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Math 211

Lecture #6

September 14, 2000

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Qualitative Analysis

- Ways to discover the properties of solutions without solving the equation.
- Works best with autonomous equations

$$y' = f(y)$$

- Example

$$y' = \sin y$$

  - ◇ Go to dfield

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Properties of Autonomous Equations

- The direction field does not depend on t
- Solution curves can be translated left and right to get other solution curves.
  - ◇ If $y(t)$ is a solution, so is $y_1 = y(t + c)$ for any constant c.

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Equilibrium Points & Solutions

$$y' = f(y) \qquad y' = \sin y$$

- Equilibrium point: $f(y_0) = 0$.
- Equilibrium solution: $y(t) = y_0$.
- $\sin y = 0 \quad \Leftrightarrow \quad y = k\pi, \quad k = 0, \pm 1, \ldots$
- $y' = \sin y$ has infinitely many equilibrium points and solutions:

$$y_k(t) = k\pi \quad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

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Between the Equilibrium Points

If $0 < y < \pi$, then $\sin y > 0$.
Thus $y'(t) = \sin y(t) > 0$, so $y(t)$ is increasing if $0 < y(t) < \pi$
By uniqueness, $y(t) < \pi$ for all t.
$\Rightarrow y(t) \nearrow \pi$ as $t \to \infty$
$\Rightarrow y(t) \searrow 0$ as $t \to -\infty$

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Between the Equilibrium Points

If $-\pi < y < 0$, then $\sin y < 0$.
Thus $y'(t) = \sin y(t) < 0$, so $y(t)$ is decreasing if $-\pi < y(t) < 0$
By uniqueness, $y(t) > -\pi$ for all t.
$\Rightarrow y(t) \searrow -\pi$ as $t \to \infty$
$\Rightarrow y(t) \nearrow 0$ as $t \to -\infty$

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Stable & Unstable EPs

An equilibrium point y_0 is

- asymptotically stable if all solutions starting near y_0 converge to y_0 as $t \to \infty$.
- unstable if there are solutions starting arbitrarily close to y_0 which move away from y_0 as t increases.
- There are 4 possibilities:

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The Phase Line for $y' = f(y)$

- The phase line is a y-axis, showing
  - ◇ the equilibrium points and
  - ◇ the direction of the flow between the equilibrium points.
- The y-axis in the plot of $y \to f(y)$.
- The y-axis in the ty-plane where solutions are plotted.

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Terminal Velocity

- Magnitude of the resistance proportional to the square of the velocity:

$$v' = -g - k|v|v/m$$

- One equilibrium point at

$$v_{\text{term}} = -\sqrt{\frac{mg}{k}}.$$

- v_{term} is asymptotically stable.

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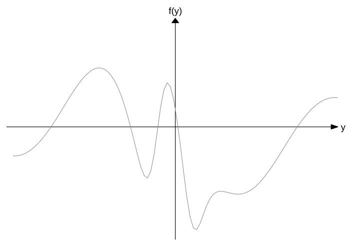
Qualitative Analysis of $y' = f(y)$.

1. Graph $y \to f(y)$.



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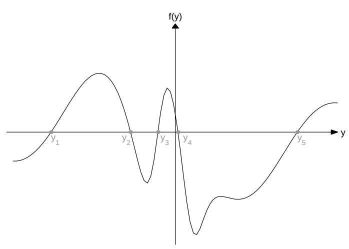
Qualitative Analysis of $y' = f(y)$.

2. Find the equilibrium points where $f(y) = 0$.



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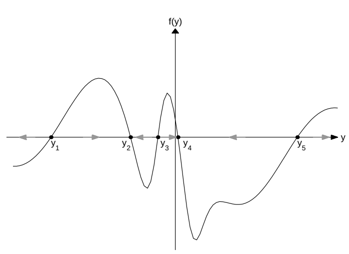
Qualitative Analysis of $y' = f(y)$.

3. Determine the behavior between eq. pts.



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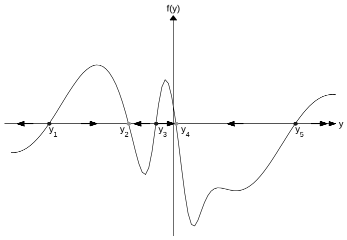
Qualitative Analysis of $y' = f(y)$.

