $A - \lambda I$ commute.

$$
\begin{aligned}
e^{tA} &= e^{t[\lambda I + (A - \lambda I)]} \\
&= e^{t\lambda I} \cdot e^{t(A - \lambda I)} \\
&= e^{\lambda t} \cdot e^{t(A - \lambda I)} \\
&= e^{\lambda t} \cdot [I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots]
\end{aligned}
$$

Matrices with One Eigenvalue

A an $n \times n$ matrix with characteristic polynomial $p(\lambda) = (\lambda - \lambda_1)^n$.

- Cayley-Hamilton Theorem: If $p(\lambda)$ is the characteristic polynomial of the matrix A then $p(A) = 0I$.
- In our case $(A - \lambda_1 I)^n = 0I$.

Matrices with One Eigenvalue (cont.)

$$
\begin{aligned}
e^{tA} = e^{\lambda_1 t} \cdot [I &+ t(A - \lambda_1 I) \\
&+ \frac{t^2}{2!}(A - \lambda_1 I)^2 + \cdots \\
&+ \frac{t^{n-1}}{(n-1)!}(A - \lambda_1 I)^{n-1}]
\end{aligned}
$$

Example 3 (a)

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- Distinct eigenvalues $\lambda_1 = -3$ & $\lambda_2 = -1$

Example 3 (b)

- Eigenspace for the eigenvalue $\lambda_1 = -3$ has dimension 1 $\Rightarrow$ one exponential solution

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1$$
$$= e^{-3t}\begin{pmatrix} -1/2 \\ 3/2 \\ 1 \end{pmatrix}$$

Example 3 (c)

- Eigenspace for the eigenvalue $\lambda_2 = -1$ has dimension 1 $\Rightarrow$ only one exponential solution

$$\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2$$
$$= e^{-t}\begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix}$$

- However, $\mathrm{null}((A - \lambda_2 I)^2)$ has dimension 2.

Example 3 (d)

- If $\mathbf{v} \in \mathrm{null}((A - \lambda_2 I)^2)$ then

$$\begin{aligned} e^{tA}\mathbf{v} &= e^{\lambda_2 t}[I + t(A - \lambda_2 I) \\ &\quad + \frac{t^2}{2!}(A - \lambda_2 I)^2 + \cdots]\mathbf{v} \\ &= e^{\lambda_2 t}[\mathbf{v} + t(A - \lambda_2 I)\mathbf{v} \\ &\quad + \frac{t^2}{2!}(A - \lambda_2 I)^2\mathbf{v} + \cdots] \\ &= e^{\lambda_2 t}[\mathbf{v} + t(A - \lambda_2 I)\mathbf{v}]. \end{aligned}$$

Example 3 (e)

- $\text{null}(A + I)^2$ has basis

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Third solution:

$$\mathbf{x}_3(t) = e^{tA}\mathbf{v}_3 = e^{-t}[\mathbf{v}_3 + t(A + I)\mathbf{v}_3]$$

Example 3 (f)

$$
\begin{aligned}
\mathbf{x}_3(t) &= e^{-t}[\mathbf{v}_3 + t(A+I)\mathbf{v}_3] \\
&= e^{-t}[\mathbf{v}_3 - 4t\mathbf{v}_2] \\
&= e^{-t}\begin{pmatrix} 1+2t \\ -4t \\ -4t \end{pmatrix}.
\end{aligned}
$$

Summary

- For a matrix A with one eigenvalue
  - ◇ The series for $e^{t(A-\lambda I)}$ truncates to a finite sum.
- In Example 3 the matrix had two eigenvalues.
  - ◇ The series for $e^{t(A-\lambda_2 I)}$ does not truncate.
  - ◇ The series for $e^{t(A-\lambda_2 I)}\mathbf{v}$ does truncate if $(A-\lambda_2 I)^2\mathbf{v} = \mathbf{0}$.

Generalized Eigenvectors

Definition: If λ is an eigenvalue of A and $(A-\lambda I)^p\mathbf{v}=\mathbf{0}$ for some integer $p \geq 1$, then $\mathbf{v}$ is called a generalized eigenvector associated with λ.

- The series for $e^{t(A-\lambda I)}\mathbf{v}$ truncates to a finite sum if $\mathbf{v}$ is a generalized eigenvector associated with λ.
- We can compute $e^{tA}\mathbf{v}$.

Generalized Eigenvectors

Theorem: If λ is an eigenvalue of A with algebraic multiplicity q, then there is an integer $p \leq q$ such that $\mathrm{null}((A - \lambda I)^p)$ has dimension q.

- For each generalized eigenvector $\mathbf{v}$ we can compute $e^{tA}\mathbf{v}$.
- We can find q linearly independent solutions this way.

Procedure (a)

To find q linearly independent solutions associated with an eigenvalue λ of algebraic multiplicity q.

- Find the smallest integer p such that $\mathrm{null}((A - \lambda I)^p)$ has dimension q.
- Find a basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q$ of $\mathrm{null}((A - \lambda I)^p)$.

Procedure (b)

- For $j = 1, 2, \ldots, q$

$$\begin{aligned}\mathbf{x}_j(t) &= e^{tA}\mathbf{v}_j \\ &= e^{\lambda t}[\mathbf{v}_j + t(A - \lambda I)\mathbf{v}_j \\ &\quad + \frac{t^2}{2!}(A - \lambda I)^2\mathbf{v}_j + \cdots \\ &\quad + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}\mathbf{v}_j]\end{aligned}$$

Example

Procedure (c)

If λ is complex of algebraic multiplicity q. Then $\overline{\lambda}$ also has multiplicity q.

