Math 211

Lecture #22

November 14, 2000

Second Order Equations

$$y'' + py' + qy = 0$$

- Equivalent system: $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$

Definition: Two functions $u(t)$ and $v(t)$ are linearly independent if neither is a constant multiple of the other.

Return

General Solution

Theorem: Suppose that $y_1(t)$ & $y_2(t)$ are linearly independent solutions to the equation

$$y'' + py' + qy = 0.$$

Then the general solution is

$$y(t) = C_1 y_1(t) + C_2 y_2(t).$$

Definition: A set of two linearly independent solutions is called a fundamental set of solutions.

Return

Solutions to $y'' + py' + qy = 0$.

- Look for exponential solutions $y(t) = e^{\lambda t}$.
- Characteristic equation: $\lambda^2 + p\lambda + q = 0$.
- Characteristic polynomial: $\lambda^2 + p\lambda + q$.
- Compare

$$\begin{aligned} y'' + py' + qy &= 0 \quad \text{ODE} \\ \lambda^2 + p\lambda + q &= 0 \quad \text{ch. poly.} \end{aligned}$$

Real Roots

- If λ is a root to the characteristic polynomial then $y(t) = e^{\lambda t}$ is a solution.
- If λ is a root to the characteristic polynomial of multiplicity 2, then $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$ are linearly independent solutions.

Complex Roots

- If $\lambda = \alpha + i\beta$ is a complex root of the characteristic equation, then so is $\overline{\lambda} = \alpha - i\beta$.
- A complex valued fundamental set of solutions is $z(t) = e^{\lambda t}$ and $\overline{z}(t) = e^{\overline{\lambda} t}$.
- A real valued fundamental set of solutions is

$$x(t) = e^{\alpha t} \cos \beta t \quad \text{and} \quad y(t) = e^{\alpha t} \sin \beta t.$$

Examples

- $y'' - 5y' + 6y = 0.$
- $y'' + 25y = 0.$
- $y'' + 4y' + 13y = 0.$

The Vibrating Spring

Newton's second law: $ma =$ total force.

- Forces acting:
  - ◇ Gravity mg.
  - ◇ Restoring force $R(x)$.
  - ◇ Damping force $D(v)$.
  - ◇ External force $F(t)$.

The Vibrating Spring (2)

$$ma = mg + R(x) + D(v) + F(t)$$

- Hooke's law: $R(x) = -kx$. $k > 0$ is the spring constant.
- Spring-mass equilibrium $x_0 = mg/k$.
- Set $y = x - x_0$. Equation becomes

$$my'' = -ky + D(y') + F(t).$$

The Vibrating Spring (3)

- Damping force $D(y') = -\mu y'$.
- Equation becomes

$$my'' = -ky - \mu y' + F(t), \quad \text{or}$$
$$my'' + \mu y' + ky = F(t), \quad \text{or}$$
$$y'' + \frac{\mu}{m}y' + \frac{k}{m}y = \frac{1}{m}F(t).$$

RLC Circuit

$$LI'' + RI' + \frac{1}{C}I = E'(t), \quad \text{or}$$
$$I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t).$$

Harmonic Motion (1)

- Spring: $y'' + \frac{\mu}{m}y' + \frac{k}{m}y = \frac{1}{m}F(t)$.
- Circuit: $I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t)$.
- Essentially the same equation. Use

$$x'' + 2cx' + \omega_0^2 x = f(t).$$

- The equation for harmonic motion.

Harmonic Motion (2)

$$x'' + 2cx' + \omega_0^2 x = f(t).$$

- ω_0 is the natural frequency.
  - ◇ Spring: $\omega_0 = \sqrt{k/m}$.
  - ◇ Circuit: $\omega_0 = \sqrt{1/LC}$.
- c is the damping constant.
- $f(t)$ is the forcing term.

Simple Harmonic Motion

No forcing, and no damping.

$$x'' + \omega_0^2 x = 0$$

- $p(\lambda) = \lambda^2 + \omega_0^2$, $\lambda = \pm i\omega_0$.

  Fundamental set of solutions

$$x_1(t) = \cos\omega_0 t \quad \& \quad x_2(t) = \sin\omega_0 t.$$

Simple Harmonic Motion (2)

General solution

$$x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t.$$

- Every solution is periodic with period ω_0.
  - ◇ ω_0 is the natural frequency.
  - ◇ The period is $T = 2\pi/\omega_0$.