Math 211

Lecture #2

Autonomous Equations

General equation:

$$\frac{dy}{dt} = f(t, y) \qquad \frac{dy}{dt} = t - y^2$$

Autonomous equation:

$$\frac{dy}{dt} = f(y) \qquad \frac{dy}{dt} = y(1 - y)$$

In an autonomous equation the right hand side has no explicit dependence on the independent variable.

Equilibrium Points

Autonomous equation:

$$\frac{dy}{dt} = f(y) \qquad \frac{dy}{dt} = y(1-y)$$

Equilibrium point:

$$f(y_0) = 0 \quad y_0 = 0 \quad \text{or} \quad 1$$

Equilibrium solution:

$$y(t) = y_0 \quad y(t) = 0 \quad \text{and} \quad y(t) = 1$$

Between Equilibrium Points

$$\frac{dy}{dt} = f(y) > 0 \Rightarrow \; y(t) \text{ is increasing.}$$

$$\frac{dy}{dt} = f(y) < 0 \Rightarrow \; y(t) \text{ is decreasing.}$$

Separable Equations

General equation:

$$\frac{dy}{dt} = f(t, y) \qquad \frac{dy}{dt} = t - y^2$$

Separable equation:

$$\frac{dy}{dt} = g(y)h(t) \qquad \frac{dy}{dt} = t \sec y$$

In a separable equation the right hand side is a product of a function of the independent variable (t) and the unknown function (y).

Solving Separable Equations

$$\frac{dy}{dt} = t \sec y$$

Separate the variables:

$$\frac{dy}{\sec y} = t\,dt \quad \text{or} \quad \cos y\,dy = t\,dt$$

We have to worry about dividing by 0, but $\sec y$ is never equal to 0.

Integrate both sides:

$$\int \cos y \, dy = \int t \, dt$$

$$-\sin(y) + C_1 = \frac{1}{2}t^2 + C_2 \quad \text{or}$$

$$\sin(y) = -\frac{1}{2}t^2 + C$$

where $C = C_1 - C_2$.

$$\sin(y) = -\frac{1}{2}t^2 + C$$

Solve for y.

$$y(t) = \arcsin\left(C - \frac{1}{2}t^2\right).$$

This is the general solution to $\frac{dy}{dt} = t\sec y$.

Solving Separable Equations

The three step process to solve

$$\frac{dy}{dt} = g(y)h(t)$$

- Separate the variables. $\dfrac{dy}{g(y)} = h(t)\,dt$
- Integrate both sides. $\displaystyle\int \frac{dy}{g(y)} = \int h(t)\,dt$
- Solve for y.

Examples

- $y' = ry$
- $R' = \dfrac{\sin t}{1+R}$ with $R(0) = 1, -2, -1$
- $x' = \dfrac{3t^2 x}{1+2x^2}$ with $x(0) = 1, 0$
- $y' = 1 + y^2$ with $y(0) = -1, 0, 1$

Why It Works

$$\frac{dy}{dt} = g(y)h(t)$$

$$\frac{1}{g(y)}\frac{dy}{dt} = h(t) \text{ if } g(y) \neq 0$$

$$\int \frac{1}{g(y)}\frac{dy}{dt}\,dt = \int h(t)\,dt$$

$$\int \frac{1}{g(y)}\,dy = \int h(t)\,dt$$