- Find the smallest integer p such that $\text{null}((A - \lambda I)^p)$ has dimension q.
- Find a basis $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q$ of $\text{null}((A - \lambda I)^p)$.

Procedure (d)

- For $j = 1, 2, \ldots, q$

$$\begin{aligned} \mathbf{z}_j(t) = e^{\lambda t}[\mathbf{w}_j &+ t(A - \lambda I)\mathbf{w}_j \\ &+ \frac{t^2}{2!}(A - \lambda I)^2\mathbf{w}_j + \cdots \\ &+ \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}\mathbf{w}_j] \end{aligned}$$

- For $j = 1, 2, \ldots, q$ set $\mathbf{x}_j(t) = \mathrm{Re}(\mathbf{z}_j(t))$ and $\mathbf{y}_j(t) = \mathrm{Im}(\mathbf{z}_j(t))$.

Stability of Solutions

What happens to solutions as $t \to \infty$?

- $D = 1$ phase line.
- $D = 2$ phase plane.
  - Sinks, sources, saddles, and centers.
  - Project.
- $D > 2$

Theorem: Let A be an $n \times n$ real matrix.

- Suppose the real part of every eigenvalue of A is negative. Then every solution to $\mathbf{x}' = A\mathbf{x}$ tends to the equilibrium point at $\mathbf{0}$ as $t \longrightarrow \infty$.

- Suppose A has at least one eigenvalue with positive real part. Then there are solutions starting arbitrarily close to $\mathbf{0}$ and get arbitrarily large as $t \longrightarrow \infty$.

Stability

Autonomous system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ with an equilibrium point at $\mathbf{x}_0$.

- $\mathbf{x}_0$ is stable if for every $\epsilon > 0$ there is a $\delta > 0$ such that a solution $\mathbf{x}(t)$ with $|\mathbf{x}(0) - \mathbf{x}_0| < \delta$ $\Rightarrow$ $|\mathbf{x}(t) - \mathbf{x}_0| < \epsilon$ for all $t \geq 0$.

Asymptotic Stability

Autonomous system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ with an equilibrium point at $\mathbf{x}_0$.

- $\mathbf{x}_0$ is asymptotically stable if it is stable and there is an $\eta > 0$ such that if $\mathbf{x}(t)$ is a solution with $|\mathbf{x}(0) - \mathbf{x}_0| < \eta$, then $\mathbf{x}(t) \longrightarrow \mathbf{x}_0$ as $t \longrightarrow \infty$.
  - ◇ Also called a sink.

Unstable

Autonomous system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ with an equilibrium point at $\mathbf{x}_0$.

- $\mathbf{x}_0$ is unstable if there is an $\epsilon > 0$ such that for any $\delta > 0$ there is a solution $\mathbf{x}(t)$ with $|\mathbf{x}(0) - \mathbf{x}_0| < \delta$ with the property that there are values of $t > 0$ such that $|\mathbf{x}(t) - \mathbf{x}_0| > \epsilon$.

Theorem: Let A be an $n \times n$ real matrix.

- Suppose the real part of every eigenvalue of A is negative. Then $\mathbf{0}$ is an asymptotically stable equilibrium point.
- Suppose A has at least one eigenvalue with positive real part. Then $\mathbf{0}$ is an unstable equilibrium point.

Examples

- $D = 2$
  - ◇ Sources are unstable.
  - ◇ Saddles are unstable.
  - ◇ Centers are stable but not asymptotically stable.
  - ◇ $T^2 - 4D = 0.$
    - ⋆ $T < 0 \Rightarrow$ sink. $T > 0 \Rightarrow$ source.
- $D > 2$ Example

Higher Order Equations

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

- Second order: $y'' + py' + qy = 0$.
- Equivalent system: $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$

- A fundamental set of solutions for the system consists of two linearly independent solutions.

Linear Independence

Definition: Two functions $u(t)$ and $v(t)$ are linearly independent if neither is a constant multiple of the other.

- $\Leftrightarrow \begin{pmatrix} u \\ u' \end{pmatrix}$ & $\begin{pmatrix} v \\ v' \end{pmatrix}$ are linearly independent.

General Solution

Theorem: Suppose that $y_1(t)$ & $y_2(t)$ are linearly independent solutions to the equation

$$y'' + py' + qy = 0.$$

Then the general solution is

$$y(t) = C_1 y_1(t) + C_2 y_2(t).$$

Definition: A set of two linearly independent solutions is called a fundamental set of solutions.

Solutions to $y'' + py' + qy = 0$.

- Equivalent system: $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$

- Look for exponential solutions $y(t) = e^{\lambda t}$.
- Characteristic equation: $\lambda^2 + p\lambda + q = 0$.
- Characteristic polynomial: $\lambda^2 + p\lambda + q$.
- Same for the 2^{nd} order equation and the system.

Return

Real Roots

- If λ is a root to the characteristic polynomial then $y(t) = e^{\lambda t}$ is a solution.
- If λ is a root to the characteristic polynomial of multiplicity 2, then $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$ are linearly independent solutions.

Complex Roots

- If $\lambda = \alpha + i\beta$ is a complex root of the characteristic equation, then so is $\overline{\lambda} = \alpha - i\beta$.
- A complex valued fundamental set of solutions is

$$z(t) = e^{\lambda t} \quad \text{and} \quad \overline{z}(t) = e^{\overline{\lambda} t}.$$

- A real valued fundamental set of solutions is

$$x(t) = e^{\alpha t} \cos \beta t \quad \text{and} \quad y(t) = e^{\alpha t} \sin \beta t.$$

Examples

- $y'' - 5y' + 6y = 0.$
- $y'' + 25y = 0.$
- $y'' + 4y' + 13y = 0.$