Math 211

Lecture #5

September 12, 2000

Existence of Solutions

- Initial value problem:

$$\sin(t)y' = \cos(t)y + \sin^2(t) \quad \text{with} \quad y(0) = 1.$$

- Every solution to the differential equation has the form

$$y(t) = t\sin t + C\sin t.$$

- Hence $y(0) = 0$ for every solution. The IVP with $y(0) = 1$ has *no solution.*

Existence of Solutions

- Put the equation $\sin(t)y' = \cos(t)y + \sin^2(t)$ into normal form

$$y' = \frac{\cos t}{\sin t}y + \sin t.$$

- The RHS is discontinuous at $t = 0$.

- If we require the RHS to be continuous there is always a solution to an initial value problem.

Existence Theorem

Suppose the function $f(t, y)$ is defined and continuous in the rectangle R in the ty-plane. Then given any point $(t_0, y_0) \in R$, the initial value problem

$$y' = f(t, y) \quad \text{with} \quad y(t_0) = y_0$$

has a solution $y(t)$ defined in an interval containing t_0. Furthermore the solution will be defined at least until the solution curve $t \longrightarrow (t, y(t))$ leaves the rectangle R.

Interval of Existence

- Example: $y' = 1 + y^2 \quad$ with $\quad y(0) = 0$.
- RHS $f(t, y) = 1 + y^2$ is defined and continuous on the whole ty-plane. The rectangle R can be any rectangle in the plane.
- Solution $y(t) = \tan t \qquad$ "blows up" at $t = \pm\pi/2$.
- Thus the size of the rectangle on which $f(t, y)$ is continuous does not say much about the interval of existence.

Uniqueness of Solutions

- The uniqueness of solutions to an initial value problem is the mathematical equivalent of being able to predict results in science and engineering.
- We will need slightly stronger restrictions to ensure uniqueness than we needed for existence.

Uniqueness of Solutions

- Initial value problem

$$y' = y^{1/3} \quad \text{with} \quad y(0) = 0.$$

- The constant function $y_1(t) = 0$ is a solution.
- Solve by separation of variables to find that

$$y_2(t) = \begin{cases} \left(\frac{2t}{3}\right)^{3/2} & \text{, if } t > 0 \\ = 0 & \text{, if } t \le 0. \end{cases}$$

is also a solution.

Theorem

Suppose $f(t,y)$, $\partial f/\partial y$ are continuous in the rectangle R. Let

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$$M = \max_{(t,y)\in R} \left| \frac{\partial f}{\partial y}(t,y) \right| .$$

Suppose that (t_0, x_0) and (t_0, y_0) both lie in R, and

$$x' = f(t,x),\ x(t_0) = x_0 \ \&\ y' = f(t,y),\ y(t_0) = y_0.$$

Then as long as $(t, x(t))$ and $(t, y(t))$ stay in R we have

$$|x(t) - y(t)| \le |x_0 - y_0| e^{M|t-t_0|}.$$

Uniqueness Theorem

Suppose the function $f(t, y)$ and its partial derivative $\partial f/\partial y$ are continuous in the rectangle R in the ty-plane. Suppose that $(t_0, x_0) \in R$. Suppose that

$$x' = f(t, x) \quad \text{and} \quad y' = f(t, y),$$

and that

$$x(t_0) = y(t_0) = x_0.$$

Then as long as $(t, x(t))$ and $(t, y(t))$ stay in R we have

$$x(t) = y(t).$$

Geometric Interpretation

- Solution curves cannot cross.
- They cannot even touch at one point.
- $y' = (y-1)(\cos t - y)$ and $y(0) = 2$. Show $y(t) > 1$ for all t.
- $y' = y - (1-t)^2$ and $y(0) = 0$. Show that $y(t) < 1 + t^2$ for all t.

Existence and Uniqueness for Linear Equations

$$y' = a(t)y + g(t) \quad \text{with} \quad y(t_0) = y_0$$

Suppose that a and g are continuous on an interval $I = (a, b)$. Then

$$f(t, y) = a(t)y + g(t) \quad \text{and} \quad \frac{\partial f}{\partial y} = a(t)$$

are continuous for $t \in I$ and all y.

- Given $t_0 \in I$ and any y_0, there is a unique solution $y(t)$ which exists for all $t \in I$.

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Get a geometric look at existence and uniqueness.

Continuity in Initial Conditions

Back to Theorem

$$|x(t) - y(t)| \leq |x_0 - y_0| e^{M|t-t_0|}.$$

- The good news:
  - ◇ By making sure that x_0 and y_0 are very close we can make the solutions $x(t)$ and $y(t)$ close for t in an interval containing t_0.
  - ◇ Solutions are *continuous in the initial conditions.*

Sensitivity with Respect to Initial Conditions

Back to Theorem

$$|x(t) - y(t)| \le |x_0 - y_0| e^{M|t-t_0|}.$$

- The bad news:
  - As $|t - t_0|$ increases the RHS grows exponentially.
  - Over long intervals in t the solutions can get very far apart. Solutions are *sensitive to initial conditions.*

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Target practice with the equation
$x' = x\cos x + t\sin t$.

Try to hit $(4, -5)$, starting at $t = 0$.

Use window [0,4] X [-8,0].