Math 211

Lecture #28

December 7, 2000

Predator-Prey

Lotka-Volterra system

$$x' = (a - by)x \quad \text{(prey – fish)}$$
$$y' = (-c + dx)y \quad \text{(predator – sharks)}$$

- Equilbrium points
  - ◇ $(0, 0)$ is a saddle.
  - ◇ $(x_0, y_0) = (c/d, a/b)$ is a linear center.

Return

- The axes are invariant.
- The positive quadrant is invariant.
- The solution curves appear to be closed. Is this actually true?

Along the solution curve $y = y(x)$ we have

$$\frac{dy}{dx} = \frac{y(-c + dx)}{x(a - by)}.$$

The solution is

$$H(x, y) = by - a \ln y + dx - c \ln x = C$$

- This is an implicit equation for the solution curve. $\Rightarrow$ All solution curves are closed, and represent periodic solutions.

Why Fishing Leads to More Fish

Compute the average of the fish & shark populations.

$$\frac{d}{dt} \ln x(t) = \frac{x'}{x} = a - by$$

$$0 = \frac{1}{T} \int_0^T \frac{d}{dt} \ln x(t)\, dt = a - b\overline{y}.$$

So $\overline{y} = a/b = y_0$. Similarly $\overline{x} = x_0 = c/d$.

The effect of fishing that does not distinquish between fish and sharks is the system

$$x' = (a - by)x - ex$$
$$y' = (-c + dx)y - ey$$

This is the same system with a replaced by $a - e$ and c replaced by $c + e$.

The average populations are

$$\overline{x}_1 = \frac{c+e}{d} \quad \text{and} \quad \overline{y}_1 = \frac{a-e}{b}$$

Fishing causes the average fish population to increase and the average shark population to decrease.

Averages

Cottony Cushion Scale Insect & the Ladybird Beetle

- Cottony cushion scale insect accidentally introduced from Australia in 1868.
  - ◇ Threatened the citrus industry.
- Ladybird beetle imported from Australia
  - ◇ Natural predator – reduced the insects to manageable low.

Return

DDT kills the scale insect.

- Massive spraying ordered.
  - ◇ Despite the warnings of mathematicians and biologists.
- The scale insect increased in numbers, as predicted by Volterra.

Model of the Immune System in Action

- How does the immune system develops immunity to virus caused diseases?
  - ◇ Diseases such as flu, the cold, mumps, . . .
- *Infectious Diseases of Humans* - Roy M. Anderson & Robert M. May, Oxford University Press 1992

The model includes the interactions between virus cells and lymphocytes generated by the immune system.

- $V(t)$ = number of virus cells
- Two types of lymphocytes, $E_1(t)$ & $E_2(t)$.

Lymphocytes

Both types:

- Are recruited from bone marrow at a constant rate
- Die at a rate proportional to their numbers
- Proliferate due to contact with each other

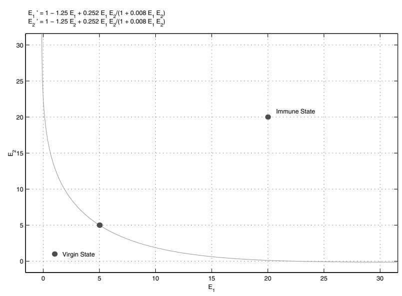
Return

Model With No Virus Present

$$E_1' = \Lambda_1 - \mu_1 E_1 + a_1 \frac{E_1 E_2}{1 + b_1 E_1 E_2}$$
$$E_2' = \Lambda_2 - \mu_2 E_2 + a_2 \frac{E_1 E_2}{1 + b_2 E_1 E_2}$$

- For `pplane5` use parameters $\Lambda_1 = \Lambda_1 = 1$, $\mu_1 = \mu_1 = 1.25$, $a_1 = a_2 = 0.252$, and $b_1 = b_2 = 0.008$.

Dynamics of the Lymphocytes



Interactions with the Virus

- Virus cells have an intrinsic growth rate r.
- Lymphocytes of type E_1:
  - ◇ kill virus because of contacts with them
  - ◇ proliferate because of contacts with virus
- Lymphocytes of type E_2:
  - ◇ do not directly interact with the virus
  - ◇ regulate cells of type E_1

Model With Virus Present

$$E_1' = \Lambda_1 - \mu_1 E_1 + a_1 \frac{E_1 E_2}{1 + b_1 E_1 E_2} + KVE_1$$
$$E_2' = \Lambda_2 - \mu_2 E_2 + a_2 \frac{E_1 E_2}{1 + b_2 E_1 E_2}$$
$$V' = rV - kVE_1$$

- For ode45 use $K = 0.5$, $k = 0.01$ and $r = 0.1$.

Equilibrium Points

- There are three realistic equilibrium points

$$\begin{pmatrix} E_1 \\ E_2 \\ V \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 5 \\ 0 \end{pmatrix}, \quad \& \quad \begin{pmatrix} 20 \\ 20 \\ 0 \end{pmatrix}$$

- The first two are unstable. The third is asymptotically stable.
- What is the global behavior? The best we can do is to check with `ode45`.