4. Analyze the equilibrium points.



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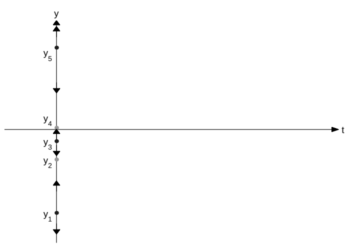
Qualitative Analysis of $y' = f(y)$.

5. Transfer the phase line to ty-space.



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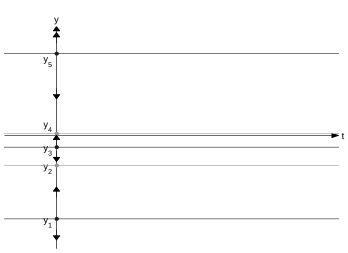
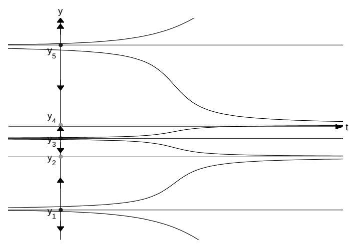
Qualitative Analysis of $y' = f(y)$.

6. Plot the equilibrium solutions.



Qualitative Analysis of $y' = f(y)$.

7. Plot other solutions approximately.



Modeling Population

Assume population changes due to births and deaths. Births are roughly proportional to population

$B = bP$ b is the *birth rate*

Deaths are roughly proportional to population

$D = dP$ d is the *death rate*

Modeling Population

Hence

$$\begin{aligned}\frac{dP}{dt} &= B - D\\ &= bP - dP\\ &= (b-d)P\\ &= rP\end{aligned}$$

$r = b - d$ is the *reproductive rate*.

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The Malthusian Model

- Birth and death rates are not necessarily constants, but can depend on P, and perhaps also on t.
- If there exist sufficient resources in term of nutrients and space, the birth and death rates will be almost constant. Then the reproductive rate $r = b - d$ is also a constant.
- This is the *Malthusian model*.

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The Malthusian Model

$$\frac{dP}{dt} = rP \quad \text{with} \quad P(0) = P_0$$

Solution:

$$P(t) = P_0 e^{rt}$$

- If $r = b - d > 0$ the population $P(t)$ grows exponentially.
- If $r = b - d < 0$ the population $P(t)$ decays exponentially.

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The Malthusian Model

Under what circumstances could the Malthusian model be a good model?

- Requires unlimited resources.
- Laboratory experiments with small populations.
- Populations always outgrow the Malthusian model. This was the point that was made by Malthus.

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The Logistic Model

- As the population increases individuals compete for resources — for food and for space.
- This causes the birth rate b to decrease , and the death rate d to increase.

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The Logistic Model

- The birth rate $b =$ probability of an individual producing offspring in a fixed period of time.
- As P increases, b decreases because of competition between individuals. Competition between individuals is a result of encounters between individuals. The number of encounters by one individual is roughly proportional to P.
- Hence the decrease in the birth rate is proportional to P, or

$$b = b_0 - b_1 P$$

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The Logistic Model

- Similarly, the increase in d is proportional to P

$$d = d_0 + d_1 P.$$

- The reproductive rate is

$$\begin{aligned} r &= b - d \\ &= (b_0 - b_1 P) - (d_0 + d_1 P) \\ &= (b_0 - d_0) - (b_1 + d_1) P \\ &= r_0 - r_1 P \end{aligned}$$

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The Logistic Model

$$\begin{aligned}\frac{dP}{dt} &= rP \\ &= (r_0 - r_1 P)P \\ &= r_0\left(1 - \frac{r_1}{r_0}P\right)P \\ &= r_0\left(1 - \frac{P}{K}\right)P\end{aligned}$$

r_0 is the *reproductive rate at small populations.*
$K = r_0/r_1$ is the *carrying capacity.*

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The Logistic Model

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P$$

- Equation is autonomous.
- Equilibrium points are 0 & K.
- 0 is unstable, K is stable.
- $P(t) \to K$ as $t \to \infty$.

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The Logistic Model

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P \quad \text{with} \quad P(0) = P_0$$

- Solution:

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$$