Math 211

Lecture #26

November 30, 2000

Linearization of a Planar System

$$x' = f(x, y)$$

$$y' = g(x, y)$$

Jacobian matrix at the equilibrium point (x_0, y_0)

$$J = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

The linearization is $\mathbf{u}' = J\mathbf{u}$.

Theorem: Consider the planar system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

where f and g are continuously differentiable. Suppose that (x_0, y_0) is an equilibrium point. If the linearization at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point at (x_0, y_0) is of the same type.

Higher Dimensional Systems

Autonomous equation $\mathbf{y}' = \mathbf{f}(\mathbf{y})$

- $\mathbf{y} = (y_1, y_2, \cdots, y_n)^T$
- $\mathbf{f}(\mathbf{y}) = (f_1(\mathbf{y}), f_2(\mathbf{y}), \cdots, f_n(\mathbf{y}))^T$
- J is the Jacobian matrix

$$\mathbf{f}(\mathbf{y}_0 + \mathbf{u}) = J(\mathbf{y}_0)\mathbf{u} + \mathbf{R}(\mathbf{u})$$

$$\text{where} \quad \lim_{\mathbf{u}\to\mathbf{0}} \frac{\mathbf{R}(\mathbf{u})}{|\mathbf{u}|} = \mathbf{0}.$$

Set $\mathbf{y} = \mathbf{y}_0 + \mathbf{u}$. The system becomes

$$\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u} + \mathbf{R}(\mathbf{u}).$$

The linearization is

$$\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u}.$$

Theorem: Suppose that $\mathbf{y}_0$ is an equilibrium point for $\mathbf{y}' = \mathbf{f}(\mathbf{y})$. Let J be the Jacobian of $\mathbf{f}$ at $\mathbf{y}_0$.

1. Suppose that the real part of every eigenvalue of J is negative. Then $\mathbf{y}_0$ is an asymptotically stable equilibrium point.
2. Suppose that J has at least one eigenvalue with positive real part. Then $\mathbf{y}_0$ is an unstable equilibrium point.

Example

$$x' = -2x - 4y + 2xy$$

$$y' = x - 6y + x^2 - y^2$$

- One eigenvalue $\lambda = -4$ of algebraic multiplicity 2.
- First theorem does not apply.
- Second theorem does apply. The origin is a sink.

The Lorenz System

$$x' = -ax + ay$$

$$y' = rx - y - xz$$

$$z' = -bz + xy$$

Equilibrium points.

- $(r \leq 1)$ $(0, 0, 0)$
- $(r > 1)$ Set $s = \sqrt{b(r-1)}$.

  $(0, 0, 0)$, $\mathbf{c}^{+} = (s, s, r{-}1)$ & $\mathbf{c}^{-} = (-s, -s, r{-}1)$

$$J = \begin{pmatrix} -a & a & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}$$

- Use $a = 10$ and $b = 8/3$.
- $(r < 1)$ $(0, 0, 0)$ is asymptotically stable.
- $(r > 1)$ $(0, 0, 0)$ is unstable.
- For $1 < r < 470/19 = 24.74$, $\mathbf{c}^+$ and $\mathbf{c}^-$ are asymptotically stable.
- For $r > 470/19 = 24.74$, $\mathbf{c}^+$ and $\mathbf{c}^-$ are unstable.

As r varies the Lorenz system displays a wide variety of behaviors.

- For $r = 28$ we have Lorenz's strange attractor.
- For $r = 100$ there is a periodic attractor.
- For $r = 200$ there is another strange attractor.

Invariant Sets

Definition: A set S is (positively) invariant for the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ if $\mathbf{y}(0) = \mathbf{y}_0 \in S$ implies that $\mathbf{y}(t) \in S$ for all $t \geq 0$.

- Examples:
  - An equilibrium point.
  - Any solution curve.

Competing Species

$$x' = (5 - 2x - y)x$$
$$y' = (7 - 2x - 3y)y$$

- The positive x- and y-axes are invariant.
- The positive quadrant is invariant.
  - Populations should remain nonnegative.
- The set $S = \{(x, y) \,|\, 0 < x < 3,\ 0 < y < 3\}$ is positively invariant.

Nullclines

$$x' = f(x, y)$$

$$y' = g(x, y)$$

Definition: The x-nullcline is the set defined by $f(x, y) = 0$. The y-nullcline is the set defined by $g(x, y) = 0$.

- Along the x-nullcline the vector field points up or down.
- Along the y-nullcline the vector field points left or right.

Competing Species

$$x' = (5 - 2x - y)x$$

$$y' = (7 - 2x - 3y)y$$

- The x-nullcline consists of the two lines $x = 0$ and $2x + y = 5$.
- The y-nullcline consists of the two lines $y = 0$ and $2x + 3y = 7$.
- The nullclines intersect at the equilibrium points.

- Two of the four regions in the positive quadrant defined by the nullclines are positively invariant.
- This information allows us to predict that all solutions in the positive quadrant $\longrightarrow (2, 1)$ as $t \longrightarrow \infty$.

Competing - Species 2^{nd} Example

$$x' = (1 - x - y)x$$
$$y' = (4 - 7x - 3y)y$$

- The equilibrium point at $(1/4, 3/4)$ is a saddle point.
- All solutions go to either $(0, 4/3)$ or $(1, 0)$.

Return

Definition: The basin of attraction of a sink $\mathbf{y}_0$ consists of all points $\mathbf{y}$ such that the solution staring at $\mathbf{y}$ approaches $\mathbf{y}_0$ as $t \to \infty$.

- In the example, the basins of attraction of the two sinks are separated by the stable orbits of the saddle point.
- The stable and unstable orbits of a saddle point are called separatrices. (Separatrices is the plural of separatrix.)