Math 211

Lecture #24

November 21, 2000

Forced Harmonic Motion

Assume an oscillatory forcing term:

$$y'' + 2cy' + \omega_0^2 y = A \cos \omega t$$

- A is the forcing amplitude
- ω is the forcing frequency

Forced Undamped Motion

$$y'' + \omega_0^2 y = A \cos \omega t$$

Homogeneous equation

$$y'' + \omega_0^2 y = 0$$

General solution

$$y(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t.$$

If $\omega = \omega_0$ we have an exceptional case.

$\omega \neq \omega_0$

$$y'' + \omega_0^2 y = A \cos \omega t$$

Look for a particular solution of the form

$$x_p(t) = a \cos \omega t + b \sin \omega t.$$

We find

$$x_p(t) = \frac{A}{\omega_0^2 - \omega^2} \cos \omega t.$$

General solution

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{A}{\omega_0^2 - \omega^2} \cos \omega t.$$

Initial conditions $x(0) = x'(0) = 0 \;\Rightarrow$

$$x(t) = \frac{A}{\omega_0^2 - \omega^2} [\cos \omega t - \cos \omega_0 t].$$

Example: $\omega_0 = 9$, $\omega = 8$, $A = \omega_0^2 - \omega^2 = 17$.

$$x(t) = \cos 9t - \cos 8t.$$

Set

$$\overline{\omega} = \frac{\omega_0 + \omega}{2} \quad \text{and} \quad \delta = \frac{\omega_0 - \omega}{2}.$$

Then

$$\omega = \overline{\omega} - \delta \quad \text{and} \quad \omega_0 = \overline{\omega} + \delta,$$

and

$$\begin{aligned} x(t) &= \frac{A}{\omega_0^2 - \omega^2}[\cos \omega t - \cos \omega_0 t] \\ &= \frac{A \sin \delta t}{2\overline{\omega}\delta} \sin \overline{\omega} t. \end{aligned}$$

$$x(t) = \frac{A \sin \delta t}{2\overline{\omega}\delta} \sin \overline{\omega} t$$

Example:

$$\overline{\omega} = 8.5 \quad \text{and} \quad \delta = 0.5.$$

- Envelope: Slow oscillation with frequency δ.
- Fast oscillation with frequency $\overline{\omega}$ and varying amplitude.
- Beats.

$\omega = \omega_0$

$$y'' + \omega_0^2 y = A\cos\omega_0 t$$

We have an exceptional case. Try

$$x_p(t) = t[a\cos\omega t + b\sin\omega t].$$

We find

$$x_p(t) = \frac{A}{2\omega_0} t\sin\omega_0 t.$$

General solution

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{A}{2\omega_0} t \sin \omega_0 t.$$

Initial conditions $x(0) = x'(0) = 0 \;\Rightarrow$

$$x(t) = \frac{A}{2\omega_0} t \sin \omega_0 t.$$

Example: $\omega_0 = 5$, and $A = 2\omega_0 = 10$.

$$x(t) = t \sin 5t.$$

Solution

$$x(t) = \frac{A}{2\omega_0} t \sin \omega_0 t.$$

- Oscillation with increasing amplitude.
- First example of resonance.
  - Driving at the natural frequency can cause oscillations that grow out of control.

Forced, Damped Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = A\cos\omega t$$

- Assume the underdamped case, where $c < \omega_0$.

Homogeneous equation

$$x'' + 2cx' + \omega_0^2 x = 0$$

- Charateristic polynomial:

$$P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$$

- Roots $\lambda = -c \pm \sqrt{c^2 - \omega_0^2} = -c \pm i\eta$ where $\eta = \sqrt{\omega_0^2 - c^2}$.
- Fundamental set of solutions $x_1(t) = e^{-ct} \cos \eta t$ and $x_2(t) = e^{-ct} \sin \eta t$

Inhomogeneous equation

$$x'' + 2cx' + \omega_0^2 x = A\cos\omega t$$

- Use the complex method to find a particular solution. Solve

$$z'' + 2cz' + \omega_0^2 z = Ae^{i\omega t}.$$

Try $z(t) = ae^{i\omega t}$.

$$\begin{aligned} z'' + 2cz' + \omega_0^2 z &= [(i\omega)^2 + 2c(i\omega) + \omega_0^2]ae^{i\omega t} \\ &= P(i\omega)z \end{aligned}.$$

Particular Solution

- Solution if $P(i\omega)z = Ae^{i\omega t}$.
- Solution is

$$z(t) = \frac{1}{P(i\omega)} Ae^{i\omega t}$$
$$= H(i\omega) Ae^{i\omega t}.$$

- $H(i\omega) = \dfrac{1}{P(i\omega)}$ is called the transfer function.

$$
\begin{aligned}
P(i\omega) &= (i\omega)^2 + 2c(i\omega) + \omega_0^2 \\
&= [\omega_0^2 - \omega^2] + 2ic\omega \\
&= Re^{i\phi} \\
&= R\cos\phi + iR\sin\phi.
\end{aligned}
$$

We need $R\cos\phi = \omega_0^2 - \omega^2$ and $R\sin\phi = 2c\omega$. Thus

$$
R = \sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}
$$

$$
\phi = \operatorname{arccot}\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right).
$$

Transfer Function

$$H(i\omega) = \frac{1}{P(i\omega)}$$
$$= \frac{1}{R}e^{-i\phi}$$
$$= G(\omega)e^{-i\phi}.$$

- $G(\omega) = \frac{1}{R}$ is called the gain.

$$G(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}.$$

- ϕ is the phase shift.

Particular Solution

The complex particular solution is

$$\begin{aligned} z(t) &= H(i\omega)Ae^{i\omega t} \\ &= G(\omega)e^{-i\phi} \cdot Ae^{i\omega t} \\ &= G(\omega)Ae^{i(\omega t-\phi)}. \end{aligned}$$

The real particular solution is

$$\begin{aligned} x_p(t) &= \mathrm{Re}(z(t)) \\ &= G(\omega)A\cos(\omega t - \phi). \end{aligned}$$

General Solution

$$
\begin{aligned}
x(t) &= x_p(t) + x_h(t) \\
&= G(\omega) A \cos(\omega t - \phi) \\
&\qquad + e^{-ct}[C_1 \cos \eta t + C_2 \sin \eta t].
\end{aligned}
$$

- Transient term.
  - ◇ $x_h(t) = e^{-ct}[C_1 \cos \eta t + C_2 \sin \eta t].$
- Steady-state solution.
  - ◇ $x_p(t) = G(\omega) A \cos(\omega t - \phi).$

Example

$\omega_0 = 1$, $c = 0.8$, $A = 1$, and $\omega = 3$.

$x(0) = 0$ and $x'(0) = 1$.

Steady-State Solution

$$x_p(t) = G(\omega) A \cos(\omega t - \phi).$$

- The forcing function is $A \cos \omega t$.
- The amplitude of the steady-state response is $G(\omega)$ times the amplitude of the forcing term.
- The steady-state response is oscillatory, but with a phase which is shifted by ϕ/ω.

Gain

$$G(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}$$

Set $\omega = s\omega_0$ and $c = D\omega_0/2$. Then

$$\omega_0^2 G(\omega) = \frac{1}{\sqrt{(1 - s^2)^2 + D^2 s^2}}$$

where

$$s = \frac{\omega}{\omega_0} \quad \text{and} \quad D = \frac{2c}{\omega_0}.$$