Math 211

Lecture #3

September 5, 2000

Models of Motion

History of models of planetary motion

- Babylonians - 3000 years ago
    - Initiated the systematic study of astronomy.

Greeks

- Descriptive model
  - Geocentric model
  - Epicycles
- Enabled predictions
- No causal explanation

Nicholas Copernicus (1543)

- Shifted the center of the universe to the sun.
- Less epicycles required.
- Still descriptive and not causal.
- Major change in human understanding of their place in the universe.

Johann Kepler (1609)

- Based on experimental work of Tycho Brahe.
- Ellipses instead of epicycles.
  - Sun at a focus of the ellipse.
- Three laws of planetary motion.
- Still descriptive and not causal.

Isaac Newton

- Three major contributions.
  - Fundamental theorem of calculus.
    - Invention of calculus.
  - Laws of mechanics.
    - Second law — $F = ma$.
  - Universal law of gravity.
  - *Principia Mathematica* 1687

Isaac Newton

- Laws of mechanics and gravitation were based on his own experiments and his understanding of the experiments of others.
- Derived Kepler's three laws of planetary motion.
- Causal explanation.
    - For any mechanical motion.

Isaac Newton

- Problems
  - Force of gravity was action at a distance.
  - Physical anomalies.
- *The Life of Isaac Newton* by Richard Westfall, Cambridge University Press 1993.

Albert Einstein

- Special theory of relativity – 1905.
- General theory of relativity – 1916.
  - Gravity is due to curvature of space-time.
  - Curvature is caused by mass.
  - Explains action at a distance.
- All known anomalies explained.

Unified Theories

- Four fundamental forces.
  - Gravity, electromagnetism, strong nuclear, and weak nuclear.
- Last three unified by quantum mechanics.
  - Quantum chromodynamics.
- Attempts to include gravity.
  - String theory.

Unified Theories

- String theory.
  - ◇ *The elegant universe : superstrings, hidden dimensions, and the quest for the ultimate theory* by Brian Greene, W.W.Norton, New York 1999.

Linear Motion

- Motion in one dimension
  - ◇ Example – motion of a ball in the earth's gravity.
- $x(t)$ is the distance from a reference position.
  - ◇ $x(t)$ is the height of the ball above the surface of the earth.
- Velocity: $v = x'$
- Acceleration: $a = v' = x''$.

Force of gravity is (approximately) constant near the surface of the earth

$$F = -mg \quad g = 9.8m/s^2$$

Newton's second law

$$F = ma$$

Equation of motion

$$ma = -mg$$

$$x'' = -g \quad \text{or} \quad \begin{aligned} x' &= v, \\ v' &= -g. \end{aligned}$$

Solving the system

$$x' = v,$$

$$v' = -g$$

$$v(t) = -gt + c_1$$

$$x(t) = -\frac{1}{2}gt^2 + c_1 t + c_2.$$

Air Resistance

Force of resistance

$$R(x, v) = -r(x, v)v \quad \text{where} \quad r(x, v) \geq 0.$$

Resistance proportional to velocity.

$$R(x, v) = -rv.$$

Resistance proportional to the square of the velocity.

$$R(x, v) = -k|v|v.$$

$R(x, v) = -rv$

Total force

$$F = -mg - rv$$

Equation of motion

$$mx'' = -mg - rv \quad \text{or} \quad \begin{aligned} x' &= v, \\ v' &= -\frac{mg + rv}{m}. \end{aligned}$$

The equation for v is separable.

$$v(t) = Ce^{-rt/m} - \frac{mg}{r}.$$

$$v(t) = Ce^{-rt/m} - \frac{mg}{r}.$$

$$\lim_{t \to \infty} v(t) = -\frac{mg}{r}.$$

The *terminal velocity* is $v_{\text{term}} = -\frac{mg}{r}.$

$R(x, v) = -k|v|v$

Total force is $F = -mg - k|v|v$. Equation of motion is

$$mx'' = -mg - k|v|v \quad \text{or} \quad \begin{aligned} x' &= v, \\ v' &= -g - \frac{k|v|v}{m}. \end{aligned}$$

The equation for v is separable. If $v < 0$ it becomes

$$v' = -g + \frac{kv^2}{m}.$$

Ball is dropped from a high point. Then $v < 0$.
The equation is

$$v' = -g + \frac{kv^2}{m}.$$

Scale variables to make equations simpler.

$$v = \alpha w \quad \text{and} \quad t = \beta s.$$

Equation becomes

$$\frac{dw}{ds} = -1 + w^2.$$

The solution is

$$w(s) = -\frac{1 - Ae^{-2s}}{1 + Ae^{-2s}}.$$

In terms of t and v

$$v(t) = -\sqrt{\frac{mg}{k}}\frac{1 - Ae^{-2t\sqrt{kg/m}}}{1 + Ae^{-2t\sqrt{kg/m}}}.$$

The terminal velocity is

$$v_{\text{term}} = -\sqrt{mg/k}.$$

Linear Equations

$$x' = a(t)x + f(t)$$

Homogeneous if $f = 0$, $\qquad x' = a(t)x$. The homogeneous linear equation is separable.

$$\frac{dx}{dt} = a(t)x \quad \text{or} \quad \frac{dx}{x} = a(t)\,dt$$

$$\ln|x(t)| = \int a(t)\,dt$$

$$x(t) = Ae^{\int a(t)\,dt}$$

Example: $x' = \tan(t)x$.

$$\int \tan(t)\,dt = -\ln(\cos(t))$$

$$x(t) = \frac{A}{\cos t} = A\sec t$$