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Math 211

Lecture #4

September 7, 2000

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Linear Equations

$$x' = a(t)x + f(t)$$

Homogeneous if $f = 0$, $x' = a(t)x$. The homogeneous linear equation is separable.

$$\frac{dx}{dt} = a(t)x$$

Solution

$$x(t) = Ae^{\int a(t)\,dt}$$

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Inhomogeneous Equation

Example $x' = \tan(t)x + 3\sin^2(t)$

Rewrite as $x' - \tan(t)x = 3\sin^2(t)$

Multiply by $\cos t$.

$$\cos(t)x' - \sin(t)x = 3\sin^2(t)\cos(t)$$

The left hand side is the derivative of $\cos(t)x$. So

$$\{\cos(t)x\}' = 3\sin^2(t)\cos(t)$$

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Integrate

$$\cos(t)x(t) = 3\int \sin^2(t)\cos(t)\,dt = \sin^3(t) + C$$

Solve for x

$$x(t) = \frac{\sin^3(t) + C}{\cos(t)}.$$

How did we do that? Can we do it in general?

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Solving the linear equation $x' = a(t)x + f(t)$.

- Rewrite
$$x' - ax = f.$$
- Multiply by the integrating factor $u(t) = e^{-\int a(t)\,dt}$. Makes the LHS an exact derivative $(ux)' = ux' - aux = uf$.
- Integrate: $u(t)x(t) = \int u(t)f(t)\,dt + C$.
- Solve for $x(t)$.

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Examples

$x' = -4x + 5$.
$x' = 2tx + 4t$.
$y' = 3y - t$.
$z' = e^{-t}z$.

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Mixing Problems

A tank originally holds 500 gallons of pure water. At $t = 0$ there starts a flow of sugar water into the tank with a concentration of $\frac{1}{2}$ lbs/gal at a rate of 5 gal/min. There is also a pipe at the bottom of the tank removing 5 gal/min from the tank. Assume that the sugar is immediately and thoroughly mixed throughout the tank.

Find the amount of sugar in the tank after 10 minutes and after 2 hours.

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Balance Law

Rate of change = Rate in - Rate out

$$\text{Rate in} = 5\frac{\text{gal}}{\text{min}} \times \frac{1}{2}\frac{\text{lb}}{\text{gal}} = 2.5\frac{\text{lb}}{\text{min}}$$

$$\text{Rate out} = 5\frac{\text{gal}}{\text{min}} \times \frac{S\,\text{lb}}{500\text{gal}} = \frac{S}{100}\frac{\text{lb}}{\text{min}}$$

$$\frac{dS}{dt} = 2.5 - \frac{S}{100}.$$

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Other possible initial conditions:

- There is initially 20 lbs of sugar in the tank.
- The concentration of sugar in the tank at $t = 0$ is 1 lb/gallon.

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Mixing Problems

A tank originally holds 500 gallons of sugar water with a concentration of $\frac{1}{10}$ lb/gal. At $t = 0$ there starts a flow of sugar water into the tank with a concentration of $\frac{1}{2}$ lbs/gal at a rate of 5 gal/min. There is also a pipe at the bottom of the tank removing 10 gal/min from the tank. Assume that the sugar is immediately and thoroughly mixed throughout the tank.

Find the amount of sugar in the tank after 10 minutes and after 2 hours.