

Math 211

Lecture #27

December 5, 2000

Review of Methods

Linearization at an equilibrium point

- $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ has an equilibrium point at \mathbf{y}_0 .
- The linearization $\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u}$ has an equilibrium point at $\mathbf{u} = \mathbf{0}$.
- The linearization can sometimes predict the behavior of solutions to the nonlinear system near the equilibrium point.
- The linearization gives only local information.

Theorem: Consider the planar system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

where f and g are continuously differentiable.

Suppose that (x_0, y_0) is an equilibrium point. If the linearization at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point at (x_0, y_0) is of the same type.

Theorem: Suppose that \mathbf{y}_0 is an equilibrium point for $\mathbf{y}' = \mathbf{f}(\mathbf{y})$. Let J be the Jacobian of \mathbf{f} at \mathbf{y}_0 .

1. Suppose that the real part of every eigenvalue of J is negative. Then \mathbf{y}_0 is an asymptotically stable equilibrium point.
2. Suppose that J has at least one eigenvalue with positive real part. Then \mathbf{y}_0 is an unstable equilibrium point.

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Invariant Sets

Definition: A set S is (positively) invariant for the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ if $\mathbf{y}(0) = \mathbf{y}_0 \in S$ implies that $\mathbf{y}(t) \in S$ for all $t \geq 0$.

- Examples:
 - ◊ An equilibrium point.
 - ◊ Any solution curve.

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Nullclines

$$x' = f(x, y)$$

$$y' = g(x, y)$$

Definition: The x -nullcline is the set defined by $f(x, y) = 0$. The y -nullcline is the set defined by $g(x, y) = 0$.

- Along the x -nullcline the vector field points up or down.
- Along the y -nullcline the vector field points left or right.

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Competing Species – 2nd Example

$$x' = (1 - x - y)x$$

$$y' = (4 - 7x - 3y)y$$

- The axes are invariant. The positive quadrant is invariant.
- The equilibrium point at $(1/4, 3/4)$ is a saddle point.

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- Almost all solutions go to one of the nodal sinks $(0, 4/3)$ or $(1, 0)$.
Definition: The basin of attraction of a sink y_0 consists of all points y such that the solution starting at y approaches y_0 as $t \rightarrow \infty$.
- In the example, the basins of attraction of the two sinks are separated by the stable orbits of the saddle point.
- The stable and unstable orbits of a saddle point are called separatrices.

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Summary

- Sometimes the understanding of invariant sets can help us understand the long term behavior of all solutions.
- Nullclines can sometimes help us find informative invariant sets.
- None of this helps us understand the predator-prey system.

Limit Sets

Definition: The (forward) limit set of the solution $\mathbf{y}(t)$ that starts at \mathbf{y}_0 is the set of all limit points of the solution curve. It is denoted by $\omega(\mathbf{y}_0)$.

- $\mathbf{x} \in \omega(\mathbf{y}_0)$ if there is a sequence $t_k \rightarrow \infty$ such that $\mathbf{y}(t_k) \rightarrow \mathbf{x}$.
- What kinds of sets can be limit sets?
 - ◊ Equilibrium points.
 - ◊ Periodic orbits.

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Properties of Limit Sets

Theorem: Suppose that the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ is defined in the set U .

1. If the solution curve starting at \mathbf{y}_0 stays in a bounded subset of U , then the limit set $\omega(\mathbf{y}_0)$ is not empty.
2. Any limit set is both positively and negatively invariant.

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Example

$$x' = -y + x(1 - x^2 - y^2)$$

$$y' = x + y(1 - x^2 - y^2)$$

- In polar coordinates this is

$$r' = r(1 - r^2)$$

$$\theta' = 1$$

- Solution curves approach the unit circle.

Definition

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Limit Cycle

Definition: A limit cycle is a closed solution curve which is the limit set of nearby solution curves. If the solution curves spiral into the limit cycle as $t \rightarrow \infty$, it is an attracting limit cycle. If they spiral into the limit cycle as $t \rightarrow -\infty$, it is a repelling limit cycle.

- In the example the unit circle is a limit cycle.

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Types of Limit Set

- A limit cycle is a new type of phenomenon.
- However, the limit set is a periodic orbit, so the type of limit set is not new.
- We still have only two types.

Example

$$x' = (y + x/5)(1 - x^2)$$

$$y' = -x(1 - y^2)$$

- The limit set of any solution that starts in the unit square is the boundary of the unit square.

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Planar Graph

Definition: A planar graph is a collection of points, called vertices, and non-intersecting curves, called edges, which connect the vertices. If the edges each have a direction the graph is said to be directed.

- The boundary of the unit square in the example is a directed planar graph.

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Theorem: If S is a limit set of a solution of a planar system defined in a set $U \subset \mathbf{R}^2$, then S is one of the following:

- An equilibrium point
- A closed solution curve
- A directed planar graph with vertices that are equilibrium points, and edges which are solution curves.

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Remarks

- These are the only possibilities.
- The closed solution curve could be a limit cycle.
- If a vertex of a limiting planar graph is a generic equilibrium point, then it must be a saddle point. The edges connecting this point must be separatrices.

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Poincaré-Bendixson Theorem

Theorem: Suppose that R is a closed and bounded planar region that is positively invariant for a planar system. If R contains no equilibrium points, then there is a closed solution curve in R .

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