Math 211

Lecture #27

December 5, 2000

Review of Methods

Linearization at an equilibrium point

- $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ has an equilibrium point at $\mathbf{y}_0$.
- The linearization $\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u}$ has an equilibrium point at $\mathbf{u} = \mathbf{0}$.
- The linearization can sometimes predict the behavior of solutions to the nonlinear system near the equilibrium point.
- The linearization gives only local information.

Theorem: Consider the planar system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

where f and g are continuously differentiable. Suppose that (x_0, y_0) is an equilibrium point. If the linearization at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point at (x_0, y_0) is of the same type.

Theorem: Suppose that $\mathbf{y}_0$ is an equilibrium point for $\mathbf{y}' = \mathbf{f}(\mathbf{y})$. Let J be the Jacobian of $\mathbf{f}$ at $\mathbf{y}_0$.

1. Suppose that the real part of every eigenvalue of J is negative. Then $\mathbf{y}_0$ is an asymptotically stable equilibrium point.

2. Suppose that J has at least one eigenvalue with positive real part. Then $\mathbf{y}_0$ is an unstable equilibrium point.

Invariant Sets

Definition: A set S is (positively) invariant for the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ if $\mathbf{y}(0) = \mathbf{y}_0 \in S$ implies that $\mathbf{y}(t) \in S$ for all $t \geq 0$.

- Examples:
  - An equilibrium point.
  - Any solution curve.

Nullclines

$$x' = f(x, y)$$

$$y' = g(x, y)$$

Definition: The x-nullcline is the set defined by $f(x, y) = 0$. The y-nullcline is the set defined by $g(x, y) = 0$.

- Along the x-nullcline the vector field points up or down.
- Along the y-nullcline the vector field points left or right.

Competing Species – 2^{nd} Example

$$x' = (1 - x - y)x$$
$$y' = (4 - 7x - 3y)y$$

- The axes are invariant. The positive quadrant is invariant.
- The equilibrium point at $(1/4, 3/4)$ is a saddle point.

- Almost all solutions go to one of the nodal sinks $(0, 4/3)$ or $(1, 0)$.
  Definition: The basin of attraction of a sink $\mathbf{y}_0$ consists of all points $\mathbf{y}$ such that the solution starting at $\mathbf{y}$ approaches $\mathbf{y}_0$ as $t \to \infty$.
- In the example, the basins of attraction of the two sinks are separated by the stable orbits of the saddle point.
- The stable and unstable orbits of a saddle point are called separatrices.

Summary

- Sometimes the understanding of invariant sets can help us understand the long term behavior of all solutions.
- Nullclines can sometimes help us find informative invariant sets.
- Non of this helps us understand the predator-prey system.

Limit Sets

Definition: The (forward) limit set of the solution $\mathbf{y}(t)$ that starts at $\mathbf{y}_0$ is the set of all limit points of the solution curve. It is denoted by $\omega(\mathbf{y}_0)$.

- $\mathbf{x} \in \omega(\mathbf{y}_0)$ if there is a sequence $t_k \to \infty$ such that $\mathbf{y}(t_k) \to \mathbf{x}$.
- What kinds of sets can be limit sets?
  - ◇ Equilibrium points.
  - ◇ Periodic orbits.

Properties of Limit Sets

Theorem: Suppose that the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ is defined in the set U.

1. If the solution curve starting at $\mathbf{y}_0$ stays in a bounded subset of U, then the limit set $\omega(\mathbf{y}_0)$ is not empty.

2. Any limit set is both positively and negatively invariant.

Example

$$x' = -y + x(1 - x^2 - y^2)$$

$$y' = x + y(1 - x^2 - y^2)$$

- In polar coordinates this is

$$r' = r(1 - r^2)$$

$$\theta' = 1$$

- Solution curves approach the unit circle.

Limit Cycle

Definition: A limit cycle is a closed solution curve which is the limit set of nearby solution curves. If the solution curves spiral into the limit cycle as $t \longrightarrow \infty$, it is a attracting limit cycle. If they spiral into the limit cycle as $t \longrightarrow -\infty$, it is a repelling limit cycle.

- In the example the unit circle is a limit cycle.

Types of Limit Set

- A limit cycle is a new type of phenomenon.
- However, the limit set is a periodic orbit, so the type of limit set is not new.
- We still have only two types.

Example

$$x' = (y + x/5)(1 - x^2)$$
$$y' = -x(1 - y^2)$$

- The limit set of any solution that starts in the unit square is the boundary of the unit square.

Planar Graph

Definition: A planar graph is a collection of points, called vertices, and non-intersecting curves, called edges, which connect the vertices. If the edges each have a direction the graph is said to be directed.

- The boundary of the unit square in the example is a directed planar graph.

Theorem: If S is a limit set of a solution of a planar system defined in a set $U \subset \mathbf{R}^2$, then S is one of the following:

- An equilibrium point
- A closed solution curve
- A directed planar graph with vertices that are equilibrium points, and edges which are solution curves.

Remarks

- These are the only possibilities.
- The closed solution curve could be a limit cycle.
- If a vertex of a limiting planar graph is a generic equilibrium point, then it must be a saddle point. The edges connecting this point must be separatrices.

Poincaré-Bendixson Theorem

Theorem: Suppose that R is a closed and bounded planar region that is positively invariant for a planar system. If R contains no equilibrium points, then there is a closed solution curve in